Hamilton description of plasmas and other models of matter: structure and applications I

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Survey Hamiltonian systems that describe matter: particles, fluids, plasmas, e.g., magnetofluids, kinetic theories, ....
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“Hamiltonian systems .... are the basis of physics.” M. Gutzwiller
William Rowan Hamilton (August 4, 1805 - September 2, 1865)

I. **Today:** Finite-dimensional systems. Particles etc. ODEs

II. **Tomorrow:** Infinite-dimensional systems. Hamiltonian field theories. PDEs
Why Hamiltonian?

• Beauty, Teleology, . . .: Still a good reason!

• 20th Century framework for physics: Fluids, Plasmas, etc. too.

• Symmetries and Conservation Laws: energy-momentum . . .

• Generality: do one problem ⇒ do all.

• Approximation: perturbation theory, averaging, . . . 1 function.

• Stability: built-in principle, Lagrange-Dirichlet, $\delta W$, . . .

• Beacon: $\exists\infty$-dim KAM theorem? Krein with Cont. Spec.?

• Numerical Methods: structure preserving algorithms: symplectic, conservative, Poisson integrators, . . . e.g. GEMPIC.

• Statistical Mechanics: energy, measure . . . e.g. absolute equil.
Today

- Natural Hamiltonian systems
- “Unnatural” Hamiltonian systems
- Noncanonical Hamiltonian systems
Action Principle

Hero of Alexandria (60 AD) $\rightarrow$ Fermat (1600’s) $\rightarrow$

Hamilton’s Principle (1800’s)

The Procedure:

- Configuration Space/Manifold $Q$: $q^i(t), i = 1, 2, \ldots, N \leftarrow \#\text{DOF}$

- Lagrangian (Kinetic Potential): $L = T - V \leftarrow L : TQ \rightarrow \mathbb{R}$

- Action Functional:

  \[ S[q] = \int_{t_0}^{t_1} L(q, \dot{q}) \, dt, \quad \delta q(t_0) = \delta q(t_1) = 0 \]

  Extremal path $\Rightarrow$ Lagrange's equations
Variation Over Paths

\[ S[q_{\text{path}}] = \text{number} \]

First Variation (Fréchet derivative):

\[ \delta S[q; \delta q] = DS \cdot \delta q = \frac{d}{d\epsilon} S[q + \epsilon \delta q] \bigg|_{\epsilon=0} \equiv 0 \quad \forall \, \delta q(t) \implies \]

Lagrange's Equations:

\[ \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0. \]
Hamilton’s Equations

Canonical Momentum: \[ p_i = \frac{\partial L}{\partial \dot{q}^i} \leftarrow \text{inverse function theorem} \]

Legendre Transform: \[ H(q, p) = p_i \dot{q}^i - L(\dot{q}, q) \]

\[
\begin{align*}
\dot{p}_i &= -\frac{\partial H}{\partial q^i}, \\
\dot{q}^i &= \frac{\partial H}{\partial p_i},
\end{align*}
\]

Phase Space Coordinates: \[ z = (q, p), \quad \alpha, \beta = 1, 2, \ldots, 2N \]

\[
\dot{z}^\alpha = J^\alpha_\beta \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}, \quad (J^\alpha_\beta) = \begin{pmatrix}
0_N & I_N \\
-I_N & 0_N
\end{pmatrix},
\]

\[ J_c := \text{Poisson tensor}, \text{ Hamiltonian bi-vector, cosymplectic form} \]

Symplectic 2-form = (cosymplectic form)\(^{-1}\): \[ \omega^c_{\alpha\beta} J^\beta_\gamma = \delta^\gamma_\alpha, \]
Natural Hamiltonian Systems
Natural Hamiltonian Systems

Natural Hamiltonian:

\[ H(q,p) = T(q,p) + V(q) \]

Kinetic Energy:

\[ T(q,p) = \frac{1}{2} \sum_{i,j} m^{-1}_{ij}(q) p_i p_j \]

where \( m_{ij} \) pos. def. mass matrix (metric tensor).

Potential energy:

\[ V(q) = V(q_1, q_2, \ldots, q_N) \]

Equations of motion:

\[ \dot{q}_i = \sum_j m^{-1}_{ij}(q) p_j \quad \text{and} \quad \dot{p}_i = -\frac{\partial V}{\partial q_i} \]

for \( m_{ij} \) constant.
Natural Hamiltonian Examples

• Mass spring systems, pendula, particle in potential well, etc.

