Formalisms for Describing Dissipation

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Given,

The Dictum: Nature is Hamiltonian,

is there a 'platonic ideal' for dissipation?

Overview

1. Rayleigh Dissipation Function

2. Cahn-Hilliard Equation

3. Caldiera-Leggett Model

4. Metriplectic Dynamics

    - incomplete (Brockett, Vallis et al. (1989))

    - complete (PJM, Kaufman, Grmela (1985))

Rayleigh Dissipation Function

Introduced for study of vibrations, stable linear oscillations, in 1873 (see e.g. Rayleigh, Theory of Sound, Chap. IV §81)

Linear friction law for n-bodies, $\mathbf{F}_i = -b_i(\mathbf{r}_i)\mathbf{v}_i$, with $\mathbf{r}_i \in \mathbb{R}^3$. Rayleigh was interested in linear vibrations, $\mathcal{F} = \sum_i b_i \,||\mathbf{v}_i||^2/2$.

Coordinates $\mathbf{r}_i \to q_\nu$ etc. $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$$

Ad hoc, phenomenological, yet is generalizable, geometrizable (e.g. Bloch et al.,...)

Cahn-Hilliard Equation

Models phase separation, nonlinear diffusive dissipation, in binary fluid with 'concentrations' n, $n = 1$ one kind $n = -1$ the other

$$\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 \left(n^3 - n - \nabla^2 n \right)$$

Lyapunov Functional

$$F[n] = \int d^3x \left[\frac{1}{4} \left(n^2 - 1 \right)^2 + \frac{1}{2} |\nabla n|^2 \right]$$

$$\frac{dF}{dt} = \int d^3x \frac{\delta F}{\delta n} \frac{\partial n}{\partial t} = \int d^3x \frac{\delta F}{\delta n} \nabla^2 \frac{\delta F}{\delta n} = -\int d^3x \left| \nabla \frac{\delta F}{\delta n} \right|^2 \leq 0$$

For example in 1D

$$\lim_{t \to \infty} n(x,t) = \tanh(x/\sqrt{2})$$

Ad hoc, phenomenological, yet generalizable and very important (Otto, Ricci Flows, Poincarè conjecture on S^3, ...)

Whence Dissipation?

- Low degree-of-freedom system coupled to 'high' degree-of-freedom system? Energy transfer or entropy production.

- Combined system Hamiltonian?

Caldiera-Leggett Model

Quantum dissipation (1981) by coupling to 'bath'

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\dot{Q}^2 - \left(\Omega^2 - \Delta\Omega^2\right) Q^2\right) - Q \sum_{i=1}^{N} f_i q_i + \sum_{i=1}^{N} \frac{1}{2}\left(\dot{q}^2 - \omega_i^2 q_i^2\right)}$$

with $N >> 1$ and $\Delta\Omega^2 := \sum_{i=1}^{N} f_i^2/\omega_i^2$.

Coupling:

$$\ddot{Q} + \left(\Omega^2 - \Delta\Omega^2\right) Q = -\sum_{i=1}^{N} f_i q_i$$

Solve q_i-equation via Green's function:

$$\ddot{Q} + \left(\Omega^2 - \Delta\Omega^2\right) Q = -\int_{-\infty}^{t} d\tau\, \mathcal{G}(t-\tau) Q(\tau)$$

$$\mathcal{G} = \sum_{i=1}^{N} \frac{f_i^2}{\omega_i^2} \sin(\omega_i t)$$

Continuum Limit:

$$\mathcal{G}(t) = \frac{2}{\pi} \int_0^{\infty} d\omega\, \mathcal{N}(\omega) \sin(\omega t) \qquad \longrightarrow \qquad \gamma \dot{Q} - \underline{\text{damping!}}$$

Hamiltonian Continuum Caldiera-Leggett Model

Hamiltonian:

$$H_{CCL}[q,p;Q,P] = \frac{\Omega}{2}P^2 + \frac{1}{2}\left(\Omega + \int_{\mathbb{R}_+} dx\, \frac{f(x)^2}{2x}\right) Q^2 \\ + \int_{\mathbb{R}_+} dx\, Q q(x) f(x) + \left[\frac{x}{2}\left(p(x)^2 + q(x)^2\right)\right] ,$$

Poisson bracket:

$$\{A,B\} = \frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial B}{\partial Q}\frac{\partial A}{\partial P} + \int_{\mathbb{R}_+} dx \left(\frac{\delta A}{\delta q}\frac{\delta B}{\delta p} - \frac{\delta A}{\delta p}\frac{\delta B}{\delta q}\right)$$

Generates system with a continuous spectrum (cf. singularity vs. infinite system size - radiation (Bloch e.g.))

