Metriplectic Dynamics and Reduction

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Goal: Review metriplectic dynamics, a form of dynamical system that blends noncanonical (Poisson) Hamiltonian and dissipative systems. Investigate reduction in this setting in terms of a simple example, the rigid body with a particular kind of dissipation. The example has dissipation that aligns the rotation without using any energy!

Metriplectic Dynamics

A dynamical model of thermodynamics that 'captures':

- First Law: conservation of energy
- Second Law: entropy production

Prototypes and Examples

- Kinetic theories: Vlasov Fokker-Planck equation, Lenard-Balescu equation, etc.

- Fluid flows: various nonideal fluids, Navier-Stokes, MHD, etc.

- Finite-dimensional theories, rigid body, etc.

- Many more ...

“History”

- Rayleigh Dissipation (Theory of Sound, Ch. IV §81 1873)

- 20th Century gradient flows (Cahn-Hilliard, Otto, Ricci Flows, Poincarè conjecture on S^3, ...)

- pjm, pjm $\cup$ Kaufman 1982

- pjm, Kaufman, ... Grmela 1984

- pjm 1986 metriplectic dynamics

- Grmela $\cup$ Oetttinger 1997, generic $\equiv$ metriplectic dynamics

- Many works since ... e.g. Bloch, pjm, Ratiu 2013

Usual Geometry

Dynamics takes place in phase space, $\mathcal{Z}$ (needn't be T^*Q), a differential manifold endowed with a closed, nondegenerate 2-form ω. A patch has canonical coordinates $z = (q, p)$.

Hamiltonian dynamics $\Leftrightarrow$ flow on symplectic manifold: $i_X\omega = dH$

Poisson tensor (J_c) is bi-vector inverse of ω, defining the Poisson bracket

$$\{f, g\} = \langle df, J_c(dg)\rangle = \omega(X_f, X_g) = \frac{\partial f}{\partial z^\alpha} J_c^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,, \qquad \alpha, \beta = 1, 2, \ldots 2N$$

Flows generated by Hamiltonian vector fields $Z_H = JdH$, H a 0-form, dH a 1-form. Poisson bracket = commutator of Hamiltonian vector fields etc.

Early refs.: Jost, Mackey, Souriau, Arnold, Abraham &Marsden

Noncanonical Hamiltonian Definition

A phase space $\mathcal{P}$ diff. manifold with binary bracket operation on $C^\infty(\mathcal{P})$ functions $f, g\colon \mathcal{P} \to \mathbb{R}$, s.t. $\{\cdot\,,\cdot\}\colon C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ satisfies

- Bilinear: $\{f + \lambda g, h\} = \{f, h\} + \lambda\{g, h\}\,, \quad \forall f, g, h$ and $\lambda \in \mathbb{R}$
- Antisymmetric: $\{f, g\} = -\{g, f\}\,, \qquad \forall f, g$
- Jacobi: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \equiv 0\,, \qquad \forall f, g, h$
- Leibniz: $\{fg, h\} = f\{g, h\} + \{f, h\}g\,, \qquad \forall f, g, h.$

Above is a Lie algebra realization on functions. Take fg to be pointwise multiplication.

Eqs. Motion: $\frac{\partial \Psi}{\partial t} = \{\Psi, H\}$, Ψ an observable & H a Hamiltonian.

Example: flows on Poisson manifolds, e.g. Weinstein 1983 ....

Noncanonical Hamiltonian Dynamics

Sophus Lie (1890)

Noncanonical Coordinates:

$$\dot{z}^\alpha = J^{\alpha\beta}\frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}\,, \qquad \{f,g\} = \frac{\partial f}{\partial z^\alpha}J^{\alpha\beta}(z)\frac{\partial g}{\partial z^\beta}\,, \qquad \alpha,\beta = 1,2,\ldots M$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{f,g\} = -\{g,f\}\,,$

Jacobi identity $\longrightarrow$ $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

$$J \rightarrow J_d = \begin{pmatrix} 0_N & I_N & 0 \\ -I_N & 0_N & 0 \\ 0 & 0 & 0_{M-2N} \end{pmatrix}.$$

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{P}$ is differentiable manifold with bracket $\{\, ,\, \} : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ st $C^\infty(\mathcal{P})$ with $\{\, ,\, \}$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $Z_H = JdH$.