• N-Body problem $q_i = (q_{x_i}, q_{y_i}, q_{z_i}) \in Q \subset \mathbb{R}^3$, $i = 1, 2, \ldots N$

$$H = \sum_{i=1}^{N} \frac{||p_i||^2}{2m_i} + \sum_{i,j=1}^{N} \frac{c_{ij}}{||q_i - q_j||}$$

where depending on sign $c_{ij}$ it represents attracting gravitational interaction (satellites, planets, stars, ...), repelling electrostatic interaction (electrons), attracting electrons and ions (protons), collection of both in plasmas.
“Unnatural” Hamiltonian Systems
“Unnatural” := ¬ Natural Hamiltonian Systems

• Charged particle in given electromagnetic fields:

\[ m\ddot{q} = eE(q, t) + \frac{e}{c} \dot{q} \times B(q, t) \]

where \( E, B \) electric and magnetic fields, respectively, \( e \) charge, \( m \) mass, \( c \) speed of light.

Potentials:

\[ B = \nabla \times A \quad E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t} \]

Hamiltonian:

\[ H(q, p, t) = \frac{1}{2m} \left\| p - \frac{e}{c} A(q, t) \right\|^2 + e\phi(q, t) \]
Other Unnatural Hamiltonian Systems

• Interaction of point vortices in the plane

\[ H = c \sum_{ij=1}^{N} \kappa_i \kappa_j \ln \left( (x_i - x_j)^2 + (y_i - y_j)^2 \right) \]

• Chaotic advection in two dimensions

\[ H = \psi(x, y, t), \quad \nabla \cdot \mathbf{v} = 0 \rightarrow \mathbf{v}(x, y, t) = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right), \]

neutrally buoyant particle or dye moves with fluid. Stream function \( \psi \).

• Magnetic field line flow (integral curves of \( B(x) \))

• Other: predator-prey, etc.
Chaotic Advection


Cyclonic (eastward) jet

particle streaks
dye
Particle in $B$-Field

- Equation of motion:
  $$m\ddot{q} = \frac{e}{c} \dot{q} \times B(q, t)$$

- Solution for $B$ uniform:
Particle in $B$-Field

- Equation of motion:

$$m\ddot{q} = \frac{e}{c} \dot{q} \times B(q, t)$$

- Solution for $B$ uniform:
Particle in $B$-Field

- Equation of motion:

\[ m\ddot{\mathbf{q}} = \frac{e}{c} \dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}, t) \]

- Solution for $B$ uniform:

  - Gyroradius: $\rho_g = mc\nu_\perp / (eB)$. Gyrofrequency: $\Omega_g = eB / (mc)$
$B$-lines as Hamiltonian system

If particles are tied to field lines $\Rightarrow$ natural to look at field line flow. If interested in confinement, then if field lines escape particle will.

Use fields for confinement: Stellerator, Tokamak, etc.

Also related are particle accelerators.

Field line flow is Hamiltonian: Kruskal, Kerst, I. M. Gelfand, Morozov & Solov’ev, ... Cary and Littlejohn 1983
Tokamak Fields

Superposition of dipole + toroid $\approx$ Tokamak
**B-line Hamiltonian for Straight Torus**

For simplicity remove metrical components and consider topological torus with cylindrical coords \((r, \theta, z)\) with \(z\) ‘toroidal’ angle (long way around) \(\theta\) poloidal angle (short way around):

\[
B = B_T \hat{z} + \hat{z} \times \nabla \psi(r, \theta, z) \quad B_T = \text{const} >> B_\perp \Rightarrow 1 \text{ way}
\]

assures \(\nabla \cdot B = 0\).

Integral curves of \(B(x)\):

\[
\frac{dR}{d\sigma} = B
\]

parametrize by \(z \Rightarrow\)

\[
\frac{dX}{dz} = B_x = -\frac{\partial \psi}{\partial Y} \quad \text{and} \quad \frac{dY}{dz} = B_y = \frac{\partial \psi}{\partial X}
\]

\(\psi(X, Y, z)\) is the Hamiltonian. General system is 1.5 dof but integrable if \(\partial \psi/\partial z \equiv 0\) (becomes 1 dof) desired equilibrium state.
Surface of Section

Symmetry breaking $\Leftrightarrow k \uparrow$
Early Symplectic Map

PROJECT MATTERHORN
Forrestal Research Center
Princeton University
Princeton, New Jersey

SOME PROPERTIES OF ROTATIONAL TRANSFORMS
February 18, 1952

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MAY BE USED ONLY IN THE
READING ROOM.

Report written by:

Martin D. Kruskal
whence again from (4) it follows that $X_n$ is similarly periodic.

