A Discontinuous Galerkin Method for Vlasov Systems

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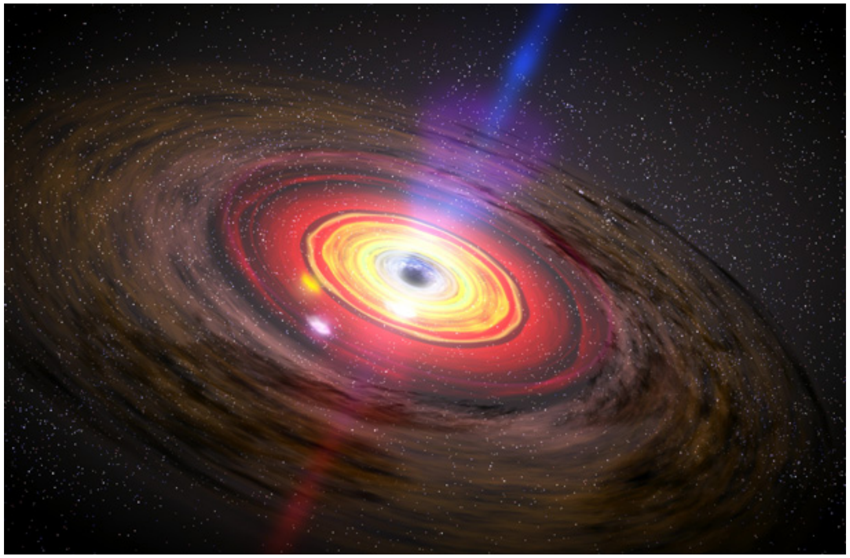
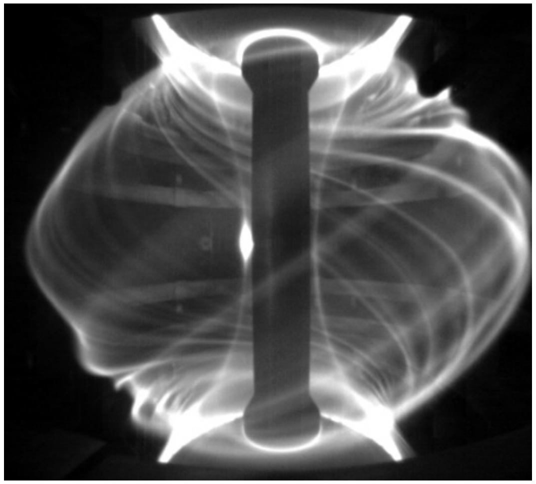
Hot Magnetized Plasmas





Hot Magnetized Plasma

Ionized gas of charged particles where

Hot $\Rightarrow$ collisions not important

Rare collisions i.e. when mean free path is very long

Magnetized $\Rightarrow$ magnetic field important

Gyroradius small compared to other scale lengths

Maxwell-Vlasov System

Vlasov Equation:

$$\frac{\partial f_\alpha(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\Big(\mathbf{E}+\mathbf{v}\times\mathbf{B}\Big)\cdot\nabla_{\mathbf{v}} f_\alpha = \frac{\partial f_\alpha}{\partial t}\Big)_c \approx 0$$

where f is phase space density, $\alpha = e, i$ is species index, and the sources, charge density and current density, are given by

$$\rho(x,t) = \sum_\alpha e_\alpha \int_{I\!R^3} d^3v\ f_\alpha\,, \qquad \mathbf{J}(x,t) = \sum_\alpha e_\alpha \int_{I\!R^3} d^3v\ \mathbf{v} f_\alpha\,,$$

which couple into

Maxwell's Equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\,, \qquad \nabla\cdot\mathbf{B} = 0$$

$$\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B} - \mu_0\mathbf{J}\,, \qquad \epsilon_0\nabla\cdot\mathbf{E} = \rho$$

Vlasov Regularity

Vlasov-Poisson:

- (1952) 3D stellar dynamics. R. Kurth local existence in time.
- (1977) spherical symmetry. J. Batt, global existence.
- (1989) 3D compact support. K. Pfaffelmoser, B. Perthame, J. Schaeffer, smooth global existence.
- ...

Maxwell-Vlasov:

- Open!

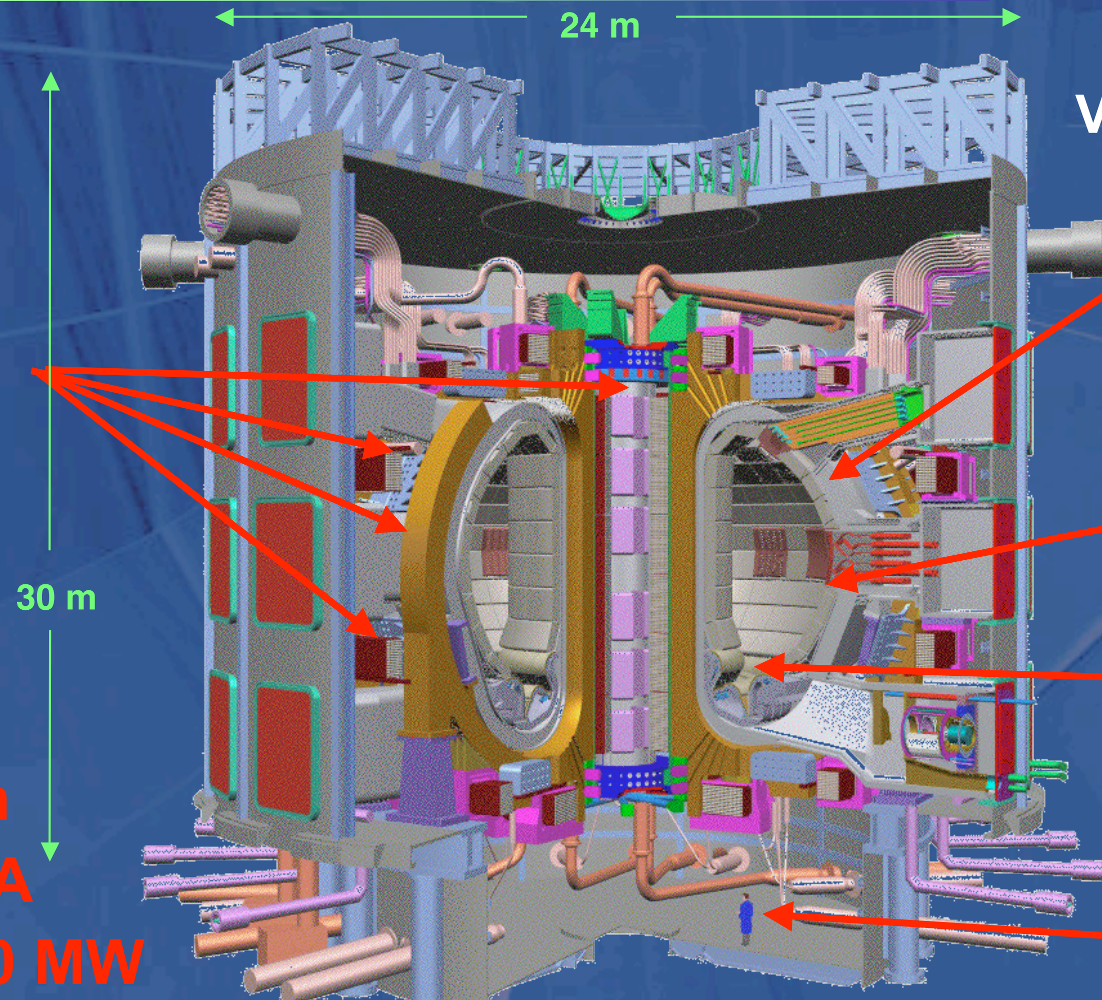
Maxwell- Vlasov Regularity

R. Glassey, J. Schaeffer, ...

"After 40 years we have precious little to show for it."

Computation?

Tokamak

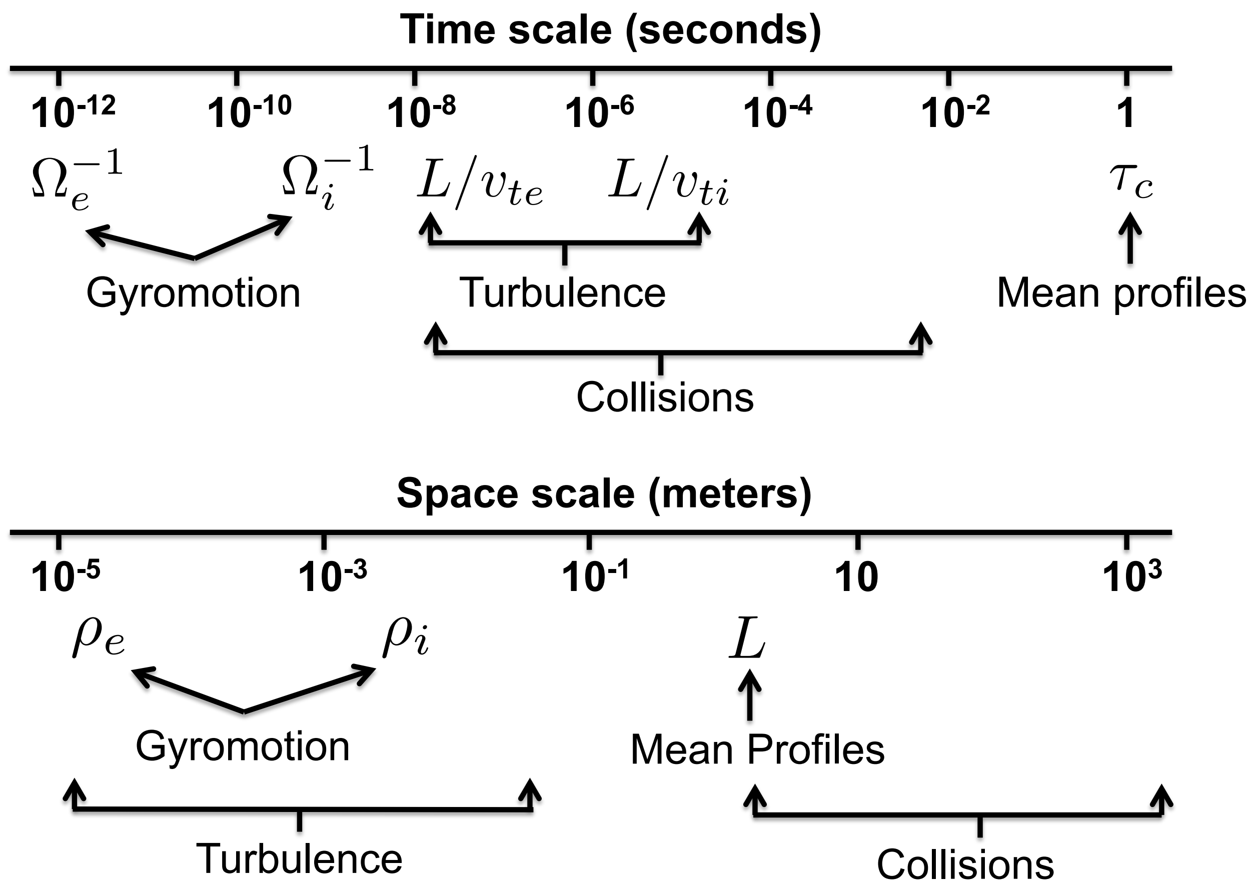


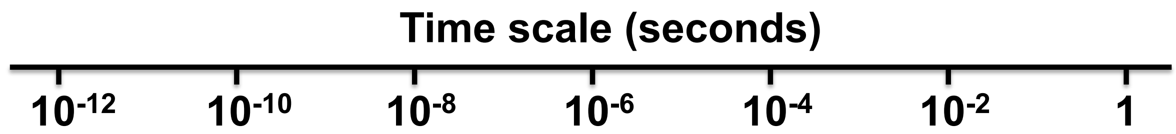
$R=6.2$ m

$I_p=15$ MA

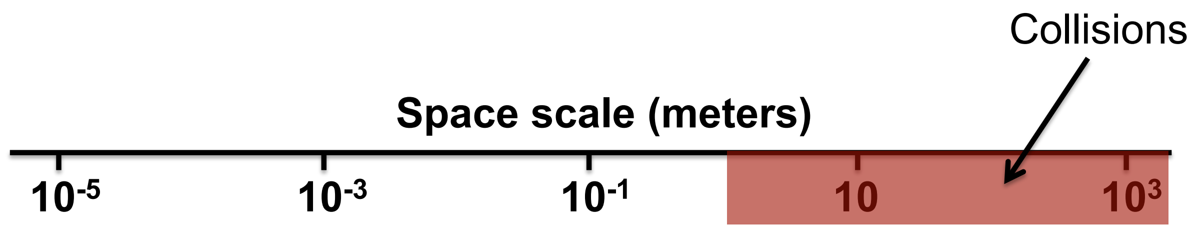
$P_{fus}=500$ MW

Vlasov-Maxwell – Multiscale Computation





Temporal grid: ~10^{13} time steps



Spatial grid: ~10^6 grid points x 3-D = 10^{18} grid points

Velocity grid: ~10 grid points x 3-D = 10^3 grid points

Total: ~10^{34} total grid points

Velocity grid: ~ 100 grid points

Total: $\sim 10^{37}$ total grid points

Feasibility

- petaflop$=10^{15}\ \frac{\text{opers}}{\text{sec}}$
- petaflop $\times\ 10^6$ in parallel $\Rightarrow 10^{21}\ \frac{\text{opers}}{\text{sec}}$
- $10^{21}\ \frac{\text{opers}}{\text{sec}} \times \pi \times 10^7 \frac{\text{sec}}{\text{year}} = 10^{28}\ \frac{\text{opers}}{\text{year}}$
- $10^{37} \div 10^{28} = 10^9 \sim$ age of solar system $<$ age of universe