Vlasov-Poisson System

Phase space density (1 + 1 + 1 field theory):

$$f(x, v, t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \phi[x, t; f]}{\partial x}\frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\phi_{xx} = 4\pi \left[e \int_{\mathbb{R}} f(x, v, t)\, dv - \rho_B \right]$$

Energy:

$$H = \frac{m}{2} \int_{\Pi} \int_{\mathbb{R}} v^2 f\, dx dv + \frac{1}{8\pi} \int_{\Pi} (\phi_x)^2\, dx$$

Noncanonical Hamiltonian Structure

Hamiltonian structure of media in Eulerian variables

Kinematic Commonality:

energy, momentum, Casimir conservation; dynamics is measure preserving rearrangement; continuous spectra; ... $\longrightarrow$ Krein's theorem

Noncanonical Poisson Bracket (K-K,L-P):

$$\{F,G\} = \int_{\mathcal{Z}} \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right] dqdp = \int_{\mathcal{Z}} \frac{\delta F}{\delta \zeta} \mathcal{J} \frac{\delta G}{\delta \zeta} dqdp$$

Cosymplectic Operator:

$$\mathcal{J}\ \cdot = -\left(\frac{\partial \zeta}{\partial q}\frac{\partial\ \cdot}{\partial p} - \frac{\partial\ \cdot}{\partial q}\frac{\partial \zeta}{\partial p}\right)$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = \mathcal{J}\frac{\delta H}{\delta \zeta} = -[\zeta, \mathcal{E}].$$

Organizing principle. Do one do all!

Linear Vlasov-Poisson System

Expand about Stable Homogeneous Equilibrium:

$$f = f_0(v) + \delta f(x,v,t)$$

Linearized EOM:

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{e}{m}\frac{\partial \delta\phi[x,t;\delta f]}{\partial x}\frac{\partial f_0}{\partial v} = 0$$

$$\delta\phi_{xx} = 4\pi e \int_{\mathbb{R}} \delta f(x,v,t)\,dv$$

Linearized Energy (Kruskal-Oberman):

$$H_L = -\frac{m}{2}\int_{\Pi}\int_{\mathbb{R}} \frac{v\,(\delta f)^2}{f_0'}\,dv dx + \frac{1}{8\pi}\int_{\Pi}(\delta\phi_x)^2\,dx$$

Linear Hamiltonian Structure

• Because noncanonical must expand f-dependent Poisson bracket as well as Hamiltonian. $\Rightarrow$

Linear Poisson Bracket:

$$\{F,G\}_L = \int f_0 \left[\frac{\delta F}{\delta\delta f}, \frac{\delta G}{\delta\delta f}\right]\, dx\, dv \ ,$$

where δf is the new dynamical variable and the Hamiltonian is the Kruskal-Oberman energy, H_L. The LVP system has the following Hamiltonian form:

$$\frac{\partial\delta f}{\partial t} = \{\delta f, H_L\}_L \, ,$$

with variables <u>noncanonical</u> and H_L <u>not diagonal</u>.

Linear Solution

Assume

$$\delta f = \sum_k f_k(v,t) e^{ikx}\,, \qquad \delta\phi = \sum_k \phi_k(t) e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikv f_k + ik\phi_k \frac{e}{m}\frac{\partial f_0}{\partial v} = 0\,, \qquad k^2\phi_k = -4\pi e \int_{\mathbb{R}} f_k(v,t)\, dv$$

Three methods:

1. Laplace Transforms (Landau and others 1946)
2. Normal Modes (Van Kampen, Case,... 1955)
3. Coordinate Change $\Longleftrightarrow$ Integral Transform (PJM, Pfirsch, Shadwick, Balmforth 1992)

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}} \right) dv$$

Linear Hamiltonian:

$$\begin{aligned} H_L &= -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f_0'} |f_k|^2 \, dv \;+\; \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2 \\ &= \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v)\, \mathcal{O}_{k,k'}(v|v')\, f_{k'}(v')\, dv dv' \end{aligned}$$