Part III

Inasmuch as it appears difficult to obtain formulas for the deviation from periodicity, computations for particular cases were carried out by Miss Edith Guertler. In the first case (for system (1) the transformation from $X_n$ to $X_{n+1}$ was taken to be

$$x_{n+1} = x_n - \frac{1}{3} g y_n \{2 + y_n^2\},$$

$$y_{n+1} = y_n + g \log_e \left(1 + \frac{1}{2} x_{n+1}\right)$$

{[11]}
George Miloshevich ← $250
Universal Symplectic Maps of Dimension Two

Standard (Twist) Map:

\[
\begin{align*}
x' &= x + y' \\
y' &= y - \frac{k}{2\pi} \sin(2\pi x)
\end{align*}
\]

Standard Nontwist Map:

\[
\begin{align*}
x' &= x + a(1 - y'^2) \\
y' &= y - b \sin(2\pi x)
\end{align*}
\]

Parameters:

\(a\) measures shear, while \(b\) and \(k\) measure ripple
Drifts: $\exists \mathbf{E}$ and $\mathbf{B}$ Not Uniform

Even for large $\mathbf{B}$, particles don’t follow field lines because of drifts.

Drift Types: $\mathbf{E} \times \mathbf{B}$, $\nabla \mathbf{B}$, curvature, polarization.

Example $\nabla \mathbf{B}$ (recall $\rho_g \sim 1/B$):

Reductions base on magnetic moment $\mu = m|v_\perp|^2/(2B)$ being an adiabatic invariant.
Drifts: \( \exists \ E \) and \( B \) Not Uniform (cont)
Drifts: $\exists E$ and $B$ Not Uniform (cont)
Drifts: $\exists E$ and $B$ Not Uniform (cont$^2$)
Drifts - \( \exists E \) and \( B \) Not Uniform (cont\(^2\))

Guiding center equations for transport. Hannes Alfvén.

→ Hamiltonian system in noncanonical coordinates
  \( \subset \) Noncanonical Hamiltonian systems.

Kinetic theories on guiding centers etc.
  → Drift kinetics and Gyrokinetics.
Noncanonical Hamiltonian Systems
Usual Geometry

Dynamics takes place in phase space, $\mathcal{Z}$ (needn’t be $T^*Q$), a differential manifold endowed with a closed, nondegenerate 2-form $\omega$. A patch has canonical coordinates $z = (q, p)$.

Hamiltonian dynamics $\Leftrightarrow$ flow on symplectic manifold: $i_X \omega = dH$

Poisson tensor $(J_c)$ is bivector inverse of $\omega$, defining the Poisson bracket

$$\{f, g\} = \langle df, J_c(dg) \rangle = \omega(X_f, X_g) = \frac{\partial f}{\partial z^\alpha} J_c^{\alpha \beta} \frac{\partial g}{\partial z^\beta}, \quad \alpha, \beta = 1, 2, \ldots, 2N$$

Flows generated by Hamiltonian vector fields $Z_H = JdH$, $H$ a 0-form, $dH$ a 1-form. Poisson bracket $=$ commutator of Hamiltonian vector fields etc.

Early refs.: Jost, Mackey, Souriau, Arnold, Abraham & Marsden
Noncanonical Hamiltonian Definition

A phase space $\mathcal{P}$ diff. manifold with binary bracket operation on $C^\infty(\mathcal{P})$ functions $f, g: \mathcal{P} \to \mathbb{R}$, s.t. $\{\cdot, \cdot\}: C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ satisfies

- **Bilinear:** $\{f + \lambda g, h\} = \{f, h\} + \lambda\{g, h\}$, $\forall f, g, h$ and $\lambda \in \mathbb{R}$
- **Antisymmetric:** $\{f, g\} = -\{g, f\}$, $\forall f, g$
- **Jacobi:** $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \equiv 0$, $\forall f, g, h$
- **Leibniz:** $\{fg, h\} = f\{g, h\} + \{f, h\}g$, $\forall f, g, h$.

Above is a Lie algebra realization on functions. Take $fg$ to be pointwise multiplication.

**Eqs. Motion:** $\frac{\partial \Psi}{\partial t} = \{\Psi, H\}$, $\Psi$ an observable & $H$ a Hamiltonian.

**Example:** flows on Poisson manifolds, e.g. Weinstein 1983 ....
Noncanonical Hamiltonian Dynamics

Sophus Lie (1890)

Noncanonical Coordinates:

\[
\dot{z}^\alpha = J^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}, \quad \{f, g\} = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta}(z) \frac{\partial g}{\partial z^\beta}, \quad \alpha, \beta = 1, 2, \ldots M
\]

Poisson Bracket Properties:

antisymmetry \[\rightarrow \quad