Maxwell-Vlasov System (to scale)

$$\frac{\partial f_\alpha(x,v,t)}{\partial t} + v \cdot \frac{\partial f_\alpha}{\partial x} + \ldots \quad (1)$$

⟵ person

3D Vlasov-Poisson

$$\frac{\partial f}{\partial t} = -\mathbf{v} \cdot \nabla f + \mathbf{E} \cdot \nabla_{\mathbf{v}} f \qquad \Omega \times (0, T]\,,$$

$$\mathbf{E} = -\nabla \Phi \qquad \Omega_{\mathbf{x}} \times (0, T]\,,$$

$$\Delta \Phi = \int_{\mathbb{R}^3} d^3 v \; f - 1 \qquad \Omega_{\mathbf{x}} \times (0, T]\,.$$

$$\Omega = \Omega_{\mathbf{x}} \times \mathbb{R}^3$$

Vlasov Computational Methods

The VP system with the electrostatic force has been studied extensively for the simulation of collisionless plasmas. Numerical methods include but not limited

- Particle-In-Cell (PIC) (Birdsall, Langdon; Hockney, Eastwood, 1981)
- Semi-Lagrangian approach (Cheng and Knorr, 1976, Sonnendr$\ddot{u}$cker, et al, F. Filbet, et al, Qiu and Christlieb)
- Fourier-Fourier Spectral methods (Klimas et al), WENO FD with Fourier collocation (Zhou et al.) , FEM, DG (see next page).

For gravitational VP system,

- 1D problems, Fujiwara, 1981, White, 1986
- Spherical stellar systems, Fujiwara, 1983
- Stella disks, Nishida et al, 1984.
- Gravitational clustering, Bouchet, 1985

Discontinuous Galerkin Method

- Invented by Reed and Hill (73) for neutron transport. Lesaint and Raviart (74).
- RKDG method by Cockburn and Shu (89, 90,...) for conservation laws.
- Elliptic and Parabolic problems, (IP methods), Babuška *et al.* (73), Wheeler (78), Arnold (79), Bassi and Rebay (97), Cockburn and Shu (98), Arnold *et al.* (02)...

<u>DG methods for VP systems in electrostatic case have been considered</u>

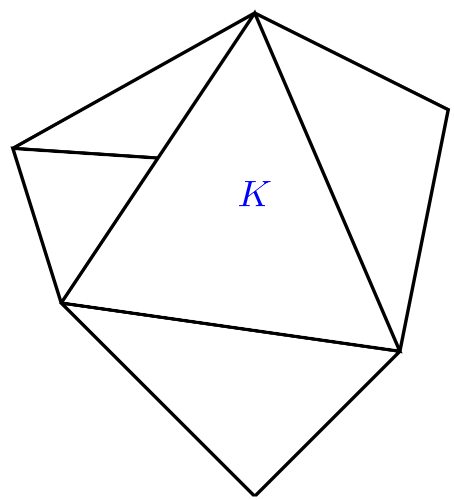
- Heath, Gamba, Morrison, Michler, JCP, 2011. Heath, 2007
- Ayuso, Carrillo, Shu, KRM, to appear; preprint.
- Qiu, Shu, JCP, 2011. Rossmanith, Seal, JCP, 2011, Crouseilles et al., preprint.

DG Method – Conservation Laws

For $u_t + \nabla \cdot \mathbf{f}(u) = 0$, the DG method is: to find $u \in \mathcal{V}(K)$, such that

$$\int_K u_t v \, dA - \int_K \mathbf{f}(u) \cdot \nabla v \, dA + \int_{\partial K} \widehat{\mathbf{f}(u) \cdot \mathbf{n}} \, v \, ds = 0$$

hold for any test function $v \in \mathcal{V}(K)$.

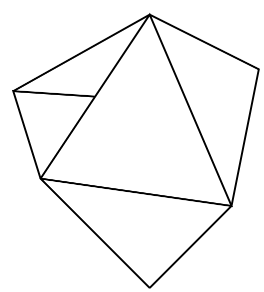


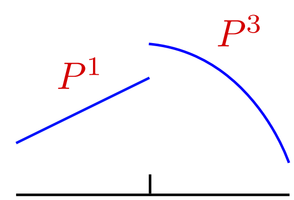
Upwinding

DG Method – Advantages

▷ Real Boundary Conditions

- Use of FVM framework, convection-dominated problems.
- Flexibility with the mesh. (hanging nodes, nonconforming mesh)
- Compact scheme, highly parallelizable.
- Polynomials of different degrees in different elements, even non-polynomial basis.





semi-discrete:

$$M\frac{df}{dt} = V(f)\,,$$

M^{-1} only once!

VP DG Error Estimates

$\mathbb{Q}^r(K)$ *: the space of polynomials on a set K of degree less than or equal to r,*
and Non-Symmetric Interior Penalty (NIPG) method for the Poisson equation

$$\|\Phi-\Phi_h\|^2_{NIPG} \quad \le \quad \lambda^{-1}\|\rho-\rho_h\|^2_{L^2(\Omega_x)} + c\frac{h^{2\mu_x-2}}{r_x^{2\bar{s}-2}}\|\tilde{\Phi}_h\|^2_{L^2(\Omega_x)},$$

$$\|\nabla\Phi-\nabla\Phi_h\|^2_{L^2(\Omega_x)} + \sum_{k_x=1}^{P_{h_x}}\frac{r_v\sigma}{|h_{j_x}|^{n/2}}\|\Phi-\Phi_h\|^2_{L^2(F_{k_x})} + \sum_{F_{k_x}\in\Omega_{x,D}}\frac{r_x\sigma}{|h_{j_x}|^{n/2}}\|\Phi-\Phi_h\|^2_{L^2(F_{k_x})}$$

$$\le \qquad \lambda^{-1}\|\rho-\rho_h\|^2_{L^2(\Omega_x)} + c\frac{h^{2\mu_x-2}}{r_x^{2\bar{s}-2}}\|\tilde{\Phi}_h\|^2_{L^2(\Omega_x)},$$

$$\|f(T)-f_h(T)\|^2_{L^2(\Omega)} + \int_0^T\sum_{k=1}^{P_h}\||\overline{\alpha_h}\cdot\nu_k|^{1/2}[f-f_h]\|^2_{L^2(F_k)}$$

$$+\int_0^T\||\alpha_h\cdot\nu_k|^{1/2}\;[f-f_h]\|^2_{0,\Gamma_0} + \int_0^T\||\alpha_h\cdot\nu_k|^{1/2}[f-f_h]\|^2_{0,\Gamma_I} \le Ch^{2\mu_v-1} + o_{\{h,\mu_x,\mu_v\}}(h^{2\mu_v-1}),$$

for $\quad \mu_x=\min\{r_x+1,\bar{s}\}\quad$ and $\quad \mu_v=\min\{r_v+1,s\}$.

and $\quad \|\theta\|^2_{NIPG}=A_{c_s}(\theta,\theta)+J_\sigma(\theta,\theta), \qquad \theta\in H^1(\mathcal{T}_h)$.

Broken Sobolev spaces H^s(mesh) etc.

1D Vlasov-Poisson & Advection Equations

Vlasov-Poisson:

$$\begin{aligned} f_t &= -v f_x + E f_v && \Omega \times (0,T] \\ E &= -\Phi_x && \Omega_x \times (0,T]\,, \\ \Phi_{xx} &= \int_{\mathbb{R}} dv\, f - 1 && \Omega_x \times (0,T] \end{aligned}$$

Linear Vlasov-Poisson:

$$\begin{aligned} (\delta f)_t &= -v(\delta f)_x + E f_0' && \Omega_x \times (0,T] \\ E &= -\Phi_x && \Omega_x \times (0,T] \\ \Phi_{xx} &= \int_{\mathbb{R}} dv\, \delta f && \Omega_x \times (0,T] \end{aligned}$$

Advection:

$$(\delta f)_t = -v(\delta f)_x \qquad \Omega \times (0,T]$$

$$\Omega_x = [0,L]\,, \qquad \Omega = \Omega_x \times I\!R$$

ICs and BCs

$$f(x,v,t) = f_0(v) + \delta f(x,v,t)$$

$$\begin{aligned}
\delta f(x,v,0) &= A\cos(kx)\, f_0(v)\,, \\
\delta f(0,v,t) &= \delta f(L,v,t)\,, \\
\Phi(0,t) &= \Phi(L,t) = 0\,,
\end{aligned}$$

Note, $\delta f(L,v,t)$ need not be small. Sample equilibria:

Maxwellian : $$f_M = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

Lorentzian : $$f_L = \frac{1}{\pi}\frac{1}{v^2+1}\,.$$

Advection

$$\begin{aligned}\rho_{tot}(x,t) &= 1 - \int_{-\infty}^{\infty} dv\, f(x,v,t) \\ &= 1 - \int_{-\infty}^{\infty} dv\, \mathring{f}(x - vt, v) \\ &= -A \int_{-\infty}^{\infty} dv\, \cos\left[k(x - vt)\right] f_0(v)\,,\end{aligned}$$

Maxwellian Advection

Choose:

$$f_0 = f_M, \quad A = 0.1, \quad k = 0.5, \quad L = 4\pi$$

$\Rightarrow$

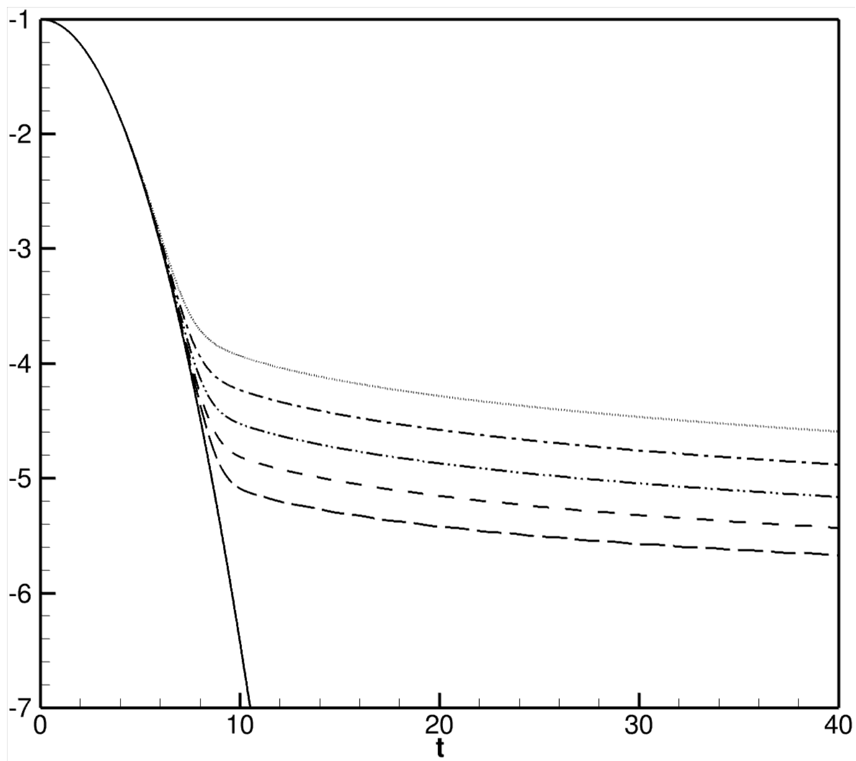
$$\boxed{\rho_{tot}(x,t) \;=\; -A\cos(kx)\, e^{-k^2t^2/2}}$$

$\Rightarrow$

$$\boxed{\max_x |\rho_{tot}(x,t)| = 0.1\, e^{-t^2/8}}$$

Maxwellian Advection

$\log_{10}(\max_x |\rho_{tot}(x,t)|)$ vs. t



exact solution *(solid)*, $(N_{h_x}, N_{h_v}) =$ (500,400) *(dot)*, (1000,800) *(dash-dot)*, (2000,1600) *(dash-dot-dot)*, (4000, 400) *(short dash)*, (8000, 400) *(long dash)*.