Canonization:

$$q_k(v,t) = f_k(v,t)\,, \qquad p_k(v,t) = \frac{m}{ikf_0'} f_{-k}(v,t) \qquad \Longrightarrow$$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k} \right) dv$$

Dynamical Accessibility

Definition *A phase space function* k *is* *dynamically accessible* *from a phase space function* h*, if* g *is an area-preserving rearrangement of* h *; i.e., in coordinates* $k(x, v) = h(X(x, v), V(x, v))$*, where* $[X, V] = 1$*. A perturbation* δh *is* *linearly dynamically accessible* *from* h *if* $\delta h = [G, h]$*, where* G *is the infinitesimal generator of the canonical transformation* $(x, v) \leftrightarrow (X, V)$*.*

Remark Dynamically accessible perturbations come about by perturbing the particle orbits under the action of some Hamiltonian; hence, dynamically accessible. For VP $\delta f = G_x f_0'$.

Lemma *Continuous rearrangements preserve the 'topology' of level sets.*

Integral Transform

Definintion:

$$\boxed{f(v) = \mathcal{G}[g](v) \;\; := \varepsilon_R(v)\, g(v) + \varepsilon_I(v)\, H[g](v)\,,}$$

where

$$\varepsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v}\,, \qquad\qquad \varepsilon_R(v) = 1 + H[\varepsilon_I](v)\,,$$

and the Hilbert transform

$$H[g](v) := \frac{1}{\pi} ⨍ \frac{g(u)}{u - v}\, du\,,$$

with $⨍$ denoting Cauchy principal value of $\int_{\mathbb{R}}$.

Transform Properties

Theorem (G1) *$\mathcal{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator; i.e.*

$$\|\mathcal{G}[g]\|_p \leq B_p \, \|g\|_p \,,$$

where B_p depends only on p.

Theorem (G2) *If $f_0' \in L^q(\mathbb{R})$, stable, Hölder decay, then $\mathcal{G}[g]$ has a bounded inverse,*

$$\mathcal{G}^{-1}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \mathcal{G}^{-1}[f](u) \\ &:= \frac{\varepsilon_R(u)}{|\varepsilon(u)|^2} \, f(u) - \frac{\varepsilon_I(u)}{|\varepsilon(u)|^2} \, H[f](u) \,. \end{aligned}$$

where $|\varepsilon|^2 := \varepsilon_R^2 + \varepsilon_I^2$.

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, \mathcal{G}[P'_k](v)\, dv$$

Canonical Coordinate changes $(q,p) \longleftrightarrow (Q', P')$:

$$p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = \mathcal{G}[P_k](v)\,, \qquad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = \mathcal{G}^{\dagger}[q_k](u)$$

New Hamiltonian:

$$H_L = \tfrac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u) \omega_k(u) \left[Q_k^2(u) + P_k^2(u) \right]$$

where $\sigma_k = -\mathrm{sgn}(u f'_0)$ and $\omega_k(u) = |ku|$

$(Q', P') \longleftrightarrow (Q, P)$ is trivial.

Note, $\sigma = 1$ for Landau problem.

Landau Damping

Landau damping is the Riemann-Lebesgue lemma

$$\lim_{t \to \infty} \rho_k(t) = \lim_{t \to \infty} \int dv\, \widehat{f}_k(v) e^{ikvt} = 0$$

Charge density $\rho_k(t)$ decays if $\widehat{f}_k \in L^1(\mathbb{R})$. If $\widehat{f}_k$ meromorphic (C^ω in strip containing $\mathbb{R}$) then exponential decay.

