Hamiltonian and Action Principle Formulations of Plasma Physics

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Review HAP formulations with plasma applications.

“Hamiltonian systems … are the basis of physics.” M. Gutzwiller

Thanks: mentors, colleagues, students ….
Finalized Course Overview

1. Review of Basics (finite $\rightarrow$ infinite)

2. Ideal Fluids and Magnetofluids A

3. Ideal Fluids and Magnetofluids B

4. Ideal Fluids and Magnetofluids C

5. Kinetic Theory – Canonization & Diagonalization, Continuous Spectra, Krein-like Theorems

6. Metriplecticism: relaxation paradigms for computation and derivation
General References

Numbers refer to items on my web page: http://www.ph.utexas.edu/~morrison/ where all can be obtained under ‘Publications’.


HAP Formulations of PP: I Basics

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Action Principle

Hero of Alexandria (75 AD) → Fermat (1600’s) →

Hamilton’s Principle (1800’s)

The Procedure:

• Configuration Space $Q$: $q^i(t)$, $i = 1, 2, \ldots, N$ ← #DOF

• Kinetic - Potential: $L = T - V : TQ \times \mathbb{R} \rightarrow \mathbb{R}$

• Action Functional: paths $\rightarrow \mathbb{R}$

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t) \, dt, \quad \delta q(t_0) = \delta q(t_1) = 0$$

Extremal path $\Rightarrow$ Lagrange’s equations
Variation Over Paths

\[ S[q_{\text{path}}] = \text{number} \]

**Functional Derivative:** \( \iff \) **vanishing first variation**

\[ \frac{\delta S[q]}{\delta q^i} = 0 \iff \]

**Lagrange’s Equations:**

\[ \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0. \]
Hamilton’s Equations

Canonical Momentum: \( p_i = \frac{\partial L}{\partial \dot{q}^i} \)

Legendre Transform: \( H(q, p) = p_i \dot{q}^i - L \)

\[
\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i},
\]

Failure of LT (not convex) \( \implies \) Dirac constraint theory

Phase Space Coordinates: \( z = (q, p) \)

\[
\dot{z}^i = J^{ij}_c \frac{\partial H}{\partial z^j}, \quad (J^{ij}_c) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},
\]

symplectic 2-form = (cosymplectic form)\(^{-1}\): \( \omega^c_{ij} J^{jk}_c = \delta^k_i \),
Phase-Space Action

Gives Hamilton’s equations directly

\[ S[q,p] = \int_{t_0}^{t_1} dt \left( p_i \dot{q}^i - H(q,p) \right) \]

Defined on paths \( \gamma \) in phase space \( \mathcal{P} \) (e.g. \( T^*Q \)) parameterized by time, \( t \), i.e., \( z_\gamma(t) = (q_\gamma(t), p_\gamma(t)) \). Then \( S : \mathcal{P} \to \mathbb{R} \). Domain of \( S \) any smooth path \( \gamma \in \mathcal{P} \).

Law of nature, set Fréchet or functional derivative, to zero. Varying \( S \) by perturbing path, \( \delta z_\gamma(t) \), gives

\[ \delta S[z_\gamma; \delta z_\gamma] = \int_{t_0}^{t_1} dt \left[ \delta p_i \left( \dot{q}^i - \frac{\partial H}{\partial p_i} \right) - \delta q^i \left( \dot{p}_i + \frac{\partial H}{\partial q^i} \right) + \frac{d}{dt} \left( p_i \delta q^i \right) \right] . \]

Under the assumption \( \delta q(t_0) = \delta q(t_1) \equiv 0 \), with no restriction on \( \delta p \), boundary term vanishes.

Admissible paths in \( \mathcal{P} \) have ‘clothesline’ boundary conditions.
Phase-Space Action Continued

\[
\delta S \equiv 0 \Rightarrow \dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2, \ldots, N,
\]

Thus, extremal paths satisfy Hamilton’s equations.
Alternatives

Rewrite action $S$ as follows:

$$S[z] = \int_{t_0}^{t_1} dt \left( \frac{1}{2} \omega^c_{\alpha\beta} z^\alpha \dot{z}^\beta - H(z) \right) =: \int_{\gamma} (d\theta - H dt)$$

where $d\theta$ is a differential one-form.

Exercise: What are boundary conditions. General $\theta$?

Exercise: Particle motion in given electromagnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$

$$S[r, p] = \int_{t_0}^{t_1} dt \left[ p \cdot \dot{r} - \frac{1}{2m} \left| p - \frac{e}{c} \mathbf{A}(r, t) \right|^2 - e\phi(r, t) \right].$$

Show Lorentz force law arises from $S$. 
Generalized Hamiltonian Structure

Sophus Lie (1890) $\rightarrow$ PJM (1980)....

Noncanonical Coordinates:

$$
\dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} = [z^i, H], \quad [A, B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}
$$

Poisson Bracket Properties:

antisymmetry $\rightarrow$ $[A, B] = -[B, A],$

Jacobi identity $\rightarrow$ $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

G. Darboux: $detJ \neq 0 \implies J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \implies$ Canonical Coordinates plus Casimirs

Matter models in Eulerian variables: $J^{ij} = c_k^{ij} z^k$ $\leftarrow$ Lie – Poisson Brackets
Definition. A Poisson manifold $\mathcal{P}$ is a differentiable manifold with bracket $[,] : \mathcal{C}^\infty(\mathcal{P}) \times \mathcal{C}^\infty(\mathcal{P}) \to \mathcal{C}^\infty(\mathcal{P})$ such that $\mathcal{C}^\infty(\mathcal{P})$ with $[,]$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) considers only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $JdH$.

Because of degeneracy, $\exists$ functions $C$ st $[f,C] = 0$ for all $f \in \mathcal{C}^\infty(\mathcal{P})$. Called Casimir invariants (Lie's distinguished functions.)
Poisson Manifold $\mathcal{P}$ Cartoon

Degeneracy in $J \Rightarrow$ Casimirs:

$$[f, C] = 0 \quad \forall f : \mathcal{P} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:

$$\nabla C(\alpha) = \text{constant}$$
Hamiltonian Reduction

Bracket Reduction:

Reduced set of variables \((q,p) \mapsto w(q,p)\)

Bracket Closure:

\[
[w,w] = c(w) \quad f(q,p) = \hat{f} \circ w = \hat{f}(w(q,p))
\]

Chain Rule \(\Rightarrow\) yields noncanonical Poisson Bracket

Hamiltonian Closure:

\[
H(q,p) = \hat{H}(w)
\]

Example: Eulerian fluid variables are noncanonical variables

(pjm & John Greene 1980)
Reduction Examples/Exercises

• Let $q \in Q = \mathbb{R}^3$ and define the angular momenta $L_i = \epsilon_{ijk} q_j p_k$, with $i, j, k = 1, 2, 3$. Show $[L_i, L_j] = f_{ij}(L)$. What is $f_{ij}$?

• Given $w_k = L^i_k(q)p_i$, with $i = 1, 2, \ldots N$, find a nontrivial condition on $L^i_k$ that ensures reduction.
Why Action/Hamiltonian?

• Beauty, Teleology, . . . : Still a good reason!

• 20th Century framework for physics: Plasma models too.

• Symmetries and Conservation Laws: energy-momentum . . .

• Generality: do one problem ⇒ do all.

• Approximation: pert theory, averaging, . . . one function.

• Stability: built-in principle, Lagrange-Dirichlet, $\delta W$, . . .

• Beacon: motivation, e.g. $\exists$ $\infty$-dim KAM theorem? . . .

• Numerical Methods: structure preserving algorithms:
  symplectic/conservative integrators, . . .

• Statistical Mechanics: energy and measure.
Functionals

Functions: \( \text{number} \mapsto \text{number} \) \quad \text{e.g.} \quad f : \mathbb{R}^n \to \mathbb{R}

example

Generalized Coordinate: \( q(t) = A \cos(\omega t + \phi) \) \quad \text{e.g. SHO}

Functionals: \( \text{function} \mapsto \text{number} \) \quad \text{e.g.} \quad F : L^2 \to \mathbb{R}

examples

General: \( F[u] = \int F(u, u_x, u_{xx}, \ldots) \, dx \).

Hamilton’s Principle: \( S[q] = \frac{1}{2} \int_{t_0}^{t_1} L(q, \dot{q}, t) \, dt \).

Vlasov Energy: \( H[f] = \frac{m}{2} \int f v^2 \, dxdv + \frac{1}{2} \int E^2 \, dx \).
# Functional Differentiation

First variation of function:

\[ \delta f(z; \delta z) = \sum_{i=1}^{n} \frac{\partial f(z)}{\partial z_i} \delta z_i =: \nabla f \cdot \delta z, \quad f(z) = f(z_1, z_2, \ldots, z_n). \]

First variation of functional:

\[ \delta F[u; \delta u] = \left. \frac{d}{d\epsilon} F[u + \epsilon \delta u] \right|_{\epsilon=0} = \int_{x_0}^{x_1} \delta u \frac{\delta F}{\delta u(x)} \, dx =: \left< \frac{\delta F}{\delta u}, \delta u \right>. \]

\[ \text{dot product} \quad \cdot \quad \leftrightarrow \quad \text{scalar product} \quad <, > \]

\[ \text{index} \quad i \quad \leftrightarrow \quad \text{integration variable} \quad x \]

\[ \text{gradient} \quad \frac{\partial f(z)}{\partial z_i} \quad \leftrightarrow \quad \text{functional derivative} \quad \frac{\delta F[u]}{\delta u(x)} \]

Vary and Isolate \( \rightarrow \) Functional Derivative
Functional Differentiation Examples/Exercises

• Given

\[ H[u] = \int_T dx \left( \frac{u^3}{6} - \frac{u^2 x}{2} \right), \quad u : T \to \mathbb{R} \]

What is \( \delta u / \delta x \)?

• Given

\[ \mathcal{E}[E] = \frac{1}{2} \int_{\mathbb{R}} d^3x \, |E|^2 \]

What is \( \delta \mathcal{E} / \delta E \)? For \( E = -\nabla \phi \), how are \( \delta \mathcal{E} / \delta E \) and \( \delta \mathcal{E} / \delta \phi \) related?
Relativistic N-Particle Action

Dynamical Variables: $q_i(t), \phi(x,t), A(x,t)$

$$S[q, \phi, A] = - \sum_{i=1}^{N} \int dt \ mc^2 \sqrt{1 - \frac{\dot{q}_i^2}{c^2}}$$

ptle kinetic energy

coupling $\rightarrow$ $-e \int dt \sum_{i=1}^{N} \int d^3x \left[ \phi(x,t) + \frac{\dot{q}_i}{c} \cdot A(x,t) \right] \delta(x - q_i(t))$

field ‘energy’ $\rightarrow$ $+ \frac{1}{8\pi} \int dt \int d^3x \left[ E^2(x,t) - B^2(x,t) \right]$.