Lorentzian Advection

Choose:

$$f_0 = f_L, \quad A = 0.01, \quad k = 1/8, 1/6, 1/4, 1/2\,, \quad L = 16\pi, 12\pi, 8\pi, 4\pi$$

$$T = 75, 75, 50, 50\,, \quad (N_x, N_v) = (1000, 2000)$$

$\Rightarrow$

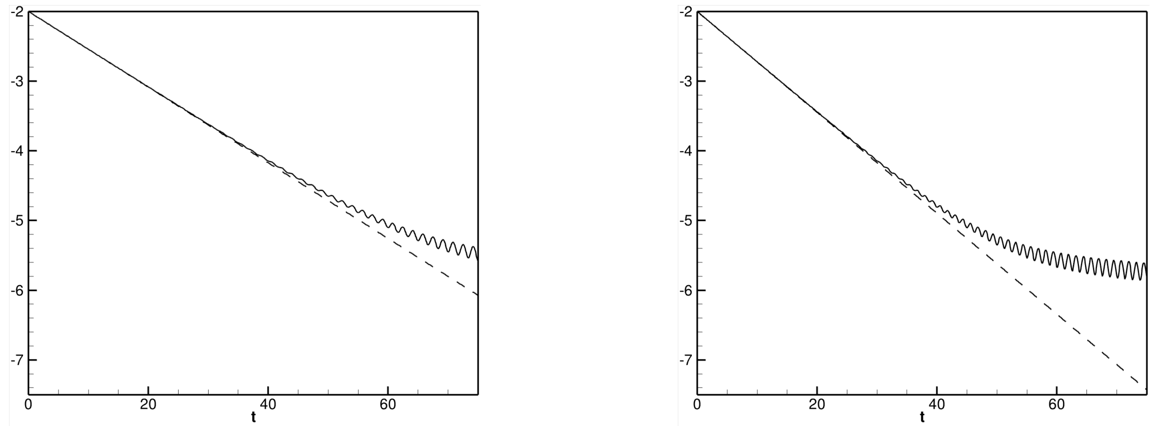
$$\boxed{\rho_{tot}(x,t) \;=\; -A\cos\left(k_x\right) e^{-kt}}$$

$\Rightarrow$

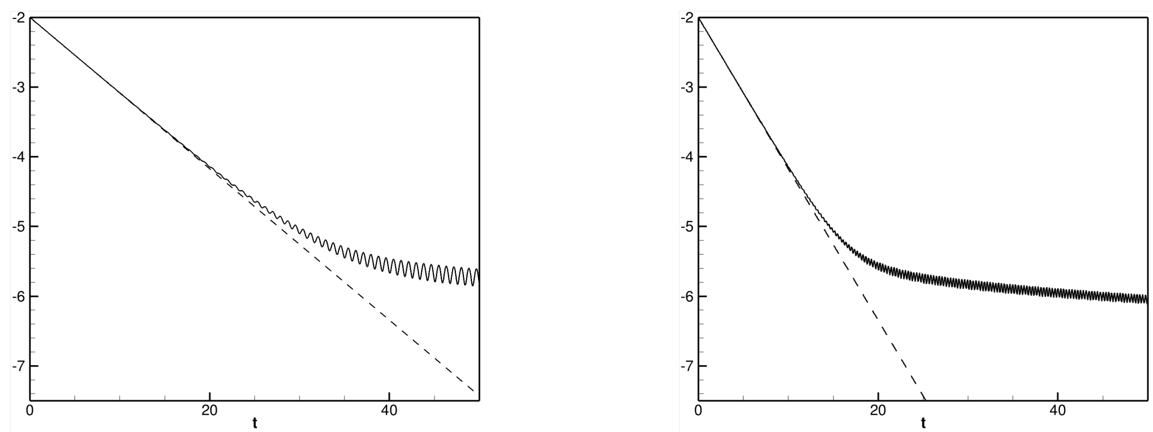
$$\boxed{\max_x |\rho_{tot}(x,t)| = 0.01\, e^{-kt}}$$

Lorentzian Advection

$\log_{10}(\max_x |\rho_{tot}(x,t)|)$ vs. t



k=1/8 *(left)* and k=1/6 *(right)*
exact solution *(dash)*, numerical solutions *(solid)*.



k=1/4 *(left)* and k=1/2 *(right)*;
exact solution *(dash)*, numerical solutions *(solid)*.

Landau Damping

Assume:

$$f(x,v,t) = f_0(v) + \delta f(x,v,t)\,, \quad \delta f(x,v,t) \sim \exp(ikx - i\omega t)$$

Plasma 'Dispersion Relation':

$$\varepsilon(k,\omega) \;=\; 1 - \frac{1}{k^2}\int_{-\infty}^{\infty} \frac{f_0'(v)}{(v-\omega/k)}\,dv\,,$$

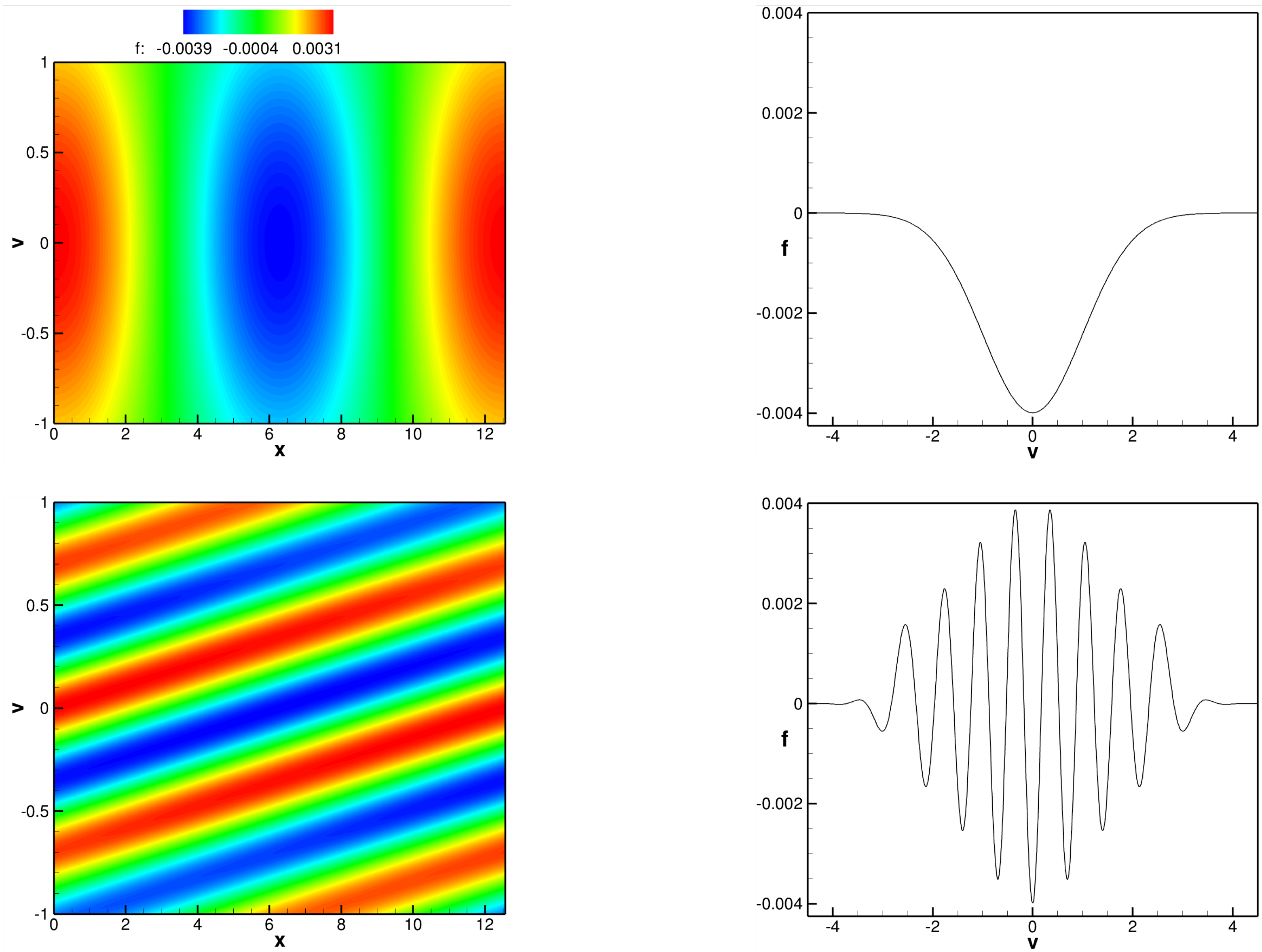
k real and positive, ω in UHP

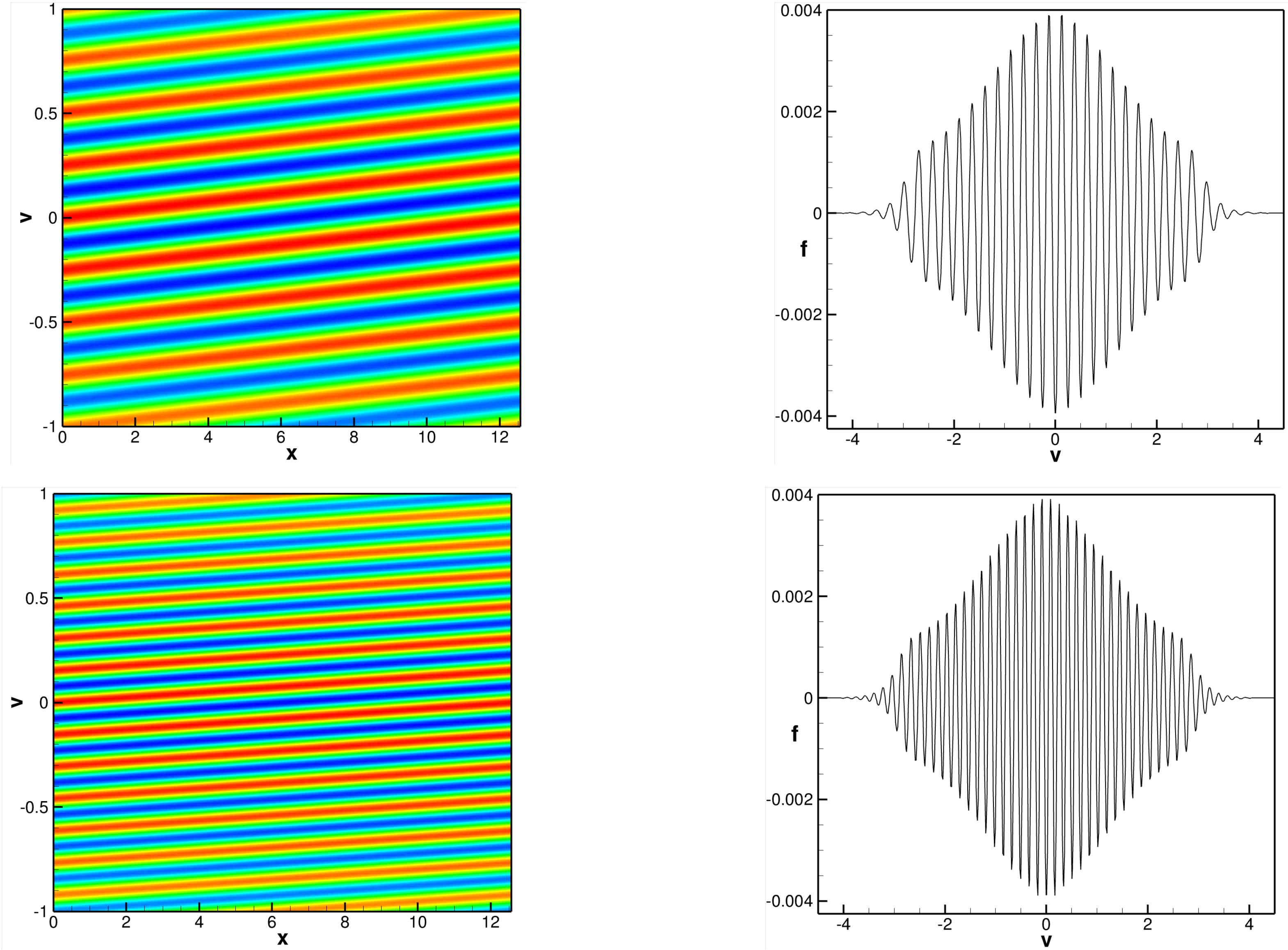
Stable and unstable eigenmodes (and embedded modes) if they exist satisfy

$$\varepsilon(k,\omega) = 0 \quad \Rightarrow \quad \omega(k) = \omega_R(k) + i\gamma(k)$$

Landau damping comes from analytically continuing into LHP (deforming the contour). Not an eigenmode! Time asymptotics.