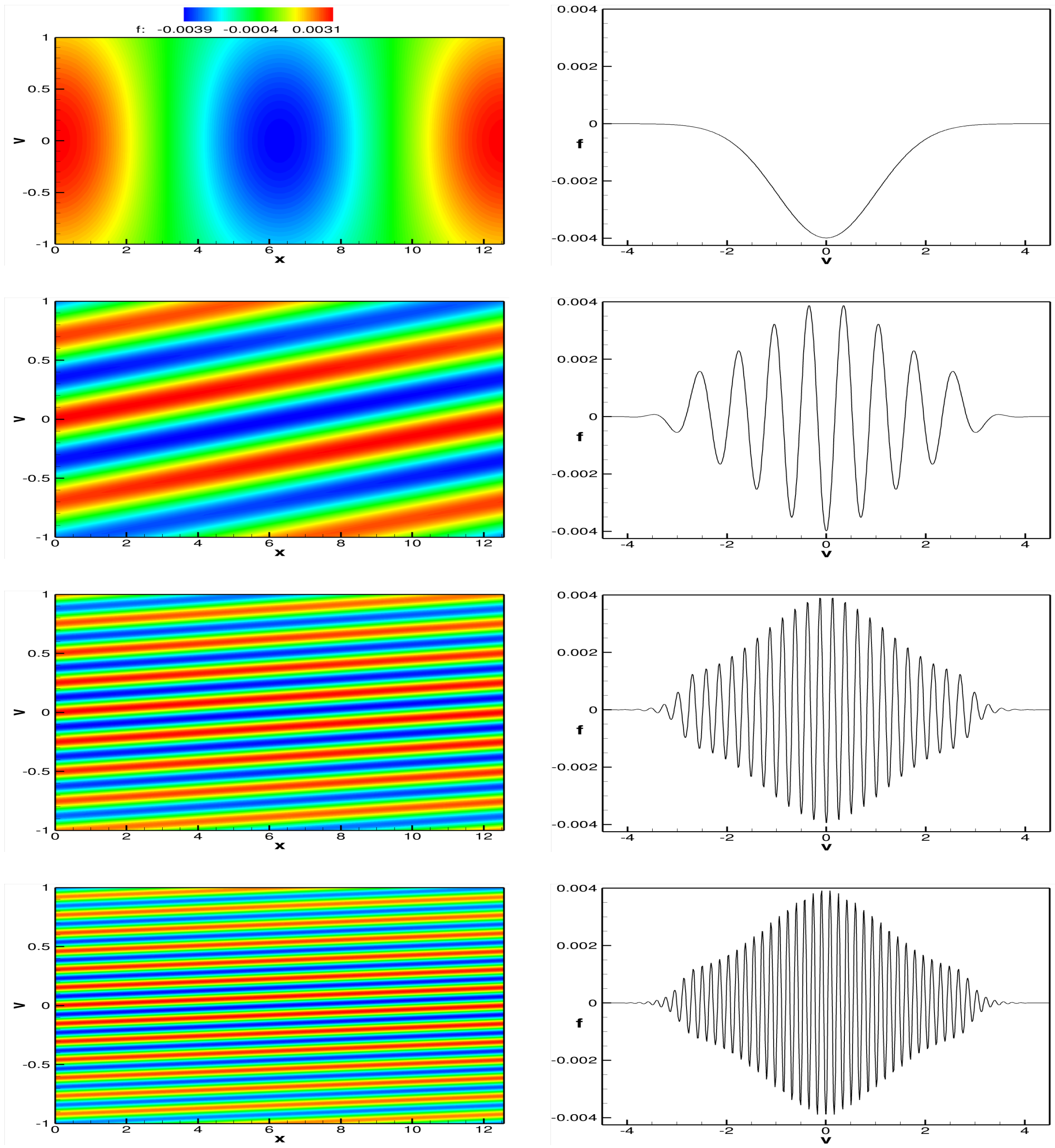

Fig. 3. (Linear Landau damping with Maxwell equilibrium) Contour plots (*left*) and cross-sectional plots (*right*), $x = 2\pi$, for δf at $t = 0$, $t = 25$, $t = 50$, $t = 75$

DG code developed with I. Gamba, et al. (2010)

Equivalent Normal Forms (with G. Hagstrom)

$$T(\text{Vlasov} - \text{Poisson}) \longrightarrow H_{VP} = \frac{1}{2}\int du\, u\left(P^2 + Q^2\right)$$

$$S(\text{Caldiera} - \text{Leggett}) \longrightarrow H_{CL} = \frac{1}{2}\int dx\, x\left(P^2 + Q^2\right)$$

Therefore

$$S(\text{Caldiera} - \text{Leggett}) = T(\text{Vlasov} - \text{Poisson})$$

$\Rightarrow$

$$(\text{Caldiera} - \text{Leggett}) = S^{-1} \circ T(\text{Vlasov} - \text{Poisson})$$

Krein-Moser (Sturrock)

Theorem (KMS) *Let H define a stable linear finite-dimensional Hamiltonian system. Then H is structurally stable if all the eigen-frequencies are nondegenerate. If there are any degeneracies, H is structurally stable if the assosciated eigenmodes have energy of the same sign. Otherwise H is structurally unstable.*

Definition The signature of the point $u \in \mathbb{R}$ is $-\mathrm{sgn}(uf_0'(u))$.