Variation:

$$\frac{\delta S}{\delta q^i(t)} = 0 \implies \text{EOM & Fields},$$

$$\frac{\delta S}{\delta \phi(x,t)} = 0, \quad \frac{\delta S}{\delta A(x,t)} = 0 \implies \text{ME & Sources}$$
All done?
Irrelevant Information

Reductions, Approximations, Mutilations, . . . :

⇒ Constraints (explicit or implicit) ⇒ Interesting!

Finite Systems

$B$-lines, ptle orbits, self-consistent models, . . .

Infinite Systems

kinetic theories, fluid models, mixed . . .

Lagrangian (material) or Eulerian (spacial) variables
Big Actions to Little Actions

**Hamiltonian**\textit{B-lines}: Set $\phi = 0$, specify $B$, let $r_G \to 0$

$$S[r] = \int A \cdot dr \quad \text{Kruskal (52)}$$

**Hamiltonian ptle orbits**: Specify $\phi$ and $B$ non-selfconsistent

Standard ptle orbit action $\implies$ tools

**Hamiltonian self-consistent models**: Specify $\phi$ and $B$ partly

**Single-Wave Model**: OWM(71), Kaufman & Mynick (79), Tennyson et al.(94), Balmforth et al. (2013), . . .


**Moment Models**: Kida, Chanell, Meacham et al. (95), Shadwick . . ., Perin et al. (2014).
Finite DOF Hamiltonian Vocabulary

Integrable 1 DOF

Poincare Section 1.5 DOF

KAM integrable limit

Invariant Tori good surfaces

Island Overlap broken surfaces

Chirkov-Taylor Map chaos

Greene’s Criterion tori far from integrable

Renormalization universality

Spectra no asymptotic stability

Stability Lagrange $\delta^2W$, Dirichlet $\delta^2H$, Energy-Casimir $\delta^2F$, ...

Normal Forms stable $\Rightarrow H = \sum \omega (q^2 + p^2)/2$, linear/nonlinear
Infinite DOF Hamiltonian Vocabulary

Integrable  KdV, ..., rare, Greene and Kruskal

KAM  active area in mathematics

Spectra  discrete, continuous

Stability  $\delta^2W, \delta^2H, \delta^2F$

Normal Forms  linear/nonlinear perturbation theories

Action Reduction  direct method of calculus of variations

Noether’s Thm  energy-momentum tensor only believable way

Hamiltonian Reduction  little systems from big, exact/approximate
HAP Formulations of PP: II Magnetofluids A

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Magnetofluid References

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General Method for Building Actions
Applied to Magnetofluids

Ex Post Facto Discovery vs. Ab Initio Construction
Senior Progeny
Computability and Intuition

Reductions $\Rightarrow$

Vlasov-Maxwell, two-fluid theory, MHD, ...

Neglect clearly identifiable dissipation $\Rightarrow$

Action principles and Hamiltonian structure

identified *ex post facto*
Simplifications: Reduced Fluid Models

Approximations:

asymptotic expansions, systematic ordering

Model Building:

Mutilations, put it what one this is important, closures etc.

Other Progeny:

Gyrokinetics, guiding-center kinetics, gyrofluids, ... .

Hamiltonian? Action?
Building Action Principles Ab Initio

Step 1: Select Domain

For fluid a spatial domain; for kinetic theory a phase space

Step 2: Select Attributes – Eulerian Variables (Observables)

L to E, map e.g. MHD \( \{v, \rho, s, B\} \). Builds in constraints!

Step 3: Eulerian Closure Principle

Terms of action must be ‘Eulerianizable’ \( \Rightarrow \) EOMs are!

Step 4: Symmetries

Traditional. Rotation, etc. via Noerther \( \Rightarrow \) invariants
Closure Principle

If closure principle is satisfied, then

i) Equations of motion obtained by variation are ‘Eulerianizable’.

ii) There exists a noncanonical Hamiltonian description.
Ideal Fluid and MHD
Fluid Action Kinematics

Giuseppe Luigi Lagrange, Mécanique analytique (1788)

Lagrangian Variables:

Fluid occupies domain $D$ e.g. $(x, y, z)$ or $(x, y)$

Fluid particle position $q(a, t), \quad q_t : D \rightarrow D$

bijective, smooth, diffeomorphism, . . .

Particle label: $a$ e.g. $q(a, 0) = a$.

Deformation: $\frac{\partial q^i}{\partial a^j} = q^i_{,j}$

Determinant: $J = \det(q^i_{,j}) \neq 0 \Rightarrow a(q, t)$

Identity: $q^i_{,k}a^k_{,j} = \delta^i_j$
Volume: \[ d^3q = \mathcal{J} d^3a \]

Area: \[ (d^2q)_i = \mathcal{J} a^j_i (d^2a)_j \]

Line: \[ (dq)_i = q^i_j (da)_j \]

Eulerian Variables:

Observation point: \( r \)

Velocity field: \( v(r, t) = ? \) Probe sees \( \dot{q}(a, t) \) for some \( a \).

What is \( a \)? \( r = q(a, t) \) \( \Rightarrow \) \( a = q^{-1}(r, t) \)

\[ v(r, t) = \dot{q}(a, t) \big|_{a=q^{-1}(r,t)} \]
IDEAL MHD

Attributes:

Entropy (1-form):

\[ s(r, t) = s_0 \big|_{a=a(r,t)} \]

Mass (3-form):

\[ \rho d^3 x = \rho_0 d^3 a \implies \rho(r, t) = \frac{\rho_0}{I} \bigg|_{a=a(r,t)} \]

B-Flux (2-form):

\[ B \cdot d^2 x = B_0 \cdot d^2 a \implies B^i(r, t) = \frac{q_{ij} B^j_0}{I} \bigg|_{a=a(r,t)} \]
**Kinetic Potential**

**Kinetic Energy:**

\[ K[q] = \frac{1}{2} \int_D d^3 a \, \rho_0 |\dot{q}|^2 = \frac{1}{2} \int_D d^3 x \, \rho |v|^2 \]

**Potential Energy:**

\[ V[q] = \int_D d^3 a \, \rho_0 \mathcal{V}(\rho_0 / \mathcal{J}, s_0, |q_{,i} B^j_0| / \mathcal{J}) = \frac{1}{2} \int_D d^3 x \, \rho \mathcal{V}(\rho, s, |B|) \]

\[ = \int_D d^3 a \, \rho_0 \mathcal{U}(\rho_0 / \mathcal{J}, s_0) + \frac{1}{2} \frac{|q_{,i} B^j_0|^2}{\mathcal{J}^2} \]

**Action:**

\[ S[q] = \int dt \, (K - V), \quad \delta S = 0 \quad \Rightarrow \quad \text{Ideal MHD} \]

**Alternative:** Lagrangian variations induce constrained Eulerian variations \( \Rightarrow \) Serrin, Newcomb, Euler-Poincaré, ...

**Stability:** \( \delta W, \) Lagrangian, Eulerian, dynamical accessible, Andreussi, Pegoraro, pjm. (2010 – 2014)
Equations of Motion and Eulerianization
Hamiltonian Structure

Legendre Transformation:

\[ p = \frac{\delta L}{\delta \dot{q}} = \rho_0 \dot{q} \quad L \rightarrow H \]

\[ H = \frac{1}{2} \int_D d^3a |p|^2 / \rho_0 + \int_D d^3a \left( \rho_0 U(\rho_0/\mathcal{J}, s_0) + \frac{1}{2} |q_{,j} B_{0j}|^2 \right) \]

Poisson Bracket:

\[ \{F, G\} = \int_D d^3a \left( \frac{\delta F}{\delta q^i} \frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta p_i} \right) \]

EOM:

\[ \dot{q} = \{q, H\} = p / \rho_0 \quad \dot{p} = \{p, H\} = \rho_0 \ddot{q} = \ldots \]

Complicated pde for \( q(a, t) \). Exercise. Derive it.
Eulerianization

Momentum:
\[ \rho \frac{\partial v}{\partial t} = -\rho v \cdot \nabla v - \nabla p + \frac{1}{c} J \times B \]

Attributes:
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v) \]
\[ \frac{\partial s}{\partial t} = -v \cdot \nabla s \]
\[ \frac{\partial B}{\partial t} = -\nabla \times E = \nabla \times (v \times B) \]

Thermodynamics:
\[ p = \rho^2 \frac{\partial U}{\partial \rho} \]
\[ s = \frac{\partial U}{\partial s} \]
**Infinite-Dimensional Hamiltonian Structure**

Field Variables: \( \psi(\mu, t) \)  
  e.g. \( \mu = x, \mu = (x, v), \ldots \)

Poisson Bracket:

\[
\{ A, B \} = \int \frac{\delta A}{\delta \psi} \mathcal{J}(\psi) \frac{\delta A}{\delta \psi} d\mu
\]

Lie-Poisson Bracket:

\[
\{ A, B \} = \left\langle \psi, \left[ \frac{\delta A}{\delta \psi}, \frac{\delta A}{\delta \psi} \right] \right\rangle
\]

Cosymplectic Operator:

\[ \mathcal{J} \cdot \sim [\psi, \cdot] \]

Form for **Eulerian theories**: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

Whence?
**Eulerian Reduction**

\[ F[q,p] = \hat{F}[v, \rho, s, B] \]

Chain Rule ⇒ yields noncanonical Poisson Bracket in terms of Eulerian variables (pjm & John Greene 1980)

It is an algorithmic process. Manipulations like calculus.