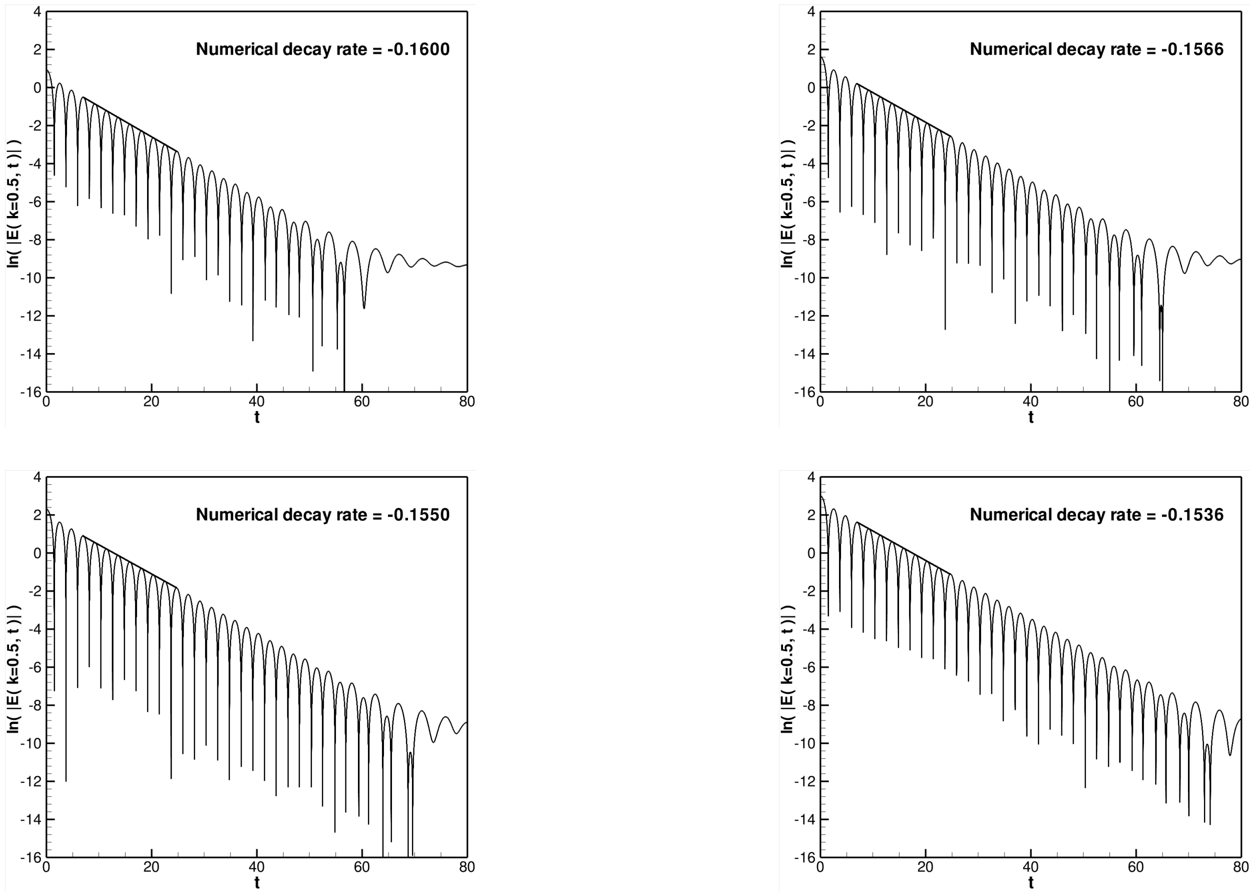
Landau Damping Maxwellian





Contour plots (*left*) and cross-sectional plots (*right*), $x = 2\pi$, for δf at $t = 0$, $t = 25$, $t = 50$, $t = 75$ (*descending order*).

Landau Damping Maxwellian Decay Rate



Time decay plots of fundamental mode under mesh refinement: (N_{h_x}, N_{h_v}) =(250, 200) *(top left)*, (500, 400) *(top right)*, (1000, 800) *(bottom left)* and (2000, 1600) *(bottom right)*. The theoretical decay rate is -0.153 to three decimal-digit accuracy.

Landau Damping with Lorentzian

Plasma Dispersion Function:

$$\varepsilon(k,\omega) = 1 + \frac{2}{\pi k^2} \int_{-\infty}^{\infty} \frac{v}{(v^2+1)^2(v-u)}\, dv\,,$$

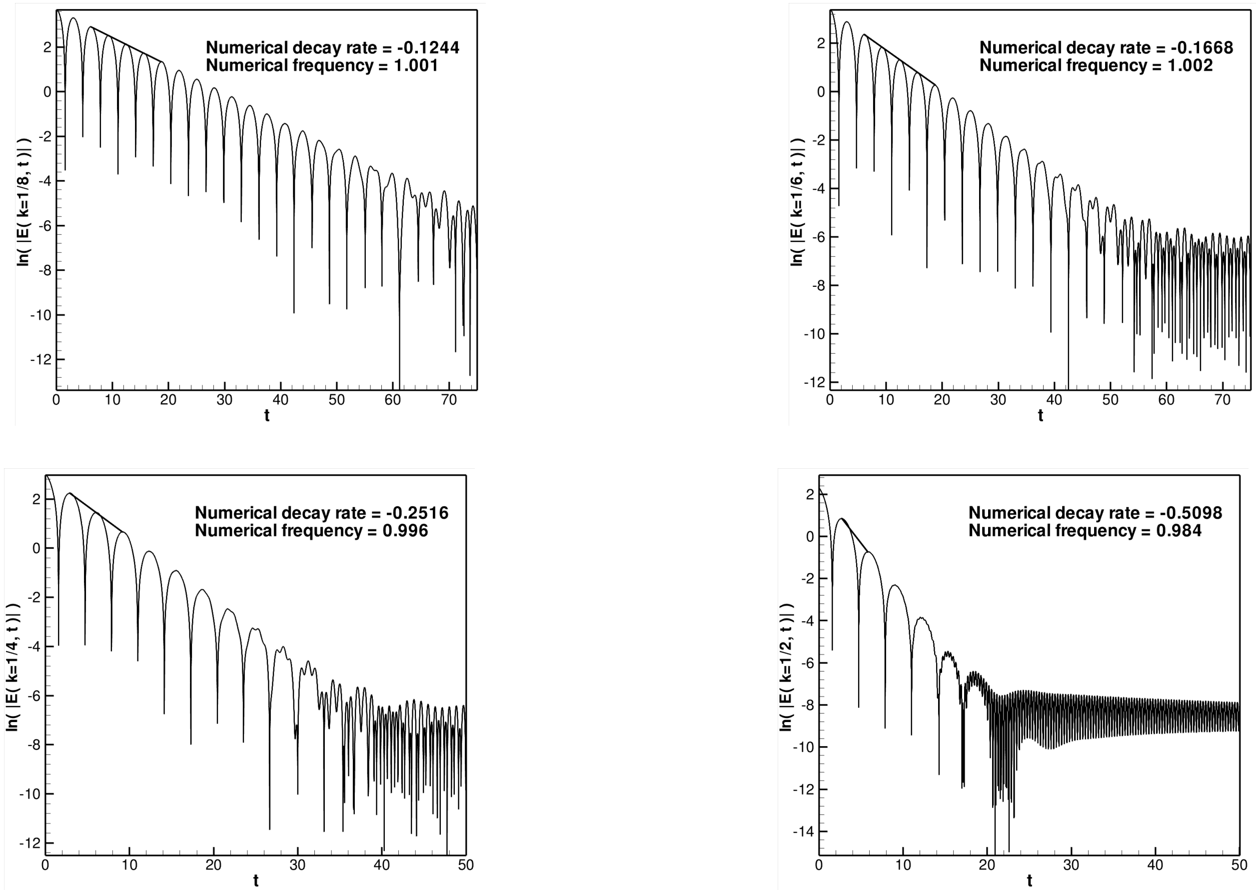
Residue calculus implies:

$$\varepsilon(k,u) = 1 - \frac{1}{k^2(u+i)^2}\,.$$

$\varepsilon = 0$ and $u = \omega/k$ implies

$$\omega = \omega_R + i\gamma = \pm 1 - ik\,,$$

Landau Damping with Lorentzian: $\gamma = k$



Decay plots of fundamental modes: k=1/8 *(top left)*, k=1/6 *(top right)*, k=1/4 *(bottom left)* and k=1/2 *(bottom right)*.

Recurrence in Advection

Given a map on a bounded domain D,

$$f_t : D \longrightarrow D\,,$$

with f measure preserving homeomorphism $\Rightarrow$ recurrence.

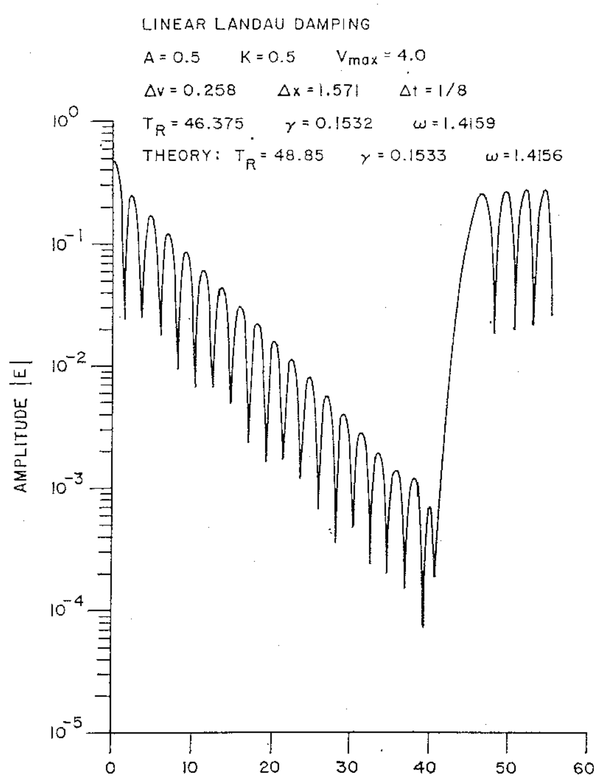


FIG. 3. Linear Landau damping with recurrence effect for the case $V_{\max} > v_p$, where v_p is the phase velocity of the wave. $k = 0.5$, $N = 8$, $M = 16$, $V_{\max} = 4.0$, and $\Delta t = \frac{1}{8}$.

Cheng-Knorr Recurrence Time

$$
\begin{aligned}
\rho(x,t) &= \sum_j f(x, v_j, t)\triangle v = \sum_j f_0(x - v_j t, v_j)\triangle v \\
&= \sum_j A f_{eq}(v_j)\cos(k(x - v_j t))\triangle v \\
&= \sum_j A f_{eq}(v_j)\cos(kx - k(j + 1/2)\triangle v t)\triangle v
\end{aligned}
$$

This is a periodic function in time with period $T_R = \frac{2\pi}{k\triangle v}$.
In this section, we consider the standard RKDG methods for this equation with upwind numerical fluxes.

DG Recurrence Time Q^1

$$f_h = f_{i-\frac{1}{4},j+\frac{1}{4}}\chi_1(x,v) + f_{i-\frac{1}{4},j-\frac{1}{4}}\chi_2(x,v)$$

$$+f_{i+\frac{1}{4},j+\frac{1}{4}}\chi_3(x,v) + f_{i+\frac{1}{4},j-\frac{1}{4}}\chi_4(x,v),$$

$$\begin{aligned}
\chi_1(x,v) &= -4\left(\frac{x-x_i}{\triangle x_i}-\frac{1}{4}\right)\left(\frac{v-v_j}{\triangle v_j}+\frac{1}{4}\right)\\
\chi_2(x,v) &= 4\left(\frac{x-x_i}{\triangle x_i}-\frac{1}{4})\right)\left(\frac{v-v_j}{\triangle v_j}-\frac{1}{4}\right)\\
\chi_3(x,v) &= 4\left(\frac{x-x_i}{\triangle x_i}+\frac{1}{4}\right)\left(\frac{v-v_j}{\triangle v_j}+\frac{1}{4}\right)\\
\chi_4(x,v) &= -4\left(\frac{x-x_i}{\triangle x_i}+\frac{1}{4}\right)\left(\frac{v-v_j}{\triangle v_j}-\frac{1}{4}\right)
\end{aligned}$$

$$f_{ij} = (f_{i-1/4,j+1/4}, f_{i-1/4,j-1/4}, f_{i+1/4,j+1/4}, f_{i+1/4,j-1/4})^T$$

DG Recurrence Time Q^1

$$\frac{df_{ij}}{dt} = \frac{\triangle v}{\triangle x}\left(S_m f_{ij} + T_m f_{i-1,j}\right) = \frac{\triangle v}{\triangle x}\left(S_m + T_m e^{-ik\Delta x}\right) f_{ij}$$

$$S_m = \begin{pmatrix} -\frac{49}{96} - \frac{7m}{8} & \frac{7}{96} & -\frac{7}{32} - \frac{3m}{8} & \frac{1}{32} \\ -\frac{7}{96} & \frac{49}{96} - \frac{7m}{8} & -\frac{1}{32} & \frac{7}{32} - \frac{3m}{8} \\ \frac{77}{96} + \frac{11m}{8} & -\frac{11}{96} & -\frac{21}{32} - \frac{9m}{8} & \frac{3}{32} \\ \frac{11}{96} & -\frac{77}{96} + \frac{m}{8} & -\frac{3}{32} & \frac{21}{32} - \frac{9m}{8} \end{pmatrix},$$

$$T_m = \begin{pmatrix} -\frac{35}{96} - \frac{5m}{8} & \frac{5}{96} & \frac{35}{32} + \frac{15m}{8} & -\frac{5}{32} \\ -\frac{5}{96} & \frac{35}{96} - \frac{5m}{8} & \frac{5}{32} & -\frac{35}{32} + \frac{15m}{8} \\ \frac{7}{96} + \frac{m}{8} & -\frac{1}{96} & -\frac{7}{32} - \frac{3m}{8} & \frac{1}{32} \\ \frac{1}{96} & -\frac{7}{96} + \frac{m}{8} & -\frac{1}{32} & \frac{7}{32} - \frac{3m}{8} \end{pmatrix},$$

with $m = 2j - N_v - 1 = 1, 3, 5 \ldots$

DG Recurrence Time Q^1

The initial condition is $f_{ij}(0) = Re(Ae^{ikx_i}\Lambda)$, where

$$\Lambda = (e^{-ik\triangle x/4} f_{eq}(v_{j+1/4}), e^{-ik\triangle x/4} f_{eq}(v_{j-1/4}), \\ e^{ik\triangle x/4} f_{eq}(v_{j+1/4}), e^{ik\triangle x/4} f_{eq}(v_{j-1/4}))^T$$

Hence the general expression for the numerical solution is

$$f_{ij}(t) = Re(e^{ikx_i}(a_1 e^{\eta_1 t} V_1 + a_2 e^{\eta_2 t} V_2 + a_3 e^{\eta_3 t} V_3 + a_4 e^{\eta_4 t} V_4))$$

where $\eta_1, \ldots \eta_4$ are eigenvalues of G_j, and $V_1, \ldots V_4$ are corresponding eigenvectors.