(Generalization of with G. Hagstrom)

Hamiltonian Spectrum

Hamiltonian Operator:

$$f_{kt} = -ikvf_k + \frac{if_0'}{k}\int_{\mathbb{R}} d\bar{v}\, f_k(\bar{v},t) =: -T_k f_k\,,$$

Complete System:

$$f_{kt} = -T_k f_k \qquad \text{and} \qquad f_{-kt} = -T_{-k} f_{-k}\,, \qquad\qquad k \in \mathbb{R}^+$$

Lemma *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f_0'(v)$, then so are $-\lambda$ and λ^* . Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Spectral Stability

Definition The dynamics of a Hamiltonian system linearized around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is spectrally stable if the spectrum $\sigma(T)$ of the time evolution operator T is purely imaginary.

Theorem *If for some $k \in \mathbb{R}^+$ and $u = \omega/k$ in the upper half plane the plasma dispersion relation*

$$\varepsilon(k,u) := 1 - k^{-2} \int_{\mathbb{R}} dv \frac{f_0'}{u-v} = 0\,,$$

then the system with equilibrium f_0 is spectrally unstable. Otherwise it is spectrally stable.

Theorem (Penrose) *If there exists a point u such that*

$$f_0'(u) = 0 \qquad \text{and} \qquad \fint dv\, \frac{f_0'(v)}{u-v} < 0\,,$$

with f_0' traversing zero at u , then the system is spectrally unstable. Otherwise it is spectrally stable.

Spectral Theorem

Set $k = 1$ and consider $T\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_{\mathbb{R}} dv(|f| + |f'|)$. Contains all functions in $L^1(\mathbb{R})$ with weak derivatives in $L^1(\mathbb{R})$. T is densely defined, closed, etc.

Definition Resolvent of T is $R(T, \lambda) = (T - \lambda I)^{-1}$ and $\lambda \in \sigma(T)$. (i) λ in point spectrum, $\sigma_p(T)$, if $R(T, \lambda)$ not injective. (ii) λ in residual spectrum, $\sigma_r(T)$, if $R(T, \lambda)$ exists but not densely defined. (iii) λ in continuous spectrum, $\sigma_c(T)$, if $R(T, \lambda)$ exists, densely defined but not bounded.

Theorem *Let* $\lambda = iu$. *(i)* $\sigma_p(T)$ *consists of all points* $iu \in \mathbb{C}$, *where* $\varepsilon = 1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(u - v) = 0$. *(ii)* $\sigma_c(T)$ *consists of all* $\lambda = iu$ *with* $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$. *(iii)* $\sigma_r(T)$ *contains all the points* $\lambda = iu$ *in the complement of* $\sigma_p(T)$ *that satisfy* $f_0'(u) = 0$.

cf. e.g. P. Degond (1986). Similar but different.

Structural Stability

Definition Consider an equilibrium solution of a Hamiltonian system and the corresponding time evolution operator T for the linearized dynamics. Let the phase space for the linearized dynamics be some Banach space $\mathcal{B}$. Suppose that T is spectrally stable. Consider perturbations δT of T and define a norm on the space of such perturbations. Then we say that the equilibrium is <u>structurally stable</u> under this norm if there is some $\delta > 0$ such that for every $\|\delta T\| < \delta$ the operator $T + \delta T$ is spectrally stable. Otherwise the system is <u>structurally unstable</u>.

Definition Consider the formulation of the linearized Vlasov-Poisson equation in the Banach space $W^{1,1}(\mathbb{R})$ with a spectrally stable homogeneous equilibrium function f_0. Let $T_{f_0+\delta f_0}$ be the time evolution operator corresponding to the linearized dynamics around the distribution function $f_0 + \delta f_0$. If there exists some ϵ depending only on f_0 such that $T_{f_0+\delta f_0}$ is spectrally stable whenever $\|T_{f_0} - T_{f_0+\delta f_0}\| < \epsilon$, then the equilibrium f_0 is structurally stable under perturbations of f_0.