Hamiltonian Closure:

\[ H = \int_D d^3x \left( \rho |v|^2/2 + \rho U(\rho, s) + |B|^2/2 \right) \]
Chain rule to density Eulerian variables, $\{\rho, \sigma, M, B\}$

\[
\{F, G\} = -\int_D d^3r \left[ M_i \left( \frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \right. \\
+ \rho \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
+ B \cdot \left[ \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
+ \left. B \cdot \left[ \nabla \left( \frac{\delta F}{\delta M} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left( \frac{\delta G}{\delta M} \right) \cdot \frac{\delta F}{\delta B} \right] \right],
\]

Eulerian Hamiltonian form:

\[
\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial s}{\partial t} = \{s, H\}, \quad \frac{\partial v}{\partial t} = \{v, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.
\]

Densities:

\[
M = \rho v \quad \sigma = \rho s
\]
HAP Formulations of PP: III Magnetofluids B

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Magnetofluid B Overview

- Complete MHD

- Other magnetofluids. More ab initio construction
Infinite-Dimensional Hamiltonian Structure

Field Variables: $\psi(\mu, t)$  e.g. $\mu = x, \mu = (x, v), \ldots$

Poisson Bracket:

$$\{A, B\} = \int \frac{\delta A}{\delta \psi} \mathcal{J}(\psi) \frac{\delta A}{\delta \psi} d\mu$$

Lie-Poisson Bracket:

$$\{A, B\} = \langle \psi, \left[ \frac{\delta A}{\delta \psi}, \frac{\delta A}{\delta \psi} \right] \rangle$$

Cosymplectic Operator:

$$\mathcal{J} \cdot \sim [\psi, \cdot]$$

Form for Eulerian theories: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

Whence?
Eulerian Reduction

\[ F[q,p] = \hat{F}[v, \rho, s, B] \]

Chain Rule ⇒ yields noncanonical Poisson Bracket in terms of Eulerian variables (pjm & John Greene 1980)

It is an algorithmic process. Manipulations like calculus.

Hamiltonian Closure:

\[ H = \int_D d^3x \left( \rho|v|^2/2 + \rho U(\rho, s) + |B|^2/2 \right) \]
Chain rule to density Eulerian variables, \{\rho, \sigma = \rho s, M = \rho v, B\}

\[\{F, G\} = -\int_D d^3r \left[ M_i \left( \frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) + \rho \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) + B \cdot \left[ \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] + B \cdot \left[ \nabla \left( \frac{\delta F}{\delta M} \right) \frac{\delta G}{\delta B} - \nabla \left( \frac{\delta G}{\delta M} \right) \frac{\delta F}{\delta B} \right]\right],\]

Eulerian Hamiltonian form

\[\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, H\}, \quad \frac{\partial M}{\partial t} = \{M, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.

What is
\[\frac{\delta \rho(x)}{\delta \rho(x')} = ?\]
Chain rule to density Eulerian variables, \(\{\rho, \sigma, M, B\}\)

\[
\{F, G\} = -\int_D d^3r \left[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i}\right) + \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho}\right) + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma}\right) + B \cdot \left[\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B}\right] \right)
\]

Eulerian Hamiltonian form

\[
\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, H\}, \quad \frac{\partial M}{\partial t} = \{M, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.
\]

What is 

\[
\frac{\delta \rho(x)}{\delta \rho(x')} = \delta(x - x')?
\]
Chain rule to density Eulerian variables, \( \{\rho, \sigma, M, B\} \)

\[
\{F, G\} = -\int_D d^3r \left[ M_i \left( \frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) + \rho \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) + B \cdot \left[ \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] \right],
\]

Eulerian Hamiltonian form

\[
\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, H\}, \quad \frac{\partial M}{\partial t} = \{M, H\}, \quad \text{and} \quad \frac{\partial B}{\partial t} = \{B, H\}.
\]

What is

\[
\frac{\delta \rho(x)}{\delta \rho(x')} = \delta(x - x') \quad \frac{\partial q^i}{\partial q^j} = \delta^i_j
\]
Explicit Eulerian Reduction

Reduce Lagrangian Hamiltonian description to Eulerian Hamiltonian description.

Recall.

Hamiltonian:

\[ H = \frac{1}{2} \int_D d^3a |p|^2 / \rho_0 + \int_D d^3a \left( \rho_0 U(\rho_0 / J, s_0) + \frac{1}{2} \frac{|q^i B^j_0|^2}{J^2} \right) \]

Poisson Bracket:

\[ \{ F, G \} = \int_D d^3a \left( \frac{\delta F}{\delta q^i} \frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta p_i} \right) \]

EOM:

\[ \dot{q} = \{ q, H \} = p / \rho_0 \quad \dot{p} = \{ p, H \} = \rho_0 \ddot{q} = -\frac{\delta V}{\delta q} \]
Suppose functionals $F$ and $G$ are restricted to Eulerian variables

$$F[q, p] = \hat{F}[\rho, s, v, B].$$

Then, variation gives

$$\delta F = \int_D d^3a \left( \frac{\delta F}{\delta q} \cdot \delta q + \frac{\delta F}{\delta p} \cdot \delta p \right) = \delta \hat{F}$$

$$= \int_D d^3x \left( \frac{\delta \hat{F}}{\delta \rho} \delta \rho + \frac{\delta \hat{F}}{\delta s} \delta s + \frac{\delta \hat{F}}{\delta v} \cdot \delta v + \frac{\delta \hat{F}}{\delta B} \cdot \delta B \right).$$

Here, \(\{\delta \rho, \delta s, \delta v, \delta B\}\) induced by \((\delta q, \delta p)\). How?

Recall

$$\rho(r, t) = \rho_0 \left| \frac{\partial}{\partial a} \right|_{a = a(r, t)} = \int_D d^3a \rho_0(a) \delta (r - q(a, t)).$$

Thus

$$\delta \rho = -\int_D d^3a \rho_0 \nabla \delta (r - q) \cdot \delta q, \quad \delta s, \delta B, \delta v = \ldots$$
Insertion of $\delta \rho$ etc. gives

$$\int_D d^3a \left( \frac{\delta F}{\delta q} \cdot \delta q + \frac{\delta F}{\delta p} \cdot \delta p \right) = -\int_D d^3x \frac{\delta \hat{F}}{\delta \rho} \int_D d^3a \rho_0 \nabla \delta (r-q) \cdot \delta q + \ldots .$$

Interchange integration order, remove $\int_D d^3a$ since $\delta q$ arbitrary gives

$$\frac{\delta F}{\delta q} = \mathcal{O}_\rho \frac{\delta \hat{F}}{\delta \rho} + \mathcal{O}_s \frac{\delta \hat{F}}{\delta s} + \mathcal{O}_v \frac{\delta \hat{F}}{\delta v} + \mathcal{O}_B \frac{\delta \hat{F}}{\delta B},$$

where the $\mathcal{O}'s$ are operators involving integration over $d^3x$ and Dirac delta functions. Upon insertion with similar expression for $\delta F/\delta p$, doing some rearrangement, and dropping the hats, yields
\[ \{F,G\} = -\int_{D} d^3x \left[ \left( \frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta v} - \frac{\delta G}{\delta \rho} \nabla \cdot \frac{\delta F}{\delta v} \right) + \left( \frac{\nabla \times v}{\rho} \cdot \frac{\delta G}{\delta v} \times \frac{\delta F}{\delta v} \right) + \frac{\nabla s}{\rho} \cdot \left( \frac{\delta F}{\delta s} \frac{\delta G}{\delta v} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta v} \right) \right. \\
+ B \cdot \left[ \frac{1}{\rho} \frac{\delta F}{\delta v} \cdot \nabla \frac{\delta G}{\delta B} - \frac{1}{\rho} \frac{\delta G}{\delta v} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
+ \left. B \cdot \left[ \nabla \left( \frac{1}{\rho} \frac{\delta F}{\delta v} \right) \right) \frac{\delta G}{\delta B} - \nabla \left( \frac{1}{\rho} \frac{\delta G}{\delta v} \right) \frac{\delta F}{\delta B} \right]. \]

Then \( M = \rho v \) and \( \sigma = \rho s \) gives Lie-Poisson form.
Other Magnetofluids
Braginskii MHD

\[ \rho \left( v_t + v \cdot \nabla v \right) = -\nabla p + J \times B + \nabla \cdot \Pi \]

Gyroviscosity Tensor: \( \Pi_{ij} = \frac{p}{B} N_{jsik} \frac{\partial v_s}{\partial x_k} \)

Action:

\[ S[q] = \int dt \left( K + G - V \right) , \]

Gyroscopic Term:

\[ G[q] = \int_D d^3a \ \Pi^* \cdot \dot{q} = \int_D d^3x \ M^* \cdot v \]

where

\[ \Pi^* = \nabla \times L^* = \frac{m}{2e} \mathcal{J} \hat{b} \times \nabla \left( \frac{p}{B} \right) \]

\( \delta S[q] = 0 \quad \Rightarrow \quad \text{Braginskii MHD} \)

pjm, Lingam, Acevedo, Wurm 2014
Inertial MHD (Tassi)

Basic Idea: Can ‘freeze-in’ anything one likes! (2-form attribute)

Choose:

\[ B_e = B + d_e^2 \nabla \times J, \]

Action:

\[ S = \int dt \int d^3x \left( \rho \frac{v^2}{2} - \rho U(\rho, s) - B_e \cdot B \right). \]

Attributes:

\[ \rho d^3x = \rho_0 d^3a, \quad B_e^i = \frac{B_{e0}^j}{J} \frac{\partial q^i}{\partial a_j} \]

\[ \delta S[q] = 0 \quad \Rightarrow \quad \text{IMHD} \]
Eulerian Reduction

\[ F[q, p] = \hat{F}[\omega, \psi] \]

Chain Rule \( \Rightarrow \) yields noncanonical Poisson Bracket in terms of Eulerian variables \((\omega, \psi)\)

It is an algorithmic process.

Example: 2D IMHD

\[ \{F, G\} = -\int d^3x (\omega[F_\omega, G_\omega] + \psi_e([F_\omega, G_\psi] - [G_\omega, F_\psi])) \]

\[ H = \int d^2x (d_e^2(\nabla^2\psi)^2 + |\nabla\psi|^2 + |\nabla\phi|^2) \]

Produces 2D incompressible IMHD (Ottaviani-Porcelli model)!

Above, \( F_\omega := \delta F/\delta \omega, [f, g] := f_yg_x - g_yf_x, \omega = \hat{z} \cdot \nabla \times v, B = \hat{z} \times \nabla \psi. \)
Two-Fluid Action
Keramidas Charidakos, Lingam, pjm, R. White and A. Wurm

\[ S[q_s, A, \phi] = \int dt \int d^3x \left[ \frac{1}{c} \frac{\partial A(x, t)}{\partial t} - \nabla \phi(x, t) \right]^2 \\
- \left| \nabla \times A(x, t) \right|^2 \frac{1}{8\pi} + \sum_s \int d^3a n_{s0}(a) \int d^3x \delta(x - q_s(a, t)) \\
\times \left[ \frac{e_s}{c} \dot{q}_s \cdot A(x, t) - e_s \phi(x, t) \right] \]

\[ + \sum_s \int d^3a n_{s0}(a) \left[ \frac{m_s}{2} |\dot{q}_s|^2 \\
- m_s U_s \left( m_s n_{s0}(a) / J_s, s_{s0} \right) \right]. \]  

Eulerian Observables:

\[ \{ n_\pm, v_\pm, A, \phi \} \]
Reduced Variables

New Lagrangian Variables:

\[ Q(a,t) = \frac{1}{\rho_{m0}(a)} (m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)) \]

\[ D(a,t) = e (n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)) \]

\[ \rho_{m0}(a) = m_i n_{i0}(a) + m_e n_{e0}(a) \]

\[ \rho_{q0}(a) = e (n_{i0}(a) - n_{e0}(a)) \].

Consistent Expansion:

\[ \frac{v_A}{c} \ll 1, \quad \frac{m_e}{m_i} \ll 1 \implies \text{quasineutrality} \]

Eulerian Closure:

\[ \{n, s, s_e, v, J\} \]
Extended MHD

Ohm’s Law:

\[ E + \frac{v \times B}{c} = m_e \frac{e^2}{2n} \left( \frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv) \right) \]
\[ - \frac{m_e}{e^2 n} (J \cdot \nabla) \left( \frac{J}{n} \right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en} . \]

Momentum:

\[ nm \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = -\nabla p + \frac{J \times B}{c} \]
\[ - \frac{m_e}{e^2} (J \cdot \nabla) \left( \frac{J}{n} \right) . \]
Extended MHD

Ohm’s Law:

\[ E + \frac{v \times B}{c} = \frac{m_e}{e^2 n} \left( \frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv) \right) \]
\[ - \frac{m_e}{e^2 n} (J \cdot \nabla) \left( \frac{J}{n} \right) + \frac{(J \times B)}{enc} \]
\[ - \frac{\nabla p_e}{en}. \]

Momentum:

\[ nm \left( \frac{\partial v}{\partial t} + (V \cdot \nabla)v \right) = -\nabla p + \frac{J \times B}{c} \]
\[ - \frac{m_e}{e^2} (J \cdot \nabla) \left( \frac{J}{n} \right). \]

Consistent with an ordering of Lüst (1958)
Extended MHD

Ohm’s Law:

\[ E + \frac{v \times B}{c} = \frac{m_e}{e^2 n} \left( \frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv) \right) \]
\[ - \frac{m_e}{e^2 n} (J \cdot \nabla) \left( \frac{J}{n} \right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en} . \]

Momentum:

\[ nm \left( \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = -\nabla p + \frac{J \times B}{c} \]
\[ -\frac{m_e}{e^2} (J \cdot \nabla) \left( \frac{J}{n} \right) . \]

Consistent with an ordering of Lüst (1958)
Extended MHD

Ohm's Law:

\[ E + \frac{v \times B}{c} = \frac{me}{e^2n} \left( \frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv) \right) \]

\[ -\frac{me}{e^2n} (J \cdot \nabla) \left( \frac{J}{n} \right) + \frac{(J \times B)}{enc} - \frac{\nabla p}{en} \]

Momentum:

\[ nm \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = -\nabla p + \frac{J \times B}{c} \]

\[ -\frac{me}{e^2} (J \cdot \nabla) \left( \frac{J}{n} \right) \]
Extended MHD

Ohm’s Law:

\[ E + \frac{v \times B}{c} = \frac{m_e}{e^2 n} \left( \frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv) \right) - \frac{m_e}{e^2 n} (J \cdot \nabla) \left( \frac{J}{n} \right) + \frac{(J \times B)}{en} - \nabla p \frac{e}{en} . \]

Momentum:

\[ nm \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = -\nabla p + \frac{J \times B}{c} - \frac{m_e}{e^2 n} (J \cdot \nabla) \left( \frac{J}{n} \right) . \]

Consistent with an ordering of Lüst (1958)
Noether → Energy Conservation

Energy:

\[ H = \int d^3 x \left[ \frac{|B|^2}{8\pi} + nU_i + nU_e + mn\frac{|v|^2}{2} + \frac{m_e |J|^2}{ne^2} \right] \]

Energy conservation requires

\[ \frac{m_e}{e^2} (J \cdot \nabla) \left( \frac{J}{n} \right) \]

in momentum equation. Otherwise inconsistent.

Physical dissipation is real. Fake dissipation is troublesome, particularly for reconnection studies. Kimura and pjm (2014).
HAP Formulations of PP: V Kinetic Theory – Canonization & Diagonalization, Continuous Spectra, Krein-like Theorems

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References

Numbers refer to items on my web page: http://www.ph.utexas.edu/~morrison/ where all can be obtained under ‘Publications’.


References (cont)


56. P. J. Morrison and D. Pfirsch, Dielectric Energy versus Plasma Energy, and Hamiltonian Action-Angle Variables for the Vlasov Equation, Physics of Fluids B 4, 3038–3057 (1992). The original transformation derives here. Here it is also shown that the usual $\omega \partial \epsilon / \partial \omega$ formula is not correct for Vlasov and the correct formal is derived.


Overview

- Solve **stable** linearized Vlasov-Poisson as a Hamiltonian system.

- **Normal Form:**

\[
H = \sum_{i}^{N} \frac{\omega_i}{2} \left( p_i^2 + q_i^2 \right) = \sum_{i}^{N} \omega_i J_i \rightarrow \sum_{k=1}^{\infty} \int_{\mathbb{R}} du \omega_k(u) \left( P_k^2(u) + Q_k^2(u) \right)
\]

When stable \( \exists \) a canonical transformation to this form. NEMs and Krein-Moser.

- **Continuous Spectrum:** Transform \( G[f] \) (generalization of Hilbert transform) that diagonalizes Vlasov.

- **General Diagonalization:** General transform for a large class of Hamiltonian systems.

- Continuous spectra and Krein bifurcations.
Phase space density $f(x, v, t)$ (1 + 1 + 1 field theory):

$$f : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e}{m} \frac{\partial \phi[x, t; f]}{\partial x} \frac{\partial f}{\partial v} = 0$$

Poisson’s equation:

$$\phi_{xx} = -4\pi \left[ e \int_{\mathbb{R}} f(x, v, t) \, dv - \rho_B \right]$$

Energy:

$$H = \frac{m}{2} \int_{T} \int_{\mathbb{R}} v^2 f \, dxdv + \frac{1}{8\pi} \int_{T} (\phi_x)^2 \, dx$$

Boundary Conditions:

periodic $\iff X = \mathbb{T} := [0, 2\pi)$
Linear Vlasov-Poisson

Linearization:

\[ f = f_0(v) + \delta f(x, v, t) \]

Linearized EOM:

\[ \frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial x} - \frac{e}{m} \frac{\partial \delta \phi [x, t; \delta f]}{\partial x} \frac{\partial f_0}{\partial v} = 0 \]

\[ \delta \phi_{xx} = -4\pi e \int_{\mathbb{R}} \delta f(x, v, t) \, dv \]

Linearized Energy (Kruskal and Oberman, 1958):

\[ H_L = -\frac{m}{2} \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{v(\delta f)^2}{f_0'} \, dv \, dx + \frac{1}{8\pi} \int_{\mathbb{T}} (\delta \phi_x)^2 \, dx \]
Solution of Linear VP by Transform

Assume

\[ \delta f = \sum_k f_k(v, t)e^{ikx}, \quad \delta \phi = \sum_k \phi_k(t)e^{ikx} \]

Linearized EOM:

\[ \frac{\partial f_k}{\partial t} + ikvf_k - ik\phi_k \frac{e}{m} \frac{\partial f_0}{\partial v} = 0 \]
\[ k^2 \phi_k = 4\pi e \int_{\mathbb{R}} f_k(v, t) \, dv \quad \text{(LVP)} \]

Three methods:

- Laplace Transforms (Landau and others 1946)
- Normal Modes (Van Kampen, Case,... 1955)
- Coordinate Change ⇔ Integral Transform (pjm, Pfirsch, Shadwick, Balmforth, Hagstrom, 1992 → 2013)
Summary

The Transform,

\[ G[g](v) := \epsilon_R(v) g(v) + \epsilon_I(v) H[g](v), \]

where \( H \) is the Hilbert transform and \( \epsilon_{R,I} \) are functions that depend on \( f_0 \), has an inverse \( \hat{G} \) that maps (LVP) into

\[ \frac{\partial g_k}{\partial t} + ik u g_k = 0 \]

whence

\[ f_k(v, t) = G[\hat{G}[\hat{f}_k]e^{-ikt}] , \]

where

\[ \hat{f}_k(v) := f_k(v, t = 0) \]
Good Equilibria $f_0$ and Initial Conditions $f_k$

**Definition (VP1).** A function $f_0(v)$ is a **good equilibrium** if $f_0'(v)$ satisfies

(i) $f_0' \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}), \ 1 < q < \infty$ and $0 < \alpha < 1$,

(ii) $\exists v^* > 0 \ st \ |f_0'(v)| < A|v|^{-\mu} \ \forall |v| > v^*, \ where \ A, \mu > 0$, and

(iii) $f_0'/v < 0 \ \forall v \in \mathbb{R}$ or $f_0$ is Penrose stable. Assume $f_0'(0) = 0$.

**Definition (VP2).** A function, $f_k(v)$, is a **good initial condition** if it satisfies

(i) $f_k(v), vf_k(v) \in L^p(\mathbb{R})$,

(ii) $\int_{\mathbb{R}} f_k(v) \ dv < \infty$. 

Hilbert Transform

Definition

\[ H[g](x) := \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{g(t)}{t-x} \, dt , \]

\( \text{P.V.} \) denotes Cauchy principal value.

\( \exists \) theorems about Hilbert transforms in \( L^p \) and \( C^{0,\alpha} \). Plemelj, M. Riesz, Zygmund, and Titchmarsh \( \cdots \) (Can be extracted from Calderón-Zygmund theory.) Recent tome by King.
Hilbert Transform Theorems

Theorem (H1).

(ii) \( H : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \ 1 < p < \infty, \) is a bounded linear operator:

\[
\|H[g]\|_p \leq A_p \|g\|_p,
\]

where \( A_p \) depends only on \( p \),

(ii) \( H \) has an inverse on \( L^p(\mathbb{R}) \), given by

\[
H[H[g]] = -g,
\]

(iii) \( H : L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}) \to L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}). \)
**Theorem (H2).** If \( g_1 \in L^p(\mathbb{R}) \) and \( g_2 \in L^q(\mathbb{R}) \) with \( \frac{1}{p} + \frac{1}{q} < 1 \), then

\[
H[g_1H[g_2] + g_2H[g_1]] = H[g_1]H[g_2] - g_1g_2.
\]

**Proof:** Based on the Hardy-Poincaré-Bertrand theorem, Tricomi.

**Lemma (H3).** If \( vg \in L^p(\mathbb{R}) \), then

\[
H[vg](u) = u H[g](u) + \frac{1}{\pi} \int_{\mathbb{R}} g \, dv.
\]

**Proof:**

\[
\frac{v}{v-u} = \frac{u+v-u}{v-u} = \frac{u}{v-u} + 1
\]
The Transform

Definition (G1). The transform is defined by

\[ f(v) = G[g](v) := \epsilon_R(v) g(v) + \epsilon_I(v) H[g](v), \]

where

\[ \epsilon_I(v) = -\pi \omega_p^2 f_0'(v)/k^2, \quad \epsilon_R(v) = 1 + H[\epsilon_I](v). \]
Remarks

1. We suppress the dependence of $\epsilon$ on $k$ throughout. Note, $\omega_p^2 := 4\pi n_0 e^2/m$ is the plasma frequency corresponding to an equilibrium of number density $n_0$.

2. $\epsilon = \epsilon_R + i\epsilon_I$ (complex extended, appropriately) is the plasma dispersion relation whose vanishing $\Rightarrow$ discrete normal eigen-modes. When $\epsilon \neq 0$ $\exists$ only continuous spectrum; there is no dispersion relation.
Transform Theorems

**Theorem (G2).** \( G: L^p(\mathbb{R}) \to L^p(\mathbb{R}), \) \( 1 < p < \infty, \) is a bounded linear operator:

\[
\|G[g]\|_p \leq B_p \|g\|_p,
\]

where \( B_p \) depends only on \( p. \)

**Theorem (G3).** If \( f_0 \) is a good equilibrium, then \( G[g] \) has an inverse, \( \hat{G}: L^p(\mathbb{R}) \to L^p(\mathbb{R}), \)

for \( 1/p + 1/q < 1, \) given by

\[
g(u) = \hat{G}[f](u)
:= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[f](u),
\]

where \( |\epsilon|^2 := \epsilon_R^2 + \epsilon_I^2. \)
Proof: First we show \( g \in L^p(\mathbb{R}) \), then \( g = \hat{G}[G[g]] \).

If \( \varepsilon_R(u)/|\varepsilon(u)|^2 \) and \( \varepsilon_I(u)/|\varepsilon(u)|^2 \) are bounded, then clearly \( g \in L^p(\mathbb{R}) \). For good equilibria the numerators are bounded and everything is Hölder, so it is only necessary to show that \( |\varepsilon| \) is bounded away from zero. Either of the conditions of (VP1)(iii) assures this. Consider the first (monotonicity) condition,

\[
|f'_0| > 0 \quad \text{for} \quad v \neq 0 \quad \text{and} \quad f'_0(0) = 0.
\]

We need only look at \( v = 0 \) and \( v = \infty \). At \( v = 0 \)

\[
\varepsilon_R(0) = 1 - \frac{\omega_P^2}{k^2} \int_{\mathbb{R}} \frac{f'_0}{v} dv > 1 > 0,
\]

while as \( v \to \infty \), \( \varepsilon_R \to 1 \).

That \( \hat{G} \) is the inverse follows directly upon inserting \( G[g] \) of (G1) into \( g = \hat{G}[G[g]] \), and using (H2) and \( \varepsilon_R(v) = 1 + H[\varepsilon_I] \). \( \square \)
That $\hat{G}$ is the inverse follows directly upon inserting $G[g]$ of (G1) into $g = \hat{G}[G[g]]$, and using (H2) and $\epsilon_R(v) = 1 + H[\epsilon_I]$. 

\[
g(u) = \hat{G}[f](u) = \frac{\epsilon_R(u)}{|\epsilon(u)|^2} f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[f](u) \\
= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} \left[ \epsilon_R(u) g(u) + \epsilon_I(u) H[g](u) \right] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H \left[ \epsilon_R(u') g(u') + \epsilon_I(u') H[g](u') \right] (u) \\
= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} g(u) + \frac{\epsilon_R(u) \epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H \left[ H[\epsilon_I] g + \epsilon_I H[g] \right] (u) \\
= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} g(u) + \frac{\epsilon_R(u) \epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} \left[ H[\epsilon_I] (u) H[g](u) - g(u) \epsilon_I(u) \right] \\
= g(u) + \frac{\epsilon_R(u) \epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[\epsilon_I] H[g] \\
= g(u) + \frac{\epsilon_R(u) \epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \left[ 1 + H[\epsilon_I](u) \right] \\
= g(u) + \frac{\epsilon_R(u) \epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \epsilon_R(u) = g(u)
\]
Lemma (G4). If $\epsilon_I$ and $\epsilon_R$ are as above, then

(i) for $vf \in L^p(\mathbb{R})$,

$$\hat{G}[vf](u) = u \hat{G}[f](u) - \frac{\epsilon_I}{|\epsilon|^2} \frac{1}{\pi} \int_{\mathbb{R}} f \, dv,$$

(ii) $\hat{G}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$

(iii) and if $f(u,t)$ and $g(v,t)$ are strongly differentiable in $t$; i.e. the mapping $t \mapsto f(t) = f(t, \cdot) \in L^p(\mathbb{R})$ is differentiable, (the usual difference quotient converges in the $L^p$ sense), then

a) $\hat{G} \left[ \frac{\partial f}{\partial t} \right] = \frac{\partial \hat{G}[f]}{\partial t} = \frac{\partial g}{\partial t}$, 

b) $G \left[ \frac{\partial g}{\partial t} \right] = \frac{\partial G[g]}{\partial t} = \frac{\partial f}{\partial t}$.

Proof: (i) goes through like (H3), (ii) follows from $\epsilon_R = 1 + H[\epsilon_I]$, and (iii) follows because $G$ is bounded and linear.
Solution

Solve like Fourier transforms: operate on EOM with $\hat{G} \Rightarrow$,

$$\frac{\partial g_k}{\partial t} + ik u g_k = 0$$

and so

$$g_k(u, t) = \hat{g}_k(u) e^{-ikut}$$

Using $\hat{g}_k = \hat{G}[\hat{f}_k]$ we obtain the solution

$$f_k(v, t) = G[g_k(u, t)] = G[\hat{g}_k(u) e^{-ikut}] = G[\hat{G}[\hat{f}_k] e^{-ikut}]$$

**Theorem (S1).** For good initial conditions and equilibria,

$$f_k(v, t) = G[\hat{G}[\hat{f}_k] e^{-ikut}]$$

is an $L^p(\mathbb{R})$ solution of (LVP).
VP Hamiltonian Structure

Energy is quadratic $\Rightarrow$ SHO? However, V-P equation is quadratically nonlinear. Canonically conjugate variables?

Noncanonical Poisson Bracket (pjm 1980):

$$\{F,G\} = \int f \left[ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right] dxdv$$

$F$ and $G$ are functionals. VP $\iff$

$$\frac{\partial f}{\partial t} = \{f, H\} = [f, \mathcal{E}].$$

where $\mathcal{E} = mv^2/2 + e\phi$ and

$$[f, \mathcal{E}] = \frac{1}{m} \left( \frac{\partial f}{\partial x} \frac{\partial \mathcal{E}}{\partial v} - \frac{\partial \mathcal{E}}{\partial x} \frac{\partial f}{\partial v} \right)$$

Organizes: VP, Euler, QG, Defect Dyn, Benny-Dirac, ....
Linear Hamiltonian Structure

Linearization:

\[ f = f_0(v) + \delta f \]

where \( f_0(v) \) assumed stable (NEMs ok) \( \Rightarrow \)

\[ \{F, G\}_L = \int f_0 \left[ \frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f} \right] dx dv , \]

which with the Kruskal and Oberman energy,

\[ H_L = -\frac{m}{2} \int_T \int_R v (\delta f)^2 dv dx + \frac{1}{8\pi} \int_T (\delta \phi_x)^2 dx , \]

LVP \( \iff \)

\[ \frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L . \]
Canonization & Diagonalization

Fourier Linear Poisson Bracket:
\[
\{ F, G \}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f'_0 \left( \frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv.
\]

Linear Hamiltonian:
\[
H_L = -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f'_0} |f_k|^2 dv + \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2
= \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v) O_{k,k'}(v|v') f_{k'}(v') dv dv'.
\]

Canonize:
\[
q_k(v, t) = \frac{m}{ik f'_0} f_k(v, t), \quad p_k(v, t) = f_{-k}(v, t)
\]

\[
\{ F, G \}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left( \frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv.
\]
Diagonalization

Mixed Variable Generating Functional:

\[ \mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v) G[P'_k](v) \, dv \]

Canonical Coordinate Change \((q, p) \leftrightarrow (Q', P'):\)

\[ p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = G[P_k](v), \quad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = G^\dagger[q_k](u) \]

New Hamiltonian:

\[ H_L = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du \, \sigma_k(u) \omega_k(u) \left[ Q'_k(u) + P'_k(u) \right] \]

where \( \omega_k(u) = |ku| \) and the signature is

\[ \sigma_k(v) := -\text{sgn}(vf'_0(v)) \]
Sample Homogeneous Equilibria

$\leftarrow$ Maxwellian

BiMaxwellian $\rightarrow$
Hamiltonian Spectrum

Hamiltonian Operator:

\[
\partial_t f_k = -i v f_k + \frac{if'_0}{k} \int_{\mathbb{R}} d\bar{v} \, f_k(\bar{v}, t) =: \mathcal{H}_k f_k ,
\]

Complete System:

\[
\partial_t f_k = \mathcal{H}_k f_k \quad \text{and} \quad \partial_t f_{-k} = \mathcal{H}_{-k} f_{-k} , \quad k \in \mathbb{R}^+ 
\]

Lemma. If \( \lambda \) is an eigenvalue of the Vlasov equation linearized about the equilibrium \( f'_0(v) \), then so are \(-\lambda\) and \( \lambda^* \). Thus if \( \lambda = \gamma + i\omega \), then eigenvalues occur in the pairs, \( \pm \gamma \) and \( \pm i\omega \), for purely real and imaginary cases, respectively, or quartets, \( \lambda = \pm \gamma \pm i\omega \), for complex eigenvalues.
Spectral Stability

**Definition** The dynamics of a Hamiltonian system linearized around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is spectrally stable if the spectrum $\sigma(\mathcal{H})$ of the time evolution operator $\mathcal{H}$ is purely imaginary.

**Theorem** If for some $k \in \mathbb{R}^+$ and $u = \omega/k$ in the upper half plane the plasma dispersion relation,

$$\varepsilon(k, u) := 1 - k^{-2} \int_{\mathbb{R}} dv \frac{f_0'}{u - v} = 0,$$

then the system with equilibrium $f_0$ is spectrally unstable. Otherwise it is spectrally stable.
Nyquist Method

\[ f_0' \in C^{0,\alpha}(\mathbb{R}) \Rightarrow \varepsilon \in C^\infty(uhp). \]

Therefore, Argument Principle \( \Rightarrow \) winding \# = \# zeros of \( \varepsilon \)
Nyquist Method Examples

Winding number of $u \in \mathbb{R} \rightarrow \varepsilon$, or

$$\lim_{u \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} dv \frac{f'_0}{v - u} = H[f'_0](u) - if'_0(u),$$

Penrose plot for $f'_0 = -2(v - \frac{3}{4}) e^{- (v-\frac{3}{4})^2} - 2(v + \frac{3}{4}) e^{- (v+\frac{3}{4})^2}$
Spectral Theorem

Set $k = 1$ and consider $\mathcal{H}: f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_\mathbb{R} dv(|f| + |f'|)$. Contains all functions in $L^1(\mathbb{R})$ with weak derivatives in $L^1(\mathbb{R})$. $\mathcal{H}$ is densely defined, closed, etc.

**Definition** Resolvent of $\mathcal{H}$ is $R(\mathcal{H}, \lambda) = (\mathcal{H} - \lambda I)^{-1}$ and $\lambda \in \sigma(\mathcal{H})$. (i) $\lambda$ in point spectrum, $\sigma_p(\mathcal{H})$, if $R(\mathcal{H}, \lambda)$ not injective. (ii) $\lambda$ in residual spectrum, $\sigma_r(\mathcal{H})$, if $R(\mathcal{H}, \lambda)$ exists but not densely defined. (iii) $\lambda$ in continuous spectrum, $\sigma_c(\mathcal{H})$, if $R(\mathcal{H}, \lambda)$ exists, densely defined but not bounded.

**Theorem** Let $\lambda = iu$. (i) $\sigma_p(\mathcal{H})$ consists of all points $iu \in \mathbb{C}$, where $\varepsilon = 1 - k^{-2} \int_\mathbb{R} dv f_0'/(u - v) = 0$. (ii) $\sigma_c(\mathcal{H})$ consists of all $\lambda = iu$ with $u \in \mathbb{R} \setminus (-i\sigma_p(\mathcal{H}) \cap \mathbb{R})$. (iii) $\sigma_r(\mathcal{H})$ contains all the points $\lambda = iu$ in the complement of $\sigma_p(\mathcal{H})$ that satisfy $f_0'(u) = 0$.

cf. e.g. P. Degond (1986). Similar but different.
**Structural Stability**

**Definition** Consider an equilibrium solution of a Hamiltonian system and the corresponding time evolution operator $H$ for the linearized dynamics. Let the phase space for the linearized dynamics be some Banach space $B$. Suppose that $H$ is spectrally stable. Consider perturbations $\delta H$ of $H$ and define a norm on the space of such perturbations. Then we say that the equilibrium is **structurally stable** under this norm if there is some $\delta > 0$ such that for every $\|\delta H\| < \delta$ the operator $H + \delta H$ is spectrally stable. Otherwise the system is **structurally unstable**.

**Definition** Consider the formulation of the linearized Vlasov-Poisson equation in the Banach space $W^{1,1}(\mathbb{R})$ with a spectrally stable homogeneous equilibrium function $f_0$. Let $H f_0 + \delta f_0$ be the time evolution operator corresponding to the linearized dynamics around the distribution function $f_0 + \delta f_0$. If there exists some $\epsilon$ depending only on $f_0$ such that $H f_0 + \delta f_0$ is spectrally stable whenever $\|H f_0 - H f_0 + \delta f_0\| < \epsilon$, then the equilibrium $f_0$ is structurally stable under perturbations of $f_0$. 
All $f_0$ are Structurally Unstable in $W^{1,1}$

True in space where Hilbert transform unbounded, e.g. $W^{1,1}$. Small perturbation $\Rightarrow$ big jump in Penrose plot.

**Theorem** A stable equilibrium distribution is structurally unstable under perturbations of $f'_0$ in the Banach spaces $W^{1,1}$ and $L^1 \cap C_0$.

Easy to make ‘bumps’ in $f_0$ that are small in norm. What to do?
**Krein-Like Theorem for VP**

**Theorem**  Let $f_0$ be a stable equilibrium distribution function for the Vlasov equation. Then $f_0$ is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f_0'(v) = 0$. If there are multiple solutions, $f_0$ is structurally unstable and the unstable modes come from the roots of $f_0'$ that satisfy $f_0''(v) < 0$.

**Remark**  A change in the signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at all points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.
Summary – Conclusions

- Described the Vlasov-Poisson system.
- Described $G$ transform and its properties.
- Canonized, diagonalized, and defined signature for $\sigma_c$.
- Variety of Krein-like theorems, e.g. valley theorem.
HAP Formulations of PP: VI Metriplecticism:
relaxation paradigms for computation and
derivation

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Ravello, September 27, 2014

Goal: Describe formal structures for dissipation and their use as a guide for deriving models and for calculating stationary
Metriplectic References

Numbers refer to items on my web page: http://www.ph.utexas.edu/~morrison/ where all can be obtained under 'Publications'.


144. G. R. Flierl and P. J. Morrison, Hamiltonian-Dirac Simulated Annealing: Application to the Calculation of Vortex States, Physica D 240, 212232 (2011). The double bracket formalism is generalized by introducing a general from for infinite-dimensional systems, introducing a metric, and incorporating Dirac constraint theory. The formalism is used to numerically obtain a variety of vortex states.

134. P. J. Morrison, Thoughts on Brackets and Dissipation: Old and New, Journal of Physics: Conference Series 169, 012006 (12pp) (2009). Brackets are revisited and ideas about open and closed systems are discussed. Attempts are made to algebraic couple the symplectic and gradient parts of the flow.

30. P. J. Morrison, A Paradigm for Joined Hamiltonian and Dissipative Systems, Physica D 18, 410419 (1986). Metriplectic dynamics is further developed with finite and infinite dimensional examples given. The name metriplectic is introduced.

13. P. J. Morrison, Bracket Formulation for Irreversible Classical Fields, Physics Letters A 100, 423427 (1984). The first reference where the full axioms of metriplectic dynamics are given. Here the idea that the sum of symplectic and symmetric brackets can effect the equilibrium variational principle is introduced. Triple brackets are introduced for the construction of the dynamics.
Overview

1. Dissipative Structures
   (a) Rayleigh, Cahn-Hilliard
   (b) Hamilton Preliminaries
   (c) Hamiltonian Based Dissipative Structures
      i. Metriplectic Dynamics
      ii. Double Bracket Dynamics

2. Computations
   (a) XXXX Contour Dynamics
   (b) 2D Euler Vortex States
Rayleigh Dissipation Function

Introduced for study of vibrations, stable linear oscillations, in
1873 (see e.g. Rayleigh, Theory of Sound, Chap. IV §81)

Linear friction law for $n$-bodies, $\mathbf{F}_i = -b_i(\mathbf{r}_i)\mathbf{v}_i$, with $\mathbf{r}_i \in \mathbb{R}^3$.

Rayleigh was interested in linear vibrations, $\mathcal{F} = \sum_i b_i ||\mathbf{v}_i||^2/2$.

Coordinates $\mathbf{r}_i \rightarrow q_{\nu}$ etc. $\Rightarrow$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} \right) - \left( \frac{\partial \mathcal{L}}{\partial q_\nu} \right) + \left( \frac{\partial \mathcal{F}}{\partial \dot{q}_\nu} \right) = 0$$

Ad hoc, phenomenological, yet is generalizable, geometrizable (e.g. Bloch et al.,...)
Cahn-Hilliard Equation

Models phase separation, nonlinear diffusive dissipation, in binary fluid with ‘concentrations’ $n$, $n = 1$ one kind $n = -1$ the other

$$\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 (n^3 - n - \nabla^2 n)$$

Lyapunov Functional

$$F[n] = \int d^3x \left[ \frac{1}{4} (n^2 - 1)^2 + \frac{1}{2} |\nabla n|^2 \right]$$

$$\frac{dF}{dt} = \int d^3x \frac{\delta F}{\delta n} \frac{\partial n}{\partial t} = \int d^3x \frac{\delta F}{\delta n} \nabla^2 \frac{\delta F}{\delta n} = - \int d^3x \left| \nabla \frac{\delta F}{\delta n} \right|^2 \leq 0$$

For example in 1D

$$\lim_{t \to \infty} n(x, t) = \tanh(x/\sqrt{2})$$

Ad hoc, phenomenological, yet generalizable and very important (Otto, Ricci Flows, Poincarè conjecture on $S^3$, ...)

Hamiltonian Preliminaries

Finite $\rightarrow$ Infinite degrees of freedom
Canonical Hamiltonian Dynamics

Hamilton’s Equations:
\[ \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \]

Phase Space Coordinates: \[ z = (q, p) \]
\[ \dot{z}^i = J_{cj}^{ij} \frac{\partial H}{\partial z^j}, \quad (J_{cj}^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}, \]

Symplectic Manifold \( \mathcal{Z}_s \):
\[ \dot{z} = Z_H = [z, H] \]
with Hamiltonian vector field generated by Poisson bracket
\[ [f, g] = \frac{\partial f}{\partial z^i} J_{cj}^{ij} \frac{\partial g}{\partial z^j} \]
symplectic 2-form = (cosymplectic form)\(^{-1}\):
\[ \omega_{ij}^c J_{cj}^{jk} = \delta^k_i, \]
Noncanonical Hamiltonian Dynamics

Noncanonical Coordinates:

\[ \dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} = [z^i, H], \quad [A, B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j} \]

Poisson Bracket Properties:

antisymmetry \[ \rightarrow \quad [A, B] = -[B, A], \]

Jacobi identity \[ \rightarrow \quad [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \]

G. Darboux: \[ \det J \neq 0 \implies J \rightarrow J_c \quad \text{Canonical Coordinates} \]

Sophus Lie: \[ \det J = 0 \implies \text{Canonical Coordinates plus Casimirs} \]

Eulerian Media: \[ J^{ij} = \dot{c}_k^i z^j \quad \leftarrow \quad \text{Lie – Poisson Brackets} \]
Poisson Manifold $\mathcal{Z}_P$

Degeneracy $\Rightarrow$ Casimir Invariants:

$$[C, g] = 0 \quad \forall g : \mathcal{Z}_P \rightarrow \mathbb{R}$$

Foliation by Casimir Invariants:

Leaf Hamiltonian vector fields:

$$Z^p_f = [z, f]$$
Noncanonical Poisson Brackets:

\[ \{F, G\} = \int dx\, dy\, \zeta \left[ \frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta} \right] = -\int dx\, dy\, \frac{\delta F}{\delta \zeta} [\zeta, \cdot] \frac{\delta G}{\delta \zeta} \]

\( \zeta = \) vorticity, \( \psi = \Delta^{-1} \zeta \) = streamfunction

\[ [f, g] = J(f, g) = f_x g_y - f_y g_x = \frac{\partial (f, g)}{\partial (x, y)} \]

Hamiltonian:

\[ H[\zeta] = \frac{1}{2} \int dx\, v^2 = \frac{1}{2} \int dx\, |\nabla \psi|^2 \]

Equation of Motion:

\[ \zeta_t = \{\zeta, H\} \]

**Dirac Constrained Hamiltonian Dynamics**

**Ingredients:**

Two functions \( D_1, D_2 : \mathcal{Z} \to \mathbb{R} \) and good Poisson bracket

**Generalized Dirac:**

\[
[f, g]_D = \frac{1}{[D_1, D_2]} \left( [D_1, D_2][f, g] - [f, D_1][g, D_2] + [g, D_1][f, D_2] \right)
\]

Degeneracy \( \Rightarrow \) \( D \)'s are Casimir Invariants:

\[
[D_1, D_2, g]_D = 0 \quad \forall \ g : \mathcal{Z}_p \to \mathbb{R}
\]

Foliation again and Dirac Hamiltonian vector fields:

\[
Z^d_f = [z, f]_D
\]
Hamiltonian Based Dissipation
Metriplectic Dynamics

A dynamical model of thermodynamics that ‘captures’:

- First Law: conservation of energy
- Second Law: entropy production

Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of \([,]\) are ‘candidate’ entropies. Election of particular \(S \in \{\text{Casimirs}\} \Rightarrow \text{thermal equilibrium (relaxed) state.}\)

- **Generator:** \(\mathcal{F} = H + S\)

- **1st Law:** identify energy with Hamiltonian, \(H\), then
  \[
  \dot{H} = [H, \mathcal{F}] + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0
  \]
  Foliate \(\mathcal{Z}\) by level sets of \(H\) with \((H, f) = 0\) \(\forall f \in C^\infty(M)\).

- **2nd Law:** entropy production
  \[
  \dot{S} = [S, \mathcal{F}] + (S, \mathcal{F}) = (S, S) \geq 0
  \]
  Lyapunov relaxation to the equilibrium state: \(\delta \mathcal{F} = 0\).
Metriplectic Dynamics

Natural hybrid Hamiltonian and dissipative flow on that embodies the first and second laws of thermodynamics;

\[ \dot{z} = (z, S) + [z, H] \]

where Hamiltonian, \( H \), is the energy and entropy, \( S \), is a Casimir.

Degeneracies:

\[(H, g) \equiv 0 \quad \text{and} \quad [S, g] \equiv 0 \quad \forall \ g \]

First and Second Laws:

\[ \frac{dH}{dt} = 0 \quad \text{and} \quad \frac{dS}{dt} \geq 0 \]

Seeks equilibria \( \equiv \) extermination of Free Energy \( F \equiv H + S \):

\[ \delta F = 0 \]
Examples

• Finite dimensional theories, rigid body, etc.

• Kinetic theories: Boltzmann equation, Lenard-Balescu equation, ...

• Fluid flows: various nonideal fluids, Navier-Stokes, MHD, etc.
5. Relaxing free rigid body

In order to illustrate the formalism outlined in the previous section we treat an example. We begin by considering the motion of a rigid body with fixed center of mass under no torques. The motion of such a free rigid body is governed by Euler’s equations

\[\begin{align*}
\dot{\omega}_1 &= \omega_3 \omega_2 (I_2 - I_1), \\
\dot{\omega}_2 &= \omega_3 \omega_1 (I_3 - I_1), \\
\dot{\omega}_3 &= \omega_1 \omega_2 (I_1 - I_2). 
\end{align*}\]  

(27)

Here we have done some scaling, but the dynamical variables \(\omega_i, i = 1, 2, 3\), are related to the three principal axis components of the angular velocity, while the constants \(I_i, i = 1, 2, 3\), are related to the three principal moments of inertia.

This system conserves the following expressions for rotational kinetic energy and squared magnitude of the angular momentum:

\[\begin{align*}
H &= \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2), \\
l^2 &= \omega_1^2 + \omega_2^2 + \omega_3^2. 
\end{align*}\]  

(28a, 28b)

The quantity \(H\) can be used to cast eqs. (27) into Hamiltonian form in terms of a noncanonical Poisson Bracket [4] that involves the three dynamical variables, \(\omega_i\). The matrix \((J^{ij})\) introduced in section 3 has a null eigenvector that is given by \(\partial l^2/\omega_i\); i.e. \(l^2\) is a Casimir. The noncanonical Poisson bracket is

\[\{f, g\} = \frac{\partial f}{\partial \omega_i} \epsilon_{ijk} \frac{\partial g}{\partial \omega_j}, \quad i, j, k = 1, 2, 3, \]  

(29)

where \(\epsilon_{ijk}\) is the Levi-Civita symbol. Evidently eqs. (27) are equivalent to

\[\dot{\omega}_i = \{\omega_i, H\}, \quad i = 1, 2, 3, \]  

(30)

and we have for an arbitrary function \(S(l^2), \{S, f\} = 0\) for all \(f\).

So far we have endowed the phase space, which has coordinates \(\omega_i\), with a cosymplectic form. Let us now add to this a metric component. In this case a dynamical constraint manifold corresponds to a surface of constant energy, i.e. an ellipsoid. We wish to construct a \((g^{ij})\) that has \(\partial H/\omega_i\) as a null eigenvector, while possessing two nonzero eigenvalues of the same sign. This is conveniently done by defining the bracket \((,\) in terms of a projection matrix; i.e.

\[\{f, h\} = -\lambda \left[ \frac{\partial H}{\partial \omega_i} \delta_{ij} \frac{\partial H}{\partial \omega_j} - \frac{\partial \delta}{\partial \omega_i} \frac{\partial H}{\partial \omega_j} \right] \frac{\partial f}{\partial \omega_i} \frac{\partial h}{\partial \omega_j}. \]  

(31)

For now we take \(\lambda\) to be constant, but it could depend upon \(\omega\). Explicitly the \((g^{ij})\) is given by

\[\begin{bmatrix}
I_1^2 \omega_1^2 + I_3^2 \omega_3^2 + I_2 \omega_2^2 & -I_1 I_2 \omega_1 \omega_2 & -I_1 I_3 \omega_1 \omega_3 \\
-I_1 I_2 \omega_1 \omega_2 & -I_1 I_2^2 + I_3^2 \omega_3^2 & -I_1 I_3 \omega_2 \omega_3 \\
-I_1 I_3 \omega_1 \omega_3 & -I_2 I_3 \omega_1 \omega_3 & -I_2 I_2^2 + I_3^2 \omega_2^2
\end{bmatrix}. \]  

(32)

We are now in a position to display a class of metriplectic flows for the rigid body; i.e.

\[\dot{\omega}_i = \{\omega_i, F\} = [\omega_i, F] + \{\omega_i, F\} = J^{ij} \frac{\partial H}{\partial \omega_j} + g^{ij} \frac{\partial S}{\partial \omega_j}, \quad i = 1, 2, 3, \]  

(33)

where \(F = H - S, H\) is given by eq. (28a) and \(S\) is an arbitrary function of \(l^2\). For the case \(i = 1\) we have

\[\dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) + 2\lambda S'(l^2) \omega_1 \times \left[ I_2 (I_2 - I_1) \omega_2^2 + I_3 (I_3 - I_1) \omega_3^2 \right]. \]  

(34)

The other two equations are obtained upon cyclic permutation of the indices. By design this system conserves energy but produces the generalized entropy \(S(l^2)\) if \(\lambda > 0\), which could be chosen to correspond to angular momentum.

It is well known that equilibria of Euler’s equations composed of pure rotation about either of the principal axes corresponding to the largest and smallest principal moments of inertia are stable. If we suppose that \(I_1 < I_2 < I_3\), then stability of an equilibrium defined by \(\omega_1 = \omega_2 = 0\) and \(\omega_3 = \omega_0\)
Generalized Vlasov-Lenard-Balescu

GVLB equation:

\[
\frac{\partial f}{\partial t}(x,v,t) = -v \cdot \nabla f + \nabla \phi(x; f) \cdot \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t}(x,v,t) \bigg|_c
\]

Energy Entropy:

\[
H = \frac{1}{2} \int dx dv m|v|^2 + \frac{1}{2} \int dx |E|^2 \quad S = \int \int dx dv s(f)
\]

Symmetric Bracket:

\[
(A, B) = -\int dx dv \int dx' dv' \left[ \frac{\partial}{\partial v_i} \frac{\delta A}{\delta f} - \frac{\partial}{\partial v_i'} \frac{\delta A}{\delta f'} \right] T_{ij} \left[ \frac{\partial}{\partial v_i} \frac{\delta B}{\delta f} - \frac{\partial}{\partial v_i'} \frac{\delta B}{\delta f'} \right]
\]

Entropy Matching:

\[
T_{ij} = w_{ij}(x,v,x',v) M(f) M(f')/2 \quad \text{with} \quad M \frac{\partial^2 s}{\partial f^2} = 1
\]
Collision Operator

Two counting dichotomies:

- Exclusion vs. Nonexclusion
- Distinguishability vs. Indistinguishability

⇒ 4 possibilities

IE → F − D
IN → B − E
DN → M − B
DE → ?
Collision Operator

Two counting dichotomies:

- Exclusion vs. Nonexclusion
- Distinguishability vs. Indistinguishability

⇒ 4 possibilities

IE → F – D
IN → B – E
DN → M – B
DE → L – B

Lynden-Bell (1967) proposed this for stars which are distinguishable.
Collision Operator

Kadomstev and Pogutse (1970) collision operator with formal $H$-theorem to F-D?

Metriplectic formalism $\rightarrow$ can do for any monotonic distribution

Conservation (mass, momentum, energy) and Lyapunov:

$$w_{ij}(z, z') = w_{ji}(z, z') \quad w_{ij}(z, z') = w_{ij}(z', z) \quad g_i w_{ij} = 0,$$

where $z = (x, v)$ and $g_i = v_i - v'_i$.

‘Entropy’ Compatibility:

$$S[f] = \int dz \, s(f) \quad \Rightarrow \quad M \frac{d^2 s}{df^2} = 1$$
Collision Operator Examples

Landau kernel:

\[ w_{ij}^{(L)} = (\delta_{ij} - g_i g_j / g^2) \delta(x - x') / g \]

Landau Entropy Compatibility

\[ S[f] = \int dz f \ln f \quad \Rightarrow \quad M \frac{d^2 s}{df^2} = 1 \Rightarrow M = f \]

Lynden-Bell Entropy Compatibility

\[ S[f] = \int dz s(f) \quad \Rightarrow \quad M \frac{d^2 s}{df^2} = 1 \Rightarrow M = f (1 - f) \]
Good Dissipative Models are Metriplectic!
Double Brackets
Double Brackets and Simulated Annealing

Good Idea:

Vallis, Carnevale, and Young; Shepherd, (1989)

‘Simulated Annealing’ Bracket:

\[
((f, g)) = [f, z^\ell][z^\ell, g] = \frac{\partial f}{\partial z^i} J^{i\ell} J^{\ell j} \frac{\partial g}{\partial z^j},
\]

Use bracket dynamics to do extremization ⇒ Relaxing Rearrangement

\[
\frac{d\mathcal{F}}{dt} = ((\mathcal{F}, H)) = ((\mathcal{F}, \mathcal{F})) \geq 0
\]

Lyapunov function, \(\mathcal{F}\), yields asymptotic stability to rearranged equilibrium.

- Maximizing energy at fixed Casimir: Works fine sometimes, but limited to circular vortex states ....
Generalized Simulated Annealing

‘Simulated Annealing’ Bracket:

\[ ((f, g))_D = [f, z^m]_D g_{mn} [z^n, g]_D = \frac{\partial f}{\partial z^i} J^i_D g_{mn} J^j_D \frac{\partial g}{\partial z^j}, \]

Relaxation Property: \( \frac{dH}{dt} = ((H, H))_D \geq 0 \) at constant Casimirs

General Geometric Construction:

Suppose manifold \( Z \) has both Riemannian and Symplectic structure: Given two vector fields \( Z_{1,2} \) the following is defined:

\[ g(Z_1, Z_2) \]

If the two vector fields are Hamiltonian, e.g., \( Z_f \), then we have the bracket

\[ ((f, g)) = g(Z_f, Z_g) \]

which produces a ‘relaxing’ flow. Such flows exist for Kähler manifolds.
Contour Dynamics Calculations
Calculation of V-States in Contour Dynamics

Goal:

CD/Waterbag Hamiltonian Reduction:

vorticity, \( \omega(x, y, t) \rightarrow X(\sigma) \), vortex patch boundary

Calculation:

V-States by simulated annealing
Contour Dynamics/Waterbags

Plane Curve:

\[ X(\sigma) = (X(\sigma), Y(\sigma)) \]

parameter \( \sigma \) arbitrary

(arc length not conserved)
V-States

→ Equilibria in rotation frame; $\delta (H + \Omega L) = 0$

Kirchoff Ellipse:

3-fold:
Hamiltonian Form

Observables are Parameterization Invariance Functionals:

\[ F[X, Y] = \oint d\sigma \mathcal{F}(X, Y, X_{\sigma}, Y_{\sigma}, Y_{\sigma\sigma}, X_{\sigma\sigma}, \ldots) \]

Invariance (equivalence relation): \( X_{\sigma} := \partial / \partial \sigma \), etc.

\[ \oint d\sigma \mathcal{F}(X, Y, X_{\sigma}, Y_{\sigma}, Y_{\sigma\sigma}, X_{\sigma\sigma}, \ldots) = \oint d\tau \mathcal{F}(X, Y, X_{\tau}, Y_{\tau}, Y_{\tau\tau}, X_{\tau\tau}, \ldots) \]

\[ \sigma = \phi(\tau), \quad d\phi(\tau)/d\tau \neq 0 \]

Lie Algebra Realization:

\( \mathcal{V} \) over \( \mathbb{R} \) is set of parameterization invariant functionals with Poisson Bracket \( \{ , \} \)

Bianchi identity:

\[ \frac{\delta F}{\delta X(\sigma)} X_{\sigma} + \frac{\delta F}{\delta Y(\sigma)} Y_{\sigma} \equiv 0, \]

Noether & isoperimetric problems
Hamiltonian Form (cont)

Poisson Bracket:

\[ \{ F, G \} = \oint d\sigma \left[ \frac{Y_\sigma \delta F}{X^2_\sigma + Y^2_\sigma} - \frac{X_\sigma \delta F}{X^2_\sigma + Y^2_\sigma} \right] \frac{\partial}{\partial \sigma} \left[ \frac{Y_\sigma \delta G}{X^2_\sigma + Y^2_\sigma} - \frac{X_\sigma \delta G}{X^2_\sigma + Y^2_\sigma} \right] \]

Area/Casimir:

\[ \Gamma = \frac{1}{2} \oint (XY_\sigma - YX_\sigma) \, d\sigma, \quad \{ \Gamma, F \} = 0 \ \forall \ F \]

Area Preservation:

\[ \{ \Gamma, F \} = \oint \frac{\partial}{\partial \sigma} \left[ \frac{Y_\sigma \delta F}{X^2_\sigma + Y^2_\sigma} - \frac{X_\sigma \delta F}{X^2_\sigma + Y^2_\sigma} \right] d\sigma = 0 \]

Dynamics of closed curves with fixed areas for any \( H \).
Contour Dynamics Clips – DSA

Built-in Invariants:

- Angular momentum:
  \[ L = \int_D (x^2 + y^2) \, d^2x \]

- Strain moment (2-fold symmetry):
  \[ K = \int_D xy \, d^2x \]

{1-Kellipse, 2-two.stationary, 3-two}
2D Euler Calculations
Four Types of Dynamics

Hamiltonian: \[ \frac{\partial F}{\partial t} = \{F, F\} \] (1)

Hamiltonian Dirac: \[ \frac{\partial F}{\partial t} = \{F, F\}_D \] (2)

Simulated Annealing: \[ \frac{\partial F}{\partial t} = \sigma\{F, F\} + \alpha((F, F)) \] (3)

Dirac Simulated Annealing: \[ \frac{\partial F}{\partial t} = \sigma\{F, F\}_D + \alpha((F, F))_D \] (4)

\( F \) an arbitrary observable, \( F \) generates time advancement. Equations (1) and (2) are ideal and conserve energy. In (3) and (4) parameters \( \sigma \) and \( \alpha \) weight ideal and dissipative dynamics: \( \sigma \in \{0, 1\} \) and \( \alpha \in \{-1, 1\} \). \( F \), can have form

\[ F = H + \sum_i C_i + \lambda^i P_i, \]

\( C \)'s Casimirs and \( P \)'s dynamical invariants.
DSA is Dressed Advection

\[
\frac{\partial \zeta}{\partial t} = -[\Psi, \zeta],
\]

\[
\Psi = \psi + A^i c_i \quad \text{and} \quad A^i = -\frac{\int dx c_j [\psi, \zeta]}{\int dx \zeta [c_i, c_j]}.
\]

with constraints

\[
C_j = \int dx c_j \zeta.
\]

“Advection” of \( \zeta \) by \( \Psi \), with \( A^i \) just right to force constraints.

Easy to adapt existing vortex dynamics codes!!
2D Euler Clip, 2-fold Symmetry – H

Initial Condition:

\[ q = e^{-(r/r_0)^{10}}, \quad r_0 = 1 + \epsilon \cos(2\theta), \quad \epsilon = 0.4 \]
Filamentation leading to ‘relaxed state’. How much? Which state?
2D Euler Clip, 2-fold Symmetry \(- \text{SA}_{\sigma=0}\)

Initial Condition:

\[ q = e^{-(r/r_0)^{10}}, \quad r_0 = 1 + \epsilon \cos(2\theta), \quad \epsilon = 0.4 \]

{(fig6)els-2-m0}
Constants vs. $t$; Kelvin’s $H$-Maximization
2-fold Symmetry – HD vs. DSA\(_{0,1}\)

Initial Condition:
\[ q = e^{-(r/r_0)^{10}}, \quad r_0 = 1 + \epsilon \cos(2\theta), \quad \epsilon = 0.4 \]

- Angular momentum:
  \[ L = \int_D (x^2 + y^2) \, d^2x \]

- Strain moment (2-fold symmetry):
  \[ K = \int_D xy \, d^2x \]

\{(fig8)els-3-m0, (fig10)els-4-m0,(fig12)els-4-m1\}
Constants vs. $t$ for $\text{DSA}_0$
Uniform positive vorticity inside circle. Net vorticity maintained. But, angular momentum not conserved? With Dirac, angular momentum conserved. Then what?
2-fold Symmetry – Minimizing SA vs. DSA₀

Initial Condition:

\[ q = e^{-\left(\frac{r}{r_0}\right)^{10}}, \quad r_0 = 1 + \epsilon \cos(2\theta), \quad \epsilon = 0.4 \]

• Angular momentum:

\[ L = \int_D \left( x^2 + y^2 \right) d^2x \]

• Strain moment (2-fold symmetry):

\[ K = \int_D xy d^2x \]

\{(fig14)els-2-p0,(fig16)els-4-p0\}
Constants vs. $t$ for $SA_0$
3-fold Symmetry and Dipole DSA

skipping details

{(fig21)tri-db2, (fig27)dip-4-m0}
Underview

1. Dissipative Structures
   (a) Rayleigh, Cahn-Hilliard
   (b) Hamilton Preliminaries
   (c) Hamiltonian Based Dissipative Structures
      i. Metriplectic Dynamics
      ii. Double Bracket Dynamics

2. Computations
   (a) XXXX Contour Dynamics
   (b) 2D Euler Vortex States