Eigenvectors independent of $m = 2j - N_v - 1 \Rightarrow$

Exact solution:

Recurrence $T_R \approx 2\pi/k\Delta v$, modulation, and decay $\mathcal{O}(k^2 \Delta x^2)$.

DG Recurrence Time Q^1

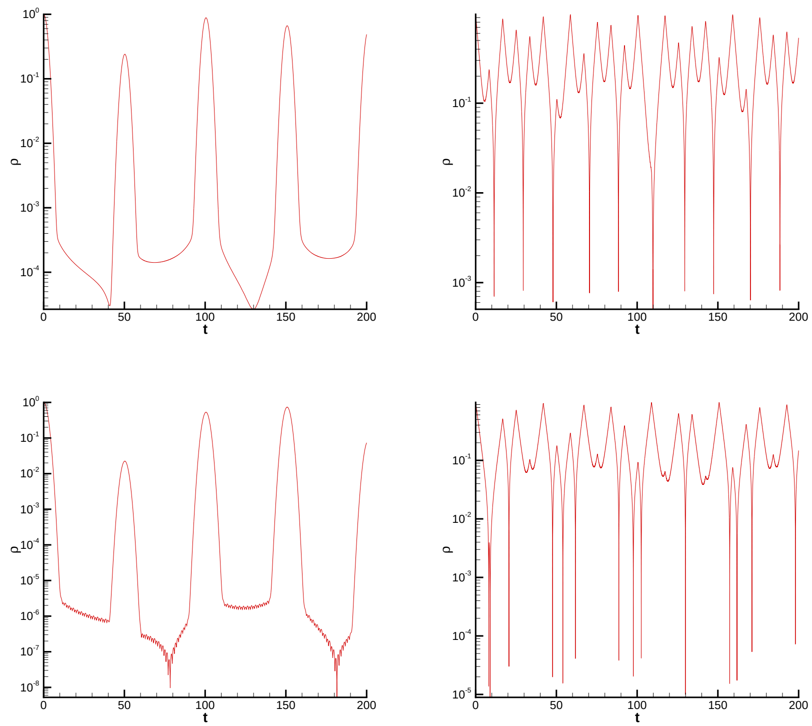
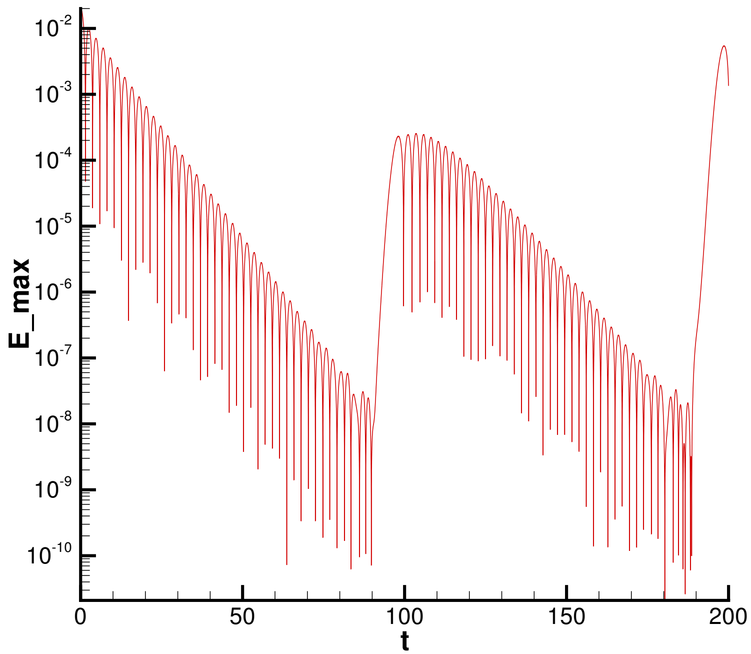
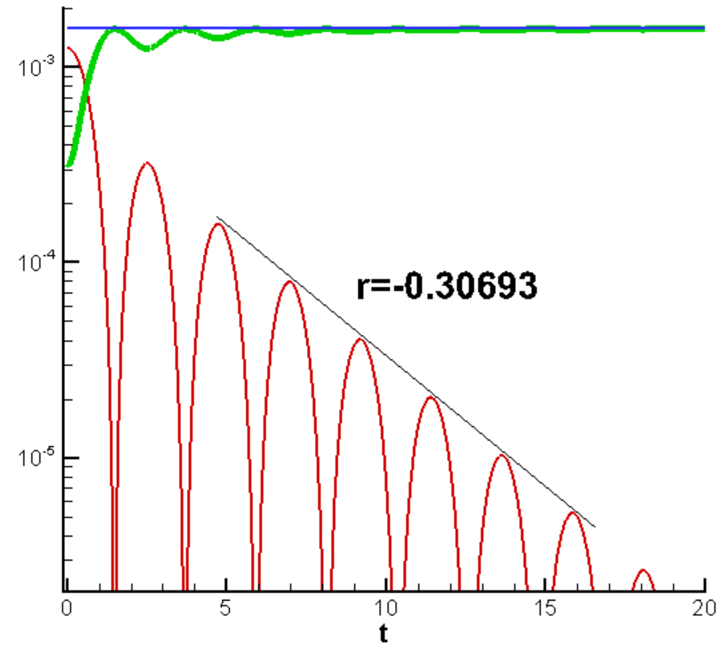


Figure: Top left: Maxwellian, Q^1. Top right: Lorentzian, Q^1. Bottom left: Maxwellian, Q^2. Bottom right: Lorentzian, Q^2.

Landau Damping – Q^2 Recurrence Time

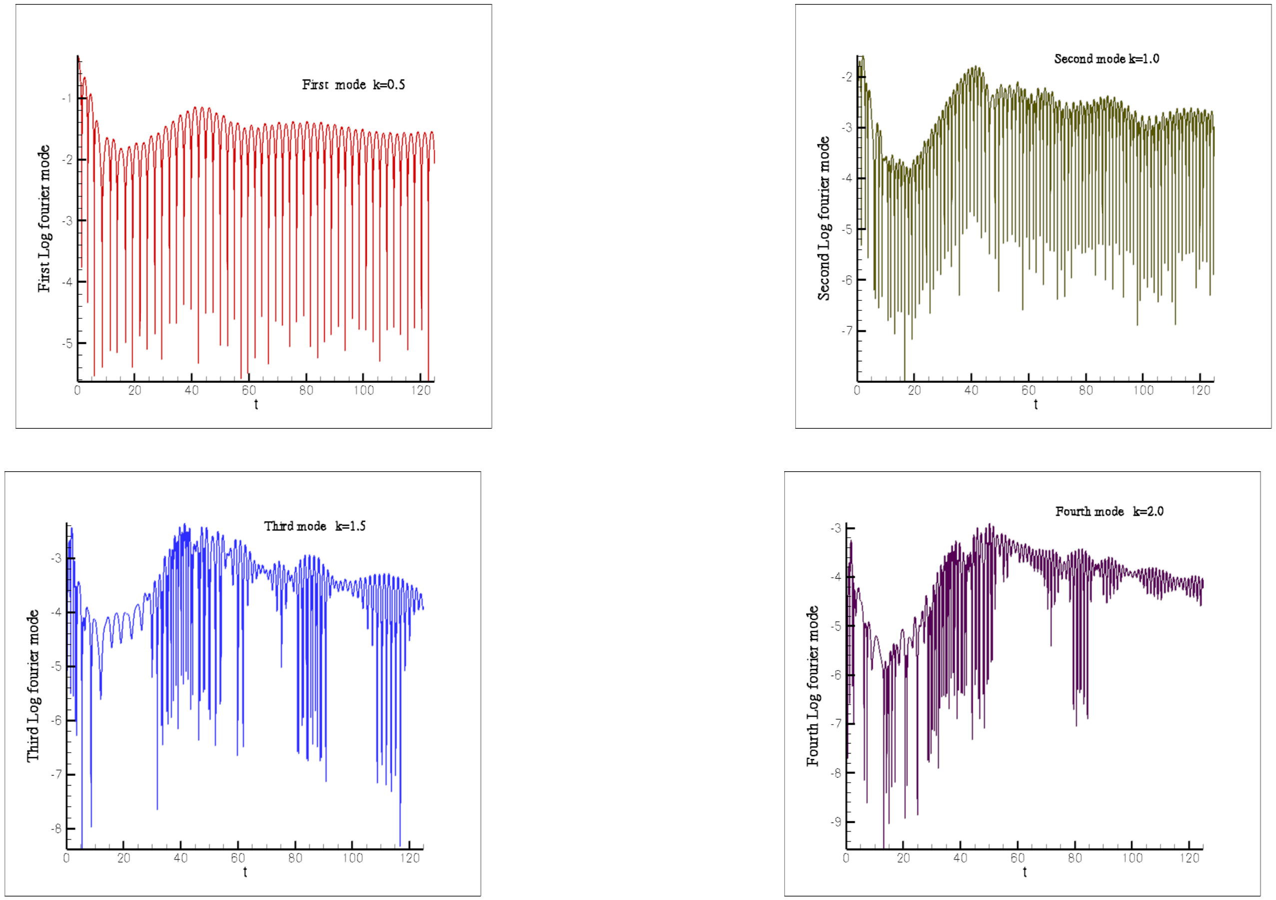




$$H_L = -\frac{1}{2}\int_0^{4\pi} dx \int_{\mathbb{R}} dv\, \frac{v\,(\delta f)^2}{f_0'} + \frac{1}{8\pi}\int_0^{4\pi} dx\; E^2 \,.$$

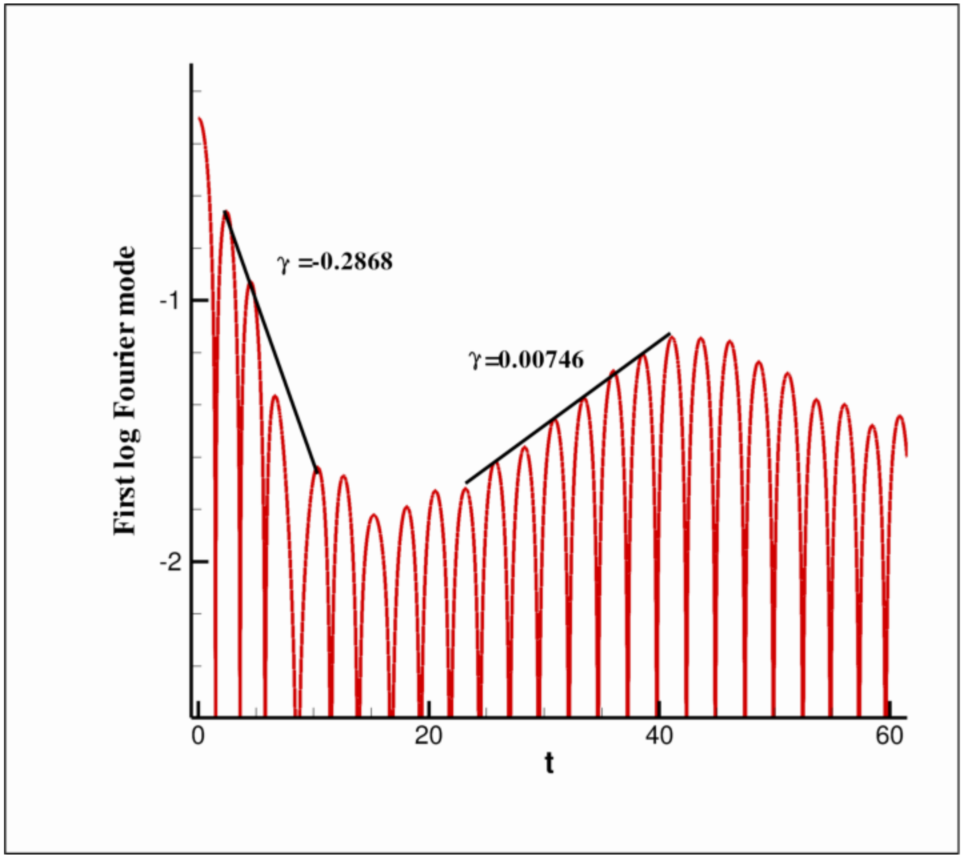
Nonlinear Computations – Analysis of Results

Nonlinear Landau Damping



Maxwellian, amplitude $A = .5$: k=.5 *(top left)*, k=1 *(top right)*, k=1.5 *(bottom left)* and k=2 *(bottom right)*. Bounce time $\approx$ 40.