All f_0 are Structurally Unstable in $W^{1,1}$

True in space where Hilbert transform unbounded, e.g. $W^{1,1}$. Small perturbation $\Rightarrow$ big jump in Penrose plot.

Theorem *A stable equilibrium distribution is structurally unstable under perturbations of f_0' in the Banach spaces $W^{1,1}$ and $L^1 \cap C_0$.*

Easy to make 'bumps' in f_0 that are small in norm. What to do?

Krein-Like Theorem for VP

Theorem *Let f_0 be a stable equilibrium distribution function for the Vlasov equation. Then f_0 is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f_0'(v) = 0$. If there are multiple solutions, f_0 is structurally unstable and the unstable modes come from the roots of f_0' that satisfy $f_0''(v) < 0$.*

Remark A change in the signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at <u>all</u> points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.

Incomplete Metriplectic Flow

Calculate stationary states using Eulerian Hamiltonian structure (noncanonical Poisson bracket) with Dirac brackets.

Example 2D Euler

Noncanonical Poisson Brackets :

$$\{F, G\} = \int dxdy\, \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right]$$

ζ = vorticity, $\psi = \triangle^{-1}\zeta$ =streamfunction

$$[f, g] = J(f, g) = f_x g_y - f_y g_x = \frac{\partial(f, g)}{\partial(x, y)}$$

Hamiltonian:

$$H[\zeta] = \frac{1}{2}\int d\mathbf{x}\, v^2 = \frac{1}{2}\int d\mathbf{x}\, |\nabla\psi|^2$$

Equation of Motion:

$$\zeta_t = \{\zeta, H\}$$

PJM (1981) and P. Olver (1982)

Hamiltonian Commonality

Dynamics is Rearrangement:

$$\zeta(x, y, t) = \zeta_0(x_0(x, y, t), y_0(x, y, t))$$

$\Rightarrow$ level set topology conservation and Casimir invariants

Casimir Invariants:

$$\{C, F\} = 0 \;\forall F \;\; \Rightarrow \;\; C[\zeta] = \int d\mathbf{x}\, \mathcal{C}(\zeta)$$

Variational Principle for Equilibria and Stability:

$$\mathcal{F}[\zeta] = H + C = \frac{1}{2}\int d\mathbf{x}\, |\nabla\psi|^2 + \int d\mathbf{x}\, \mathcal{C}(\zeta)$$

. . . , Gardner, Kruskal and Oberman, Arnold, (1960s). . .

Changing Frames:

$$\mathcal{F}_\Omega = \mathcal{F} + \Omega L$$

L = angular momentum, Ω = rotation rate

Simulated Annealing

Good Idea:

Vallis, Carnevale, and Young, Shepherd (1989)

Use bracket dynamics to do extremization $\Rightarrow$ Relaxing Rearrangement

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, H\} + ((\mathcal{F}, H)) = ((\mathcal{F}, \mathcal{F})) \geq 0$$

where

$$((F, G)) = \int d^3x \, \frac{\delta F}{\delta \chi} \mathcal{J}^2 \frac{\delta G}{\delta \chi}$$

Lyapunov function, $\mathcal{F}$, yields asymptotic stability to rearranged equilibrium.

- Maximizing energy at fixed Casimir: Works fine sometimes, but limited to circular vortex states ....

Simulated Annealing with Generalized (Noncanonical) Dirac Brackets

Dirac Bracket:

$$\{F, G\}_D = \{F, G\} + \frac{\{F, C_1\}\{C_2, G\}}{\{C_1, C_2\}} - \frac{\{F, C_2\}\{C_1, G\}}{\{C_1, C_2\}}$$

Preserves any two incipient constraints C_1 and C_2.

New Idea:

Do simulated Annealing with Generalized Dirac Bracket

$$((F, G))_D = \int d\mathbf{x} d\mathbf{x}' \, \{F, \zeta(\mathbf{x})\}_D \; \mathcal{G}(\mathbf{x}, \mathbf{x}') \; \{\zeta(\mathbf{x}'), G\}_D$$

Preserves any Casimirs of $\{F, G\}$ and Dirac constraints $C_{1,2}$

For successful implementation with contour dynamics see PJM (with Flierl) Phys. Plasmas **12** 058102 (2005).