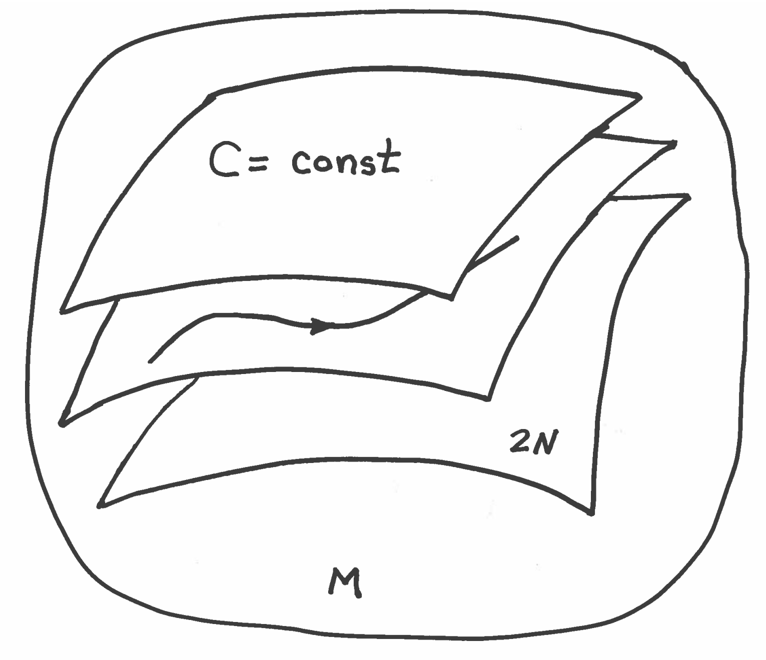
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{P})$. Called Casimir invariants (Lie's distinguished functions.)

Poisson Manifold $\mathcal{P}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{P} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Leaf vector fields, $Z_f = \{z, f\} = Jdf$ are tangent to leaves.

Lie-Poisson Brackets

Coordinates:

$$J^{\alpha\beta} = c_{\gamma}^{\alpha\beta} z^{\gamma}$$

where $c_{\gamma}^{\alpha\beta}$ are the structure constants for some Lie algebra.

<u>Examples</u>:

• free rigid body SO(3), Kida vortex SL(2,1), ...

• Infinite-dimensional theories - matter models: Ideal fluid flow, MHD, shearflow, extended MHD, Vlasov-Maxwell, BBGKY, etc.

Lie-Poisson Geometry

Lie Algebra: $\mathfrak{g}$, a vector space with

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\,,$$

antisymmetric, bilinear, satisfies Jacobi identity

Pairing:

$$\langle\ \ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$$

with $\mathfrak{g}^*$ vector space dual to $\mathfrak{g}$

Lie-Poisson Bracket:

$$\{f, g\} = \left\langle z, \left[\frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right]\right\rangle\,, \qquad z \in \mathfrak{g}^*,\ \frac{\partial f}{\partial z} \in \mathfrak{g}$$

Example $\mathfrak{so}(3)$

Lie Algebra is antisymmetric matrices, or $\mathbf{L} = (L_1, L_2, L_3)$, a vector space with

$$[\mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$$

where $\times$ is vector cross product.

Pairing between $\mathbf{L} \in \mathfrak{so}(3)^*$ and $\partial f / \partial \mathbf{L} \in \mathfrak{so}(3)$ yields the Lie-Poisson bracket:

$$\{f, g\} = \mathbf{L} \cdot \frac{\partial f}{\partial \mathbf{L}} \times \frac{\partial g}{\partial \mathbf{L}} = \epsilon_{\alpha\beta\gamma} \, L_\alpha \, \frac{\partial f}{\partial L_\beta} \frac{\partial g}{\partial L_\gamma} \, ,$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita (permutation) symbol, which denotes the structure constants for $\mathfrak{so}(3)$.

Casimirs (nested spheres S^2 foliation):

$$C = L_1^2 + L_2^2 + L_3^2$$

Note $L_i = I_i \omega_i$, not summed.   Examples: spin system, free rigid body with Euler's equations

Metriplectic Manifold $\big(\mathcal{M}, \{\,,\,\}, (\,,\,)\big)$

Two structures:

- Poisson Manifold, with associated degenerate bi-vector J
- Degenerate 'metric' g

Metriplectic Vector Field in coordinate patch:

$$V_{MP} = -\{\mathcal{F},\ \} - (\mathcal{F},\ ) = \frac{\partial \mathcal{F}}{\partial z^\alpha}\, J^{\alpha\beta}\, \frac{\partial}{\partial z^\beta} + \frac{\partial \mathcal{F}}{\partial z^\alpha}\, g^{\alpha\beta}\, \frac{\partial}{\partial z^\beta}$$

What are degeneracies? What is the 'generator' $\mathcal{F}$?

Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of $\{,\}$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator (free energy): $\mathcal{F} = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = \{H, \mathcal{F}\} + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0$$

  Degeneracy such that $(H, f) = 0 \ \forall \ f \in C^\infty(\mathcal{M})$.

- 2nd Law: entropy production

$$\dot{S} = \{S, \mathcal{F}\} + (S, \mathcal{F}) = (S, S) \geq 0$$

  Lyapunov relaxation to the equilbrium state: $\delta\mathcal{F} = 0$.

Metriplectic Dynamics

Equations of motion:

$$\dot{z} = \{z, \mathcal{F}\} + (z, \mathcal{F}) = \{z, H\} + (z, S)$$

Using degeneracies:

$$\{S, g\} \equiv 0 \quad \text{and} \quad (H, g) \equiv 0 \quad \forall\ g$$

First and Second Laws:

$$\frac{dH}{dt} = 0 \qquad \text{and} \qquad \frac{dS}{dt} \geq 0$$

Seeks equilibria $\equiv$ extermination of Free Energy $F = H + S$:

$$\delta F = 0$$

Example: Metriplectic Rigid Body

Euler's equations

$$\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3),$$
$$\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1), \quad (27)$$
$$\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2).$$

Here we have done some scaling, but the dynamical variables ω_i, $i = 1,2,3$, are related to the three principal axis components of the angular velocity, while the constants I_i, $i = 1,2,3$, are related to the three principal moments of inertia.

This system conserves the following expressions for rotational kinetic energy and squared magnitude of the angular momentum:

$$H = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2), \quad (28a)$$
$$l^2 = \omega_1^2 + \omega_2^2 + \omega_3^2. \quad (28b)$$

The quantity H can be used to cast eqs. (27) into Hamiltonian form in terms of a noncanonical Poisson Bracket [4] that involves the three dynamical variables, ω_i. The matrix (J^{ij}) introduced in section 3 has a null eigenvector that is given by $\partial l^2/\omega_i$; i.e. l^2 is a Casimir. The noncanonical Poisson bracket is

$$[f, g] = \frac{\partial f}{\partial \omega_i}\omega_k \epsilon_{ijk}\frac{\partial g}{\partial \omega_j}, \quad i, j, k = 1,2,3, \quad (29)$$

where ϵ_{ijk} is the Levi-Civita symbol. Evidently eqs. (27) are equivalent to

$$\dot{\omega}_i = [\omega_i, H], \quad i = 1,2,3, \quad (30)$$

projection matrix; i.e.

$$(f, h) = -\lambda\left[\left(\frac{\partial H}{\partial \omega_i}\frac{\partial H}{\partial \omega_j} - \delta_{ij}\frac{\partial H}{\partial \omega_l}\frac{\partial H}{\partial \omega_l}\right)\frac{\partial f}{\partial \omega_i}\frac{\partial h}{\partial \omega_j}\right]. \quad (31)$$

For now we take λ to be constant, but it could depend upon ω. Explicitly the (g^{ij}) is given by

$$(g^{ij}) = \lambda\begin{bmatrix} I_2^2\omega_2^2 + I_3^2\omega_3^2 & -I_1I_2\omega_1\omega_2 & -I_1I_3\omega_1\omega_3 \\ -I_1I_2\omega_1\omega_2 & I_1^2\omega_1^2 + I_3^2\omega_3^2 & -I_2I_3\omega_1\omega_3 \\ -I_1I_3\omega_1\omega_3 & -I_2I_3\omega_1\omega_3 & I_1^2\omega_1^2 + I_2^2\omega_2^2 \end{bmatrix}. \quad (32)$$

We are now in a position to display a class of metriplectic flows for the rigid body; i.e.

$$\dot{\omega}_i = \{\omega_i, F\} = [\omega_i, F] + (\omega_i, F)$$
$$= J^{ij}\frac{\partial H}{\partial \omega_j} + g^{ij}\frac{\partial S}{\partial \omega_j}, \quad i = 1,2,3, \quad (33)$$

where $F = H - S$, H is given by eq. (28a) and S is an arbitrary function of l^2. For the case $i = 1$ we have

$$\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) + 2\lambda S'(l^2)\omega_1$$
$$\times\left[I_2(I_2 - I_1)\omega_2^2 + I_3(I_3 - I_1)\omega_3^2\right]. \quad (34)$$

The other two equations are obtained upon cyclic permutation of the indices. By design this system conserves energy but produces the generalized entropy $S(l^2)$ if $\lambda > 0$, which could be chosen to correspond to angular momentum.

It is well known that equilibria of Euler's equations composed of pure rotation about either of

Generator $H + S$:

$$H = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)\,, \qquad C = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\,, \qquad S = S(C)$$

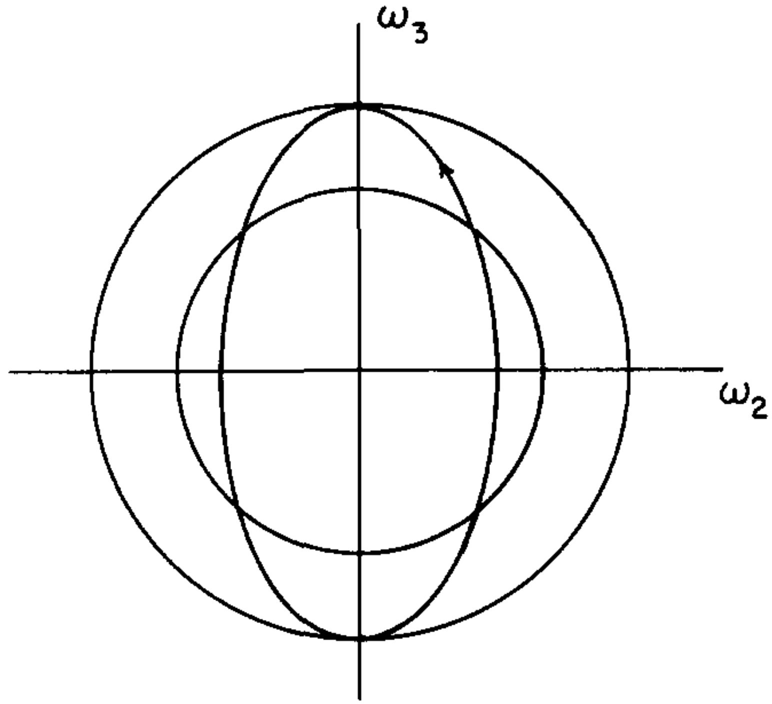
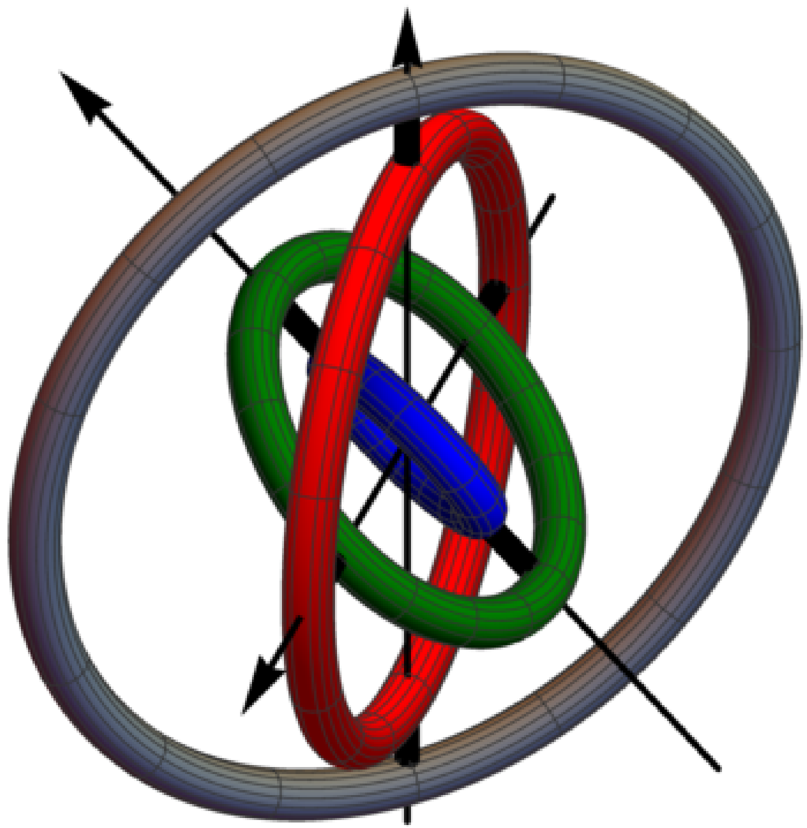


Fig. 1. Depiction of the metriplectic phase space for the relaxing free rigid body. Symplectic leaves are concentric spheres while constant energy surfaces are ellipsoids.

Metriplectic Motor



Servomotors at axles of Cardan suspension, $\dot{H} = 0 \Rightarrow$ no energy used to align angular momentum. Application? Use a very small amount of electronic energy to redirect energy from axle to axle.

Usual Reduction

Rigid Body has canonical 6D phase space T^*G where configuration space G is $SO(3)$. Coordinates can be, e.g., the Euler angles χ and conjugate momenta p_χ.

Standard Reduction: $T^*G/G \cong \mathfrak{g}^*$

Momentum map: $(\chi, p_\chi) \mapsto \mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$

Bracket closure: $\{L_\alpha, L_\beta\} = \epsilon_{\alpha\beta\gamma}L_\gamma$

Hamiltonian closure: $H(\chi, p_\chi) = \bar{H}(L)$

Dynamics via Euler's Equations: $\dot{\mathbf{L}} = \{\mathbf{L}, \bar{H}\}$

What is Mertriplectic Reduction?

As rigid body is relaxing to rotation about a single axis, the coordinates (χ, p_χ) are changing until χ has the simple time dependence of rotation.

What are the possible dynamics in the inverse image of the momentum map, that reduce to the metriplectic dynamics?

Metriplectic momentum map takes

$$\dot{\chi} = \{\chi, H\} + ? \quad \text{and} \qquad \dot{p}_\chi = \{p_\chi, H\} + ?$$

into the reduced metriplectic dynamics.

What is the unreduced relaxation? Limit cycle? Other?

Recall rigid body formulas:

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\Sigma}^{-1} \cdot \mathbf{L} \qquad \text{with} \qquad \Sigma = \mathrm{dia}(I_1, I_2, I_3) \\ \boldsymbol{\omega} &= \mathcal{A}(\chi) \cdot p_\chi \qquad \text{or} \qquad \boldsymbol{\omega} = D(\chi) \cdot \dot{\chi} \end{aligned}$$

and

$$D(\chi) = \begin{pmatrix} \cos\chi_3 & \sin\chi_1 \sin\chi_3 & 0 \\ -\sin\chi_3 & \sin\chi_1 \cos\chi_3 & 0 \\ 0 & \cos\chi_1 & 1 \end{pmatrix}, \qquad \mathcal{A}(\chi) = \Sigma^{-1} \cdot \left(D^{-1}\right)^{\mathsf{T}}$$

Example of Unreduced Dynamics:

$$\begin{aligned} \dot{\chi} &= D^{-1} \cdot \mathcal{A} \cdot p_\chi \,, \\ \dot{p}_\chi &= -p_\chi^{\mathsf{T}} \cdot \mathcal{A}^{\mathsf{T}} \cdot \Sigma \cdot \frac{\partial \mathcal{A}}{\partial \chi} \cdot p_\chi \\ &\qquad + S' \mathcal{A}^{-1} \cdot \Gamma \cdot \Sigma^2 \cdot \mathcal{A} \cdot p_\chi \,. \end{aligned}$$

S' measures relaxation time scale, Γ a matrix related to the axis of rotation. Note, dissipation is in the momentum equation where it usually is.

Conclusion

Metriplectic reduction takes special dissipation in a canonical Hamiltonian system into metriplectic dissipation.

Recall question:

What is the unreduced relaxation? Limit cycle? Other?

Answer:

Not a limit cycle, but an attracting cylinder $S \times \mathbb{R}$ of periodic orbits.

M. Materassi and pjm, Cybernetics and Physics **7**, 78–86 (2018).

Extensions: Navier-Stokes for entropy producing fluid, where Lagrange to Euler map is usual reduction; various kinetic theories. Anything with a momentum map!