Nonlinear Landau Damping



Maxwellian, amplitude $A = .5$. First mode. γ smaller than linear Landau damping because nonlinear coupling matters early.

Nonlinear Two-Stream Instability

Equilibrium:

$$f_{TS}(v) \;=\; \frac{1}{\sqrt{2\pi}}\, v^2\, e^{-v^2/2}$$

Manipulations:

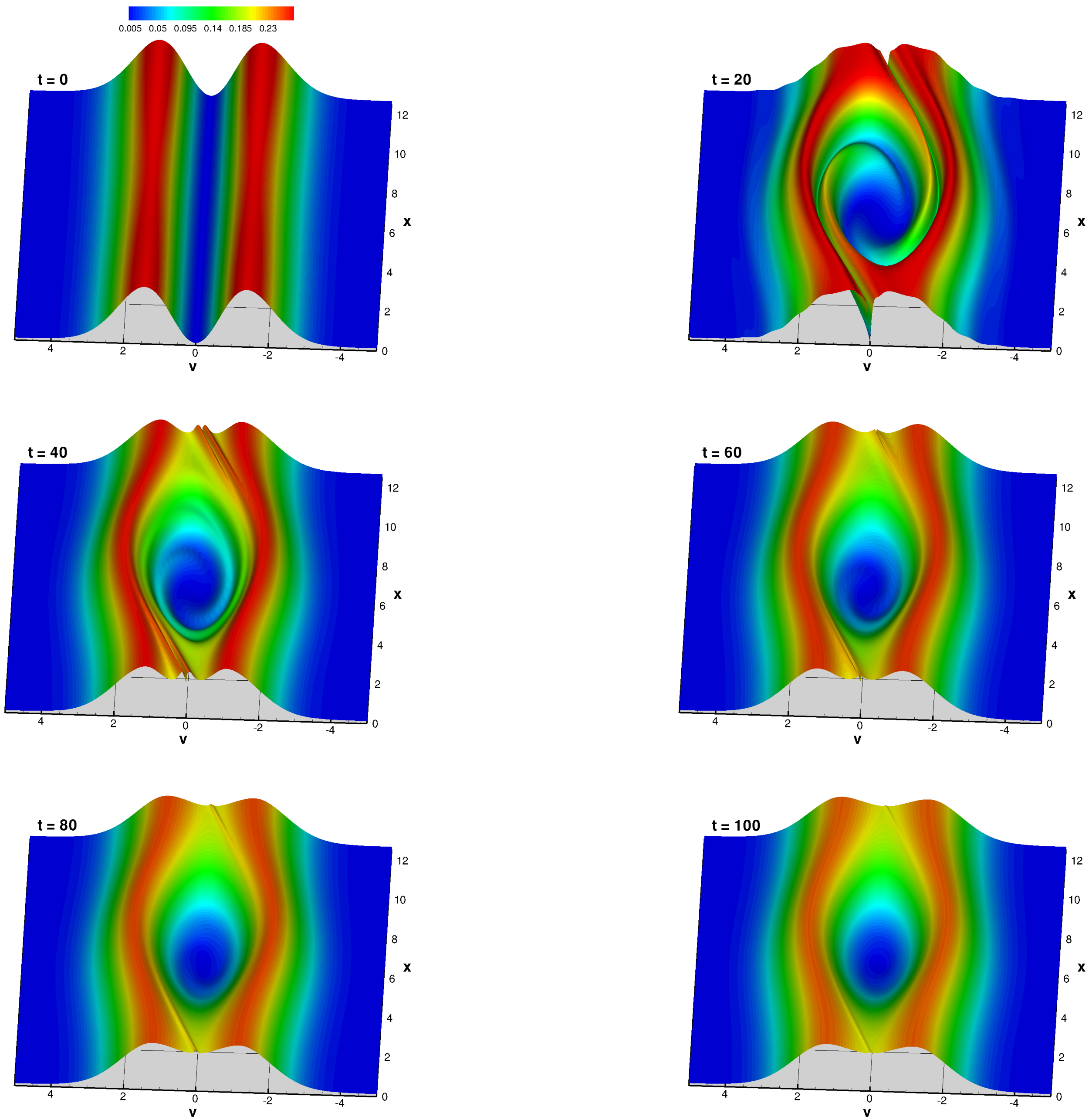
$$\varepsilon = 1 - \frac{2}{k^2}\left[1 - 2z^2 + 2zZ(z)\left(1 - z^2\right)\right] .$$

where $z = \omega/k$.

Plasma Z-function:

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-w^2} \frac{dw}{w - z} = 2ie^{-z^2} \int_{-\infty}^{iz} e^{-t^2} dt$$

first expression $\Im(z) > 0$ and the value of Z for $\Im(z) < 0$ is obtained by analytic continuation; second expression valid for all complex z good for numerics. $\varepsilon = 0$ implies instability! γ agrees!



0.005 0.05 0.095 0.14 0.185 0.23
t = 0
t = 20
t = 40
t = 60
t = 80
t = 100
x
v

Invariants

Particle Number:

$$N = \int_0^L dx \int_{\mathbb{R}} dv \, f(x, v, t) \, ,$$

Total Momentum:

$$P = \int_0^L dx \int_{\mathbb{R}} dv \, v f(x, v, t) \, ,$$

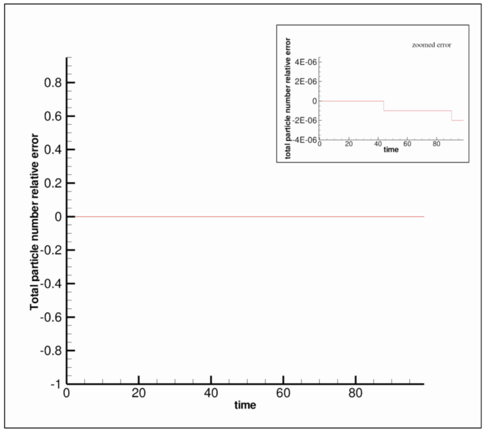
Total Energy:

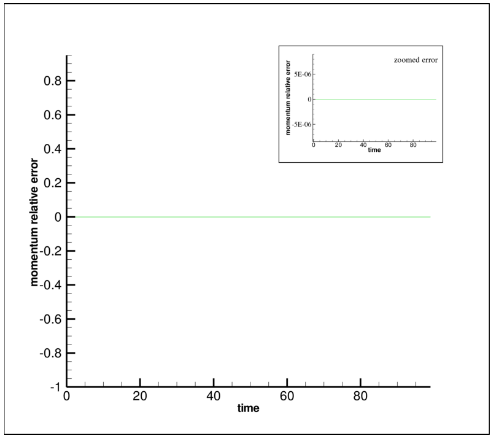
$$H = \frac{1}{2} \int_0^L dx \int_{\mathbb{R}} dv \, |v|^2 f(x, v, t) + \frac{1}{2} \int_0^L dx \, |E(x, t)|^2 \, ,$$

Casimir Invariants:

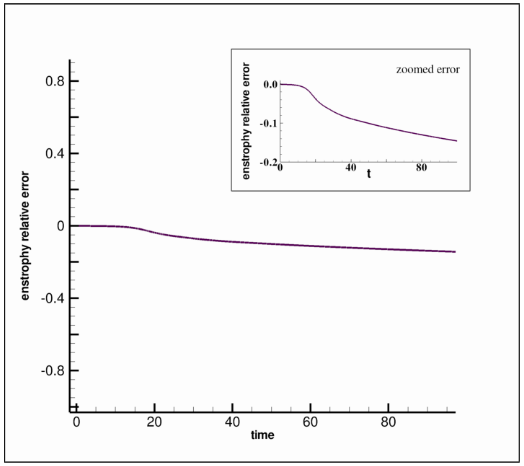
$$C = \int_0^L dx \int_{\mathbb{R}} dv \, \mathcal{C}(f) \, .$$

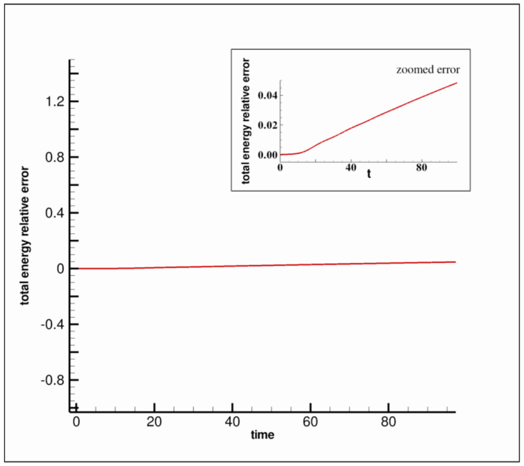
Invariants – Relative Error





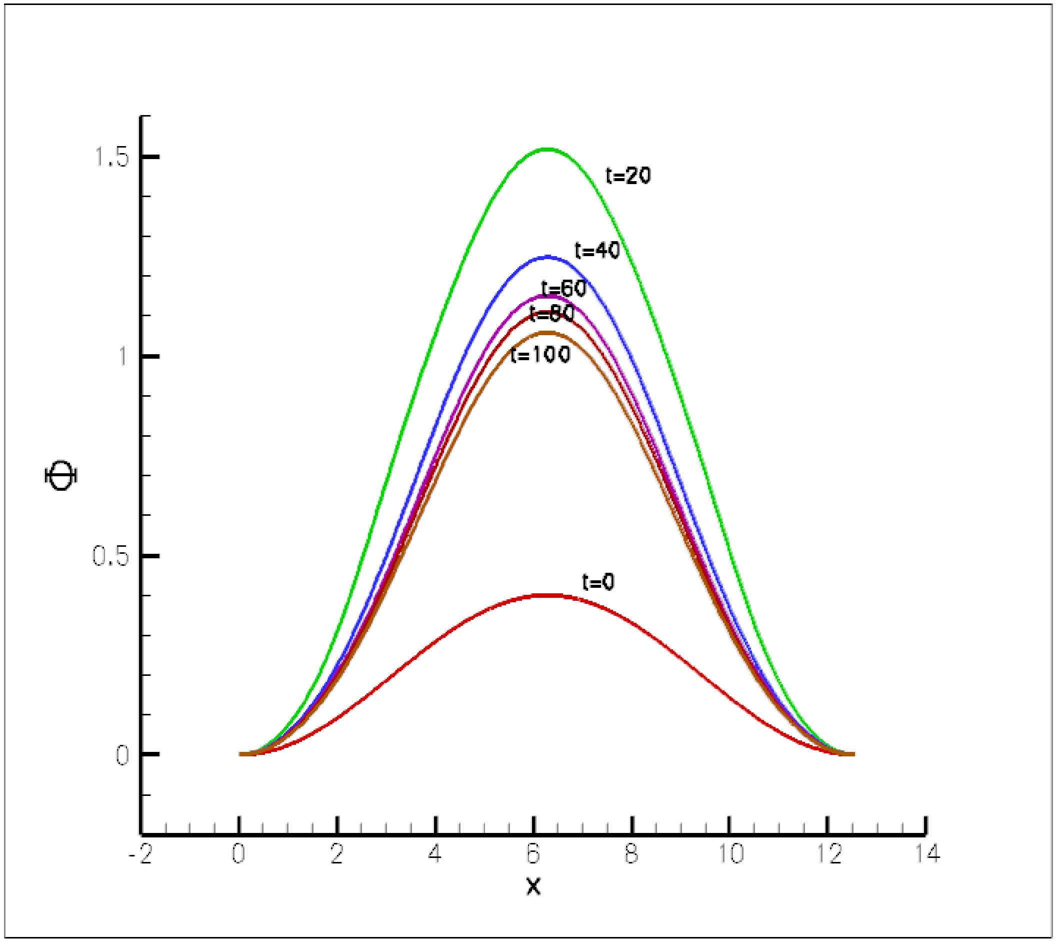
Total particle number *(left)*. Total momentum *(right)*.





Enstrophy *(left)*. Total energy *(right)*.

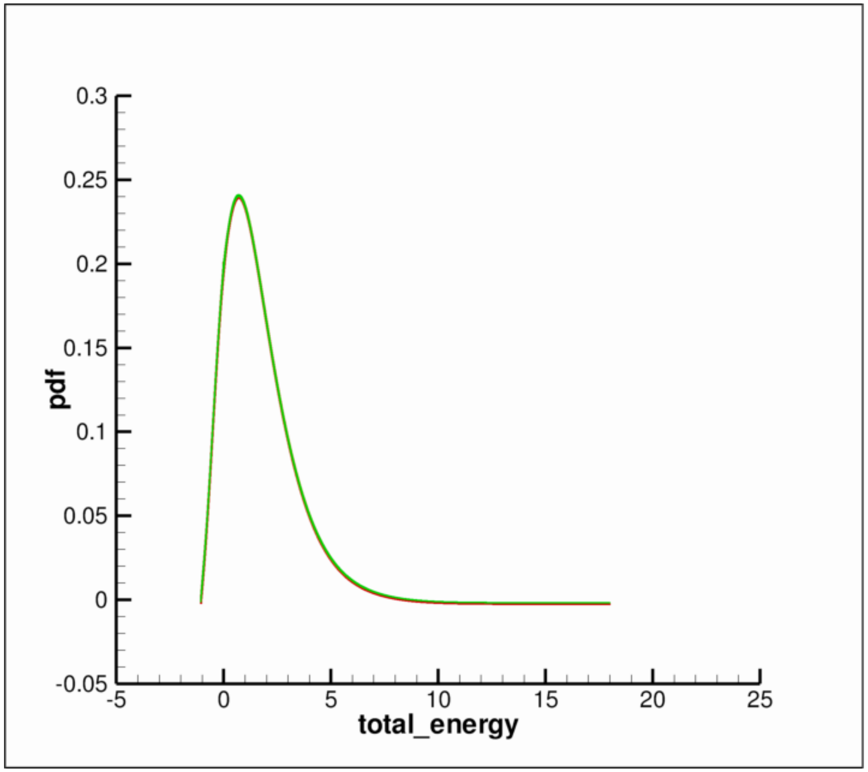
BGK Mode Potential



The electrostatic potential up to $\Phi(x, t = 100)$

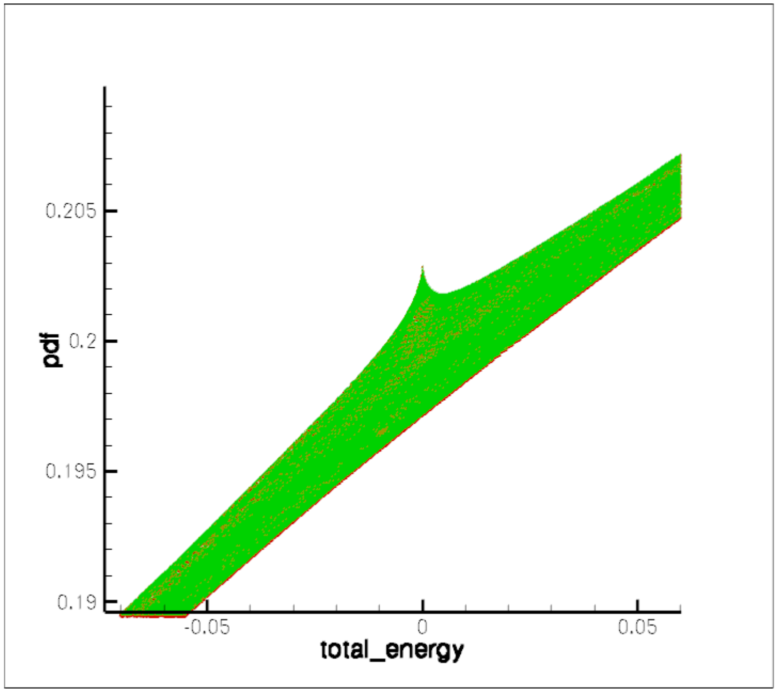
Scatter Plot f versus $\mathcal{E}(x,v)$

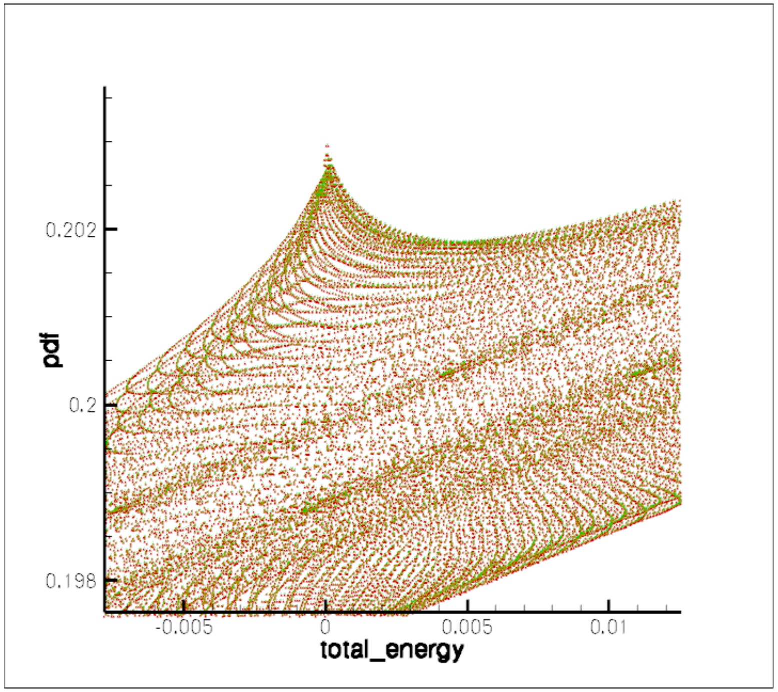
At $t = 100$ for every x, v, know $\Phi \Rightarrow \mathcal{E}(x,v) = v^2/2 - \Phi(x,100)$.
Make scatter plot of 9 million pairs (x,v) of f_{100} versus $\mathcal{E}(x,v)$:



f_{100} a graph over $\mathcal{E}(x,v)$ to within line thickness. Green positive velocities; red negative velocities.

Scatter Plot Detail





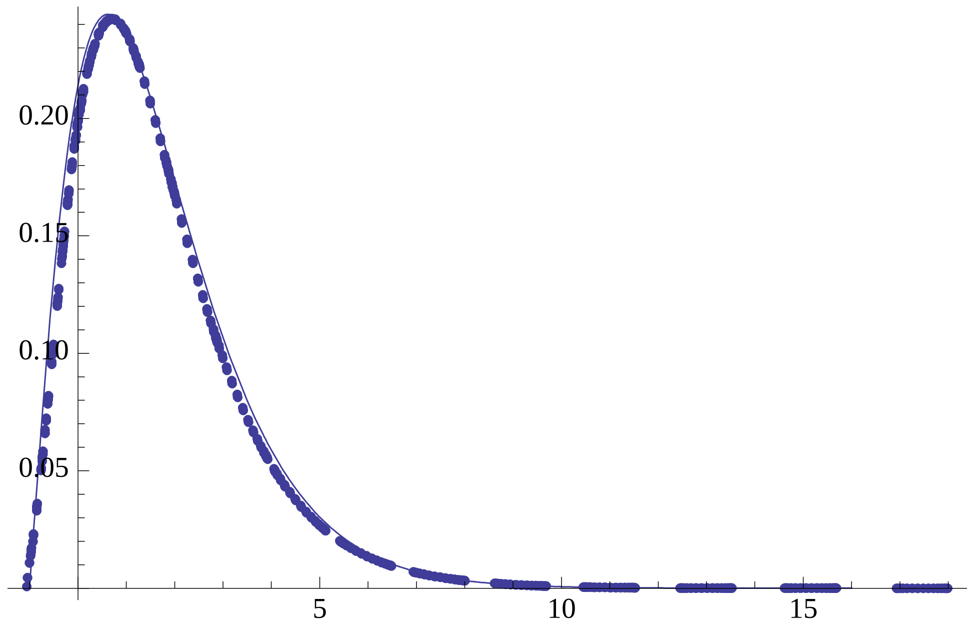
Blow-up of $f_{100}(\mathcal{E})$ near $\mathcal{E} = 0$. Is cusp universal trapping feature?

BGK Modeling

Model Distribution:

$$f_{\text{fit}} = A(\mathcal{E} + \Phi_M)(\mathcal{E} + \mathcal{E}^*)e^{-\beta\mathcal{E}}\,.$$

Here $\Phi_M = \max(\Phi)$. Since $\mathcal{E} = v^2/2 - \Phi$, $\min(\mathcal{E}) = -\Phi_M$. $f > 0 \Rightarrow \mathcal{E}^* > \Phi_M$.



Rough guess:

$\Phi_M = 1$ and $\mathcal{E}^* = 2$ uniformly good fit. $f'(\mathcal{E}_M) = 0$. For $\beta = 1$, $\mathcal{E}_M = 1/\gamma$, where γ is the golden mean!

Pseudo-potential

$$\rho(\Phi) = \int_{I\!R} dv\, f(\mathcal{E}) = \int_{-\Phi}^{\infty} \frac{d\mathcal{E}\; f_0(\mathcal{E})}{\sqrt{2(\mathcal{E} + \Phi)}}$$

Poisson's Equation:

$$\Phi_{xx} = -\rho(\Phi) = -\frac{d\mathcal{V}}{d\Phi}$$

Integrable Newton's second law: $\Phi \sim x$, $x \sim t$. Oscillation if pseudo-potential $\mathcal{V}$ has local minimum etc. Compares well.

Dynamically Accessible IC

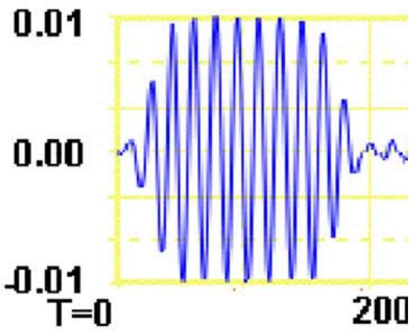
Vlasov with Drive:

$$f_t = -v f_x + \Big(E + E_d(x,t)\Big) f_v \,, \qquad E_x = 1 - \int_{\mathbb{R}} dv\, f$$

External Drive:

$$E_d(x,t) = A_d(t)\ cos(kx - \omega t)$$

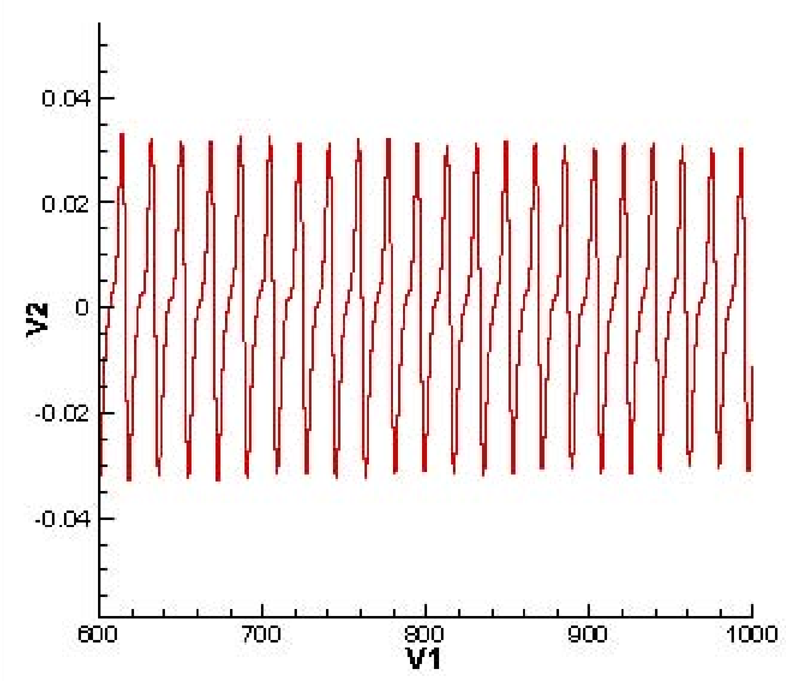
Drive Created IC:

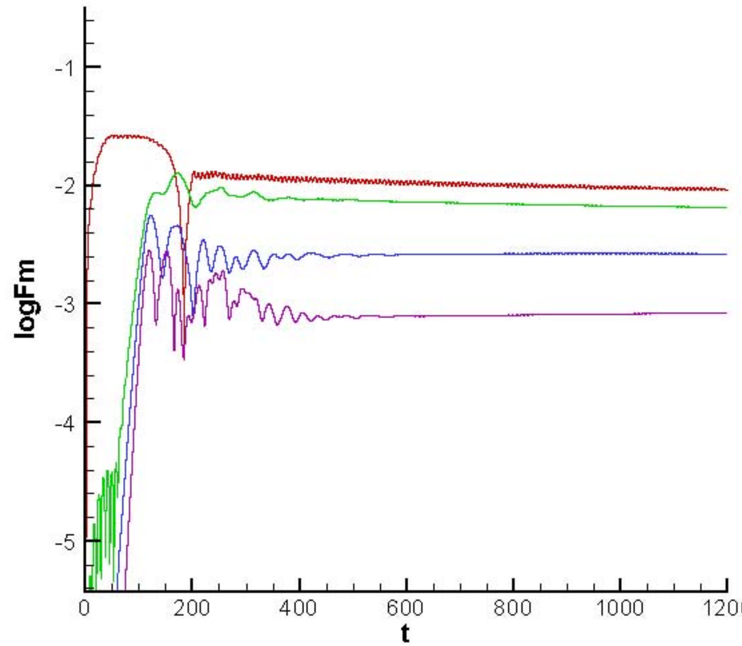


$A_d(t) = .052$ and $T_d = 200$

Johnston et al., Afeyan, Rose, PJM, ...

Weak Drive: $E(t) = E(t + T)$

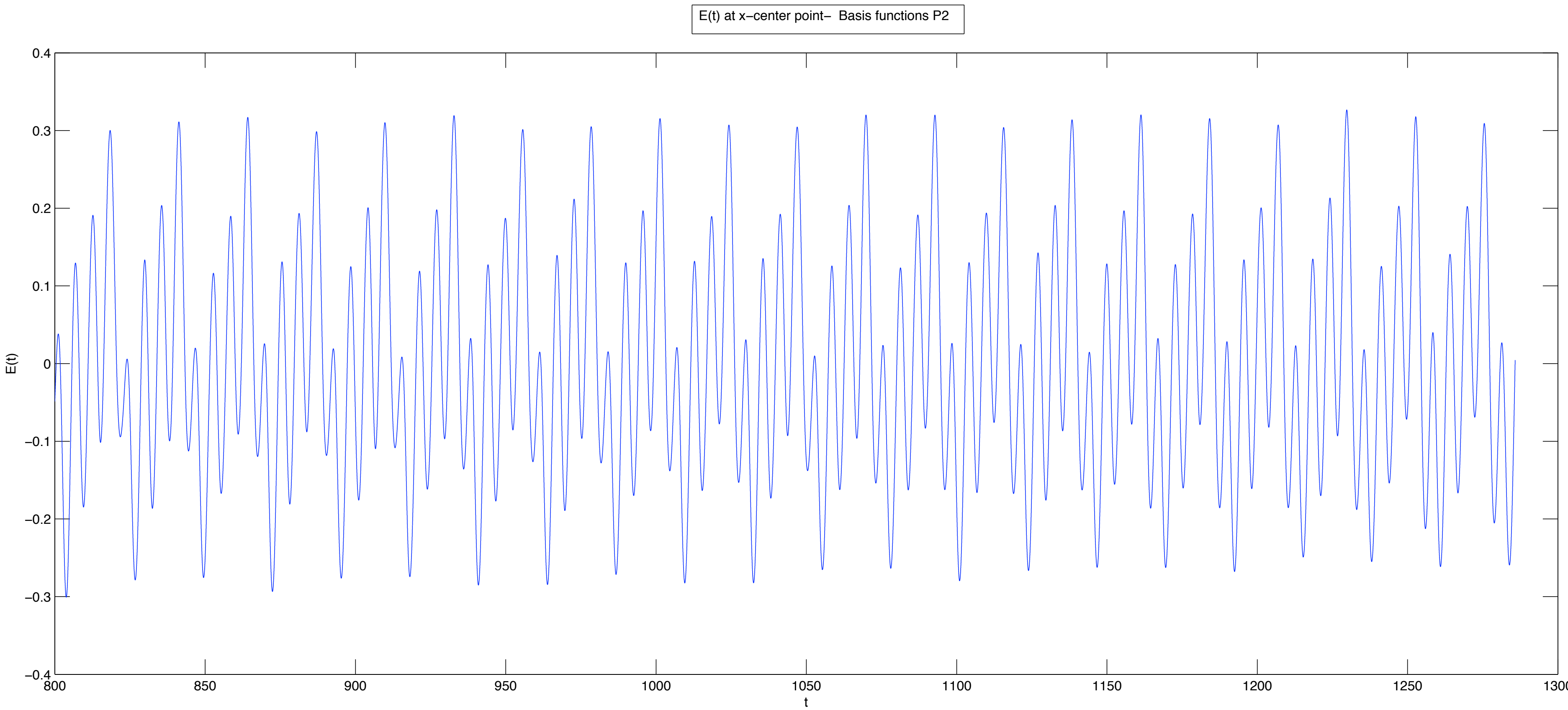




$A_d(t) = .052$ and $T_d = 200$

Appears to settle into <u>periodic orbit</u> – travelling BGK hole.

Strong Drive



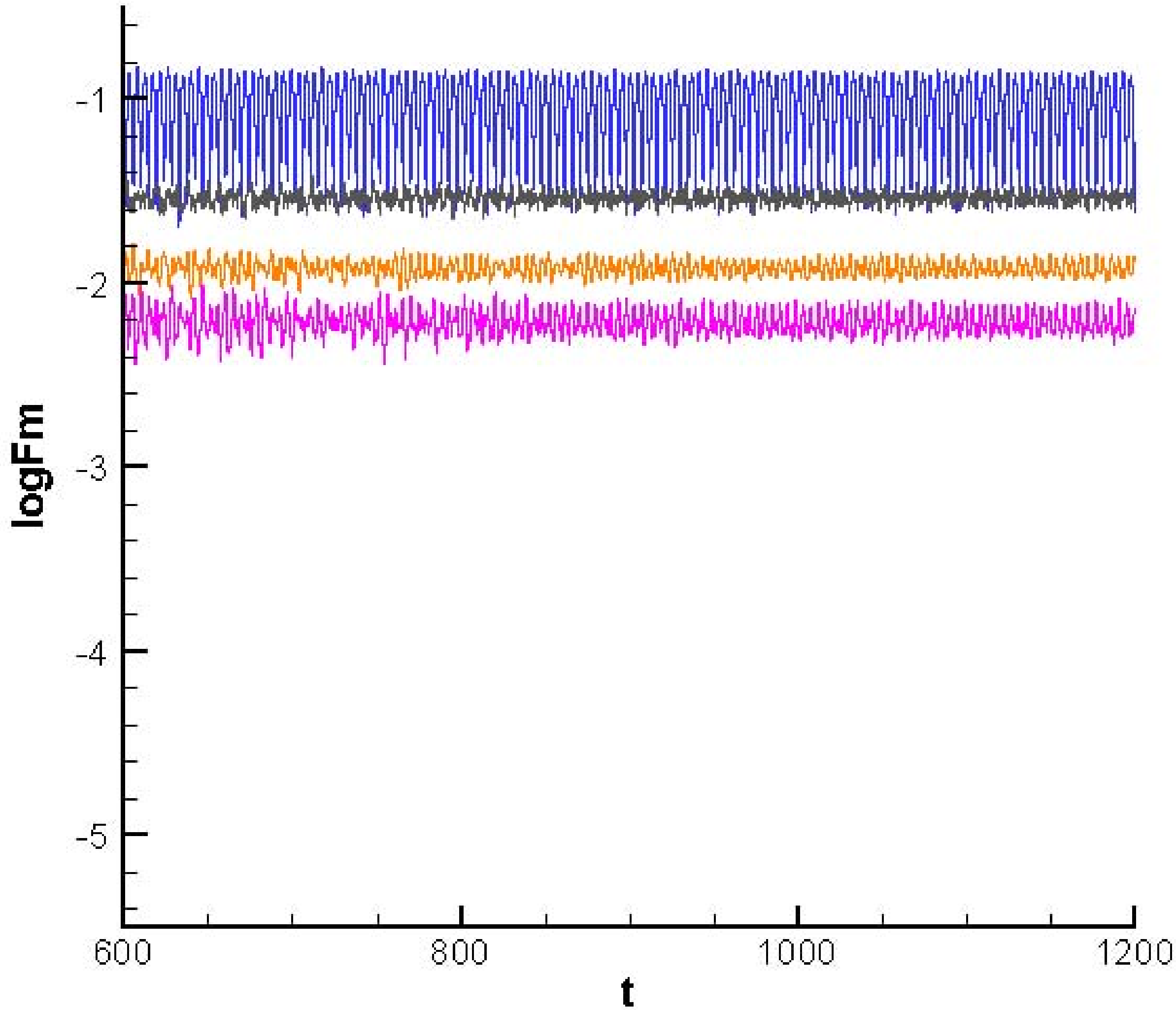
$$A_d(t) = .4 \text{ and } T_d = 200$$

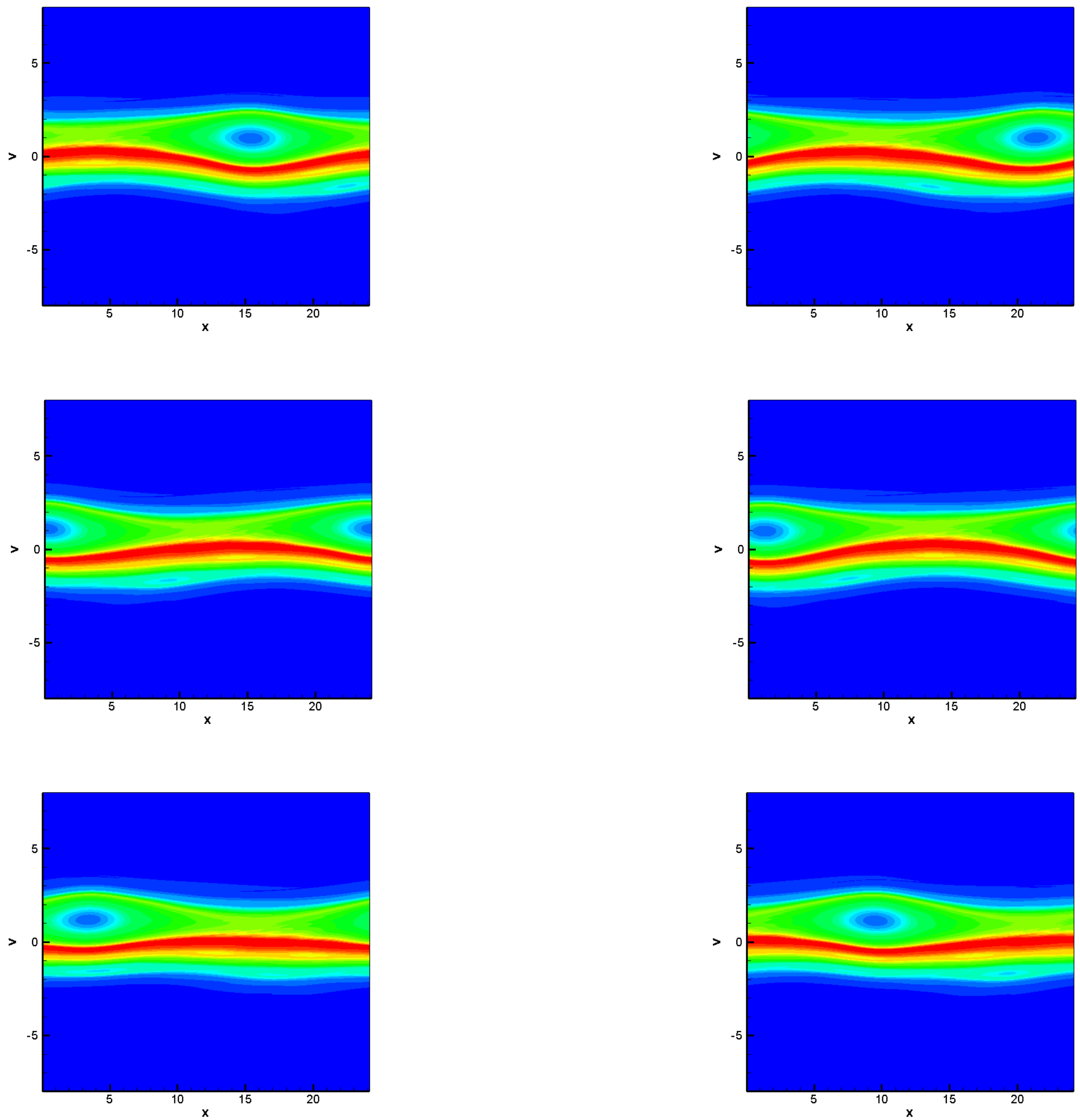
Higher Order Periodic/Quasiperiodic Orbit: $E(t) = A(t)E_0(t)$

$A(t) = A(t + T/4)$ with $E_0(t) = E_0(t + T)$

$E_0(t)$ like weak drive

Strong Drive Fourier





5
0
-5
v
5
10
15
20
x

Open Mathematics Problems

- Prove nonlinear Landau damping rate, growth, bounce – say anything about general phenomenology.

- Prove stability of any BGK mode. Mine?

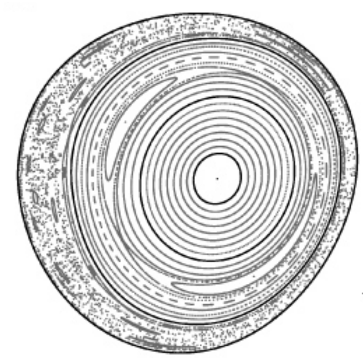
- Prove 'weak' asymptotic stability.

- Prove existence/nonexistence of cusp.

- Prove existence of weak drive periodic orbit. Stability. Weak asymptotic stability.

- Prove existence of strong drive periodic/quasiperiodic orbit. Stability. Weak asymptotic stability.

How?



- Finite-Dimensional Hamiltonian Systems:
  - ▷ ∃ periodic orbits near equilibria
    Lyapunov, Weinstein, Moser, ...
  - ▷ variational methods
    Rabinowitz, Ekland, ...

- Infinite-Dimensional Hamiltonian VP-Like Systems:
  - ▷ ∃ Hamilton-Jacobi Variational Principle for VP
    PJM, ... tutorial web page, online ICERM lecture
  - ▷ techniques: viscosity solutions, weak KAM, ...
    Villiani, Gangbo, Li, ...

Time is Ripe!