Four Types of Dynamics

$$
\begin{aligned}
\text{Hamiltonian}: \frac{\partial F}{\partial t} &= \{F, \mathcal{F}\} && (1)\\
\text{Hamiltonian Dirac}: \frac{\partial F}{\partial t} &= \{F, \mathcal{F}\}_D && (2)\\
\text{Simulated Annealing}: \frac{\partial F}{\partial t} &= \sigma\{F, \mathcal{F}\} + \alpha((F, \mathcal{F})) && (3)\\
\text{Dirac Simulated Annealing}: \frac{\partial F}{\partial t} &= \sigma\{F, \mathcal{F}\}_D + \alpha((F, \mathcal{F}))_D && (4)
\end{aligned}
$$

F an arbitrary observable, $\mathcal{F}$ generates time advancement. Equations (1) and (2) are ideal and conserve energy. In (3) and (4) parameters σ and α weight ideal and dissipative dynamics: $\sigma \in \{0, 1\}$ and $\alpha \in \{-1, 1\}$. $\mathcal{F}$, can have form

$$\mathcal{F} = H + \sum_i C_i + \lambda^i P_i \,,$$

Cs Casimirs and Ps dynamical invariants.

DSA is Dressed Advection

$$\frac{\partial \zeta}{\partial t} = -[\Psi, \zeta]\,,$$

$$\Psi = \psi + A^i c_i \quad \text{and} \quad A^i = -\frac{\int d\mathbf{x}\, c_j [\psi, \zeta]}{\int d\mathbf{x}\, \zeta [c_i, c_j]}\,.$$

with constraints

$$C_j = \int d\mathbf{x}\, c_j\, \zeta\,.$$

“Advection” of ζ by Ψ, with A^i just right to force constraints.

Easy to adapt existing vortex dynamics codes!!

Examples

Constraints:

$$C_1 = \frac{1}{2}\int d\mathbf{x}\,\zeta(\mathbf{x})(x^2+y^2) = 2 \times \text{angular momentum}$$

$$C_2 = \frac{1}{2}\int d\mathbf{x}\,xy\,\zeta(\mathbf{x})$$

Initial Condition:

$$\zeta_0 = e^{-(r/r_0)^6} \quad \text{where} \quad r_0 = 1 + .4\cos(2\theta)$$

Seven Movies: relaxation to rotating ellipses, relaxation to 3-fold symmetric states, Kelvin sponge, Dirac constrained sponge.

Complete Metriplectic Flow

A dynamical model of thermodynamics that ‘captures’:.

- First Law: conservation of energy

- Second Law: entropy production

Metriplectic Manifold

Two foliations:

- Poisson Manifold

- SubRiemannian Manifold

$$\Big(\mathcal{Z}, [\,,\,], (\,,\,)\Big)$$

use $z = (z^1, z^2, \ldots, z^N)$ for coord patch.

Metriplectic Vector Field:

$$V_{MP} = [\mathcal{F}, \cdot] + (\mathcal{F}, \cdot) = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial}{\partial z^j} + \frac{\partial \mathcal{F}}{\partial z^i} g^{ij} \frac{\partial}{\partial z^j}$$

What are degeneracies? What is 'generator' $\mathcal{F}$?

Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of $[,]$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator: $\mathcal{F} = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = [H, \mathcal{F}] + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0$$

  Foliate $\mathcal{Z}$ by level sets of H with are subRiemannian, i.e. $(H, f) = 0 \ \forall \ f \in C^{\infty}(M)$.

- 2nd Law: entropy production

$$\dot{S} = [S, \mathcal{F}] + (S, \mathcal{F}) = (S, S) \geq 0$$

  Lyapunov relaxation to the equilbrium state: $\nabla \mathcal{F} = 0$.

Examples

- Finite dimensional theories, rigid body, etc.

- Kinetic theories: Boltzmann equation, Lenard-Balescu equation, ...

- Fluid flows: various nonideal fluids, MHD, etc.

‘In Progress’

- Derivation from large system: n-body, $n >> 1$, BBGKY hierarchy, Landau damping mechanism.

- Structure theorems: Kähler generalization, etc.

- Statistical mechanics on Poisson manifold with symplectic leaves in bath contact (with Bouchet, Thalabard, Zaboronski). Liouville's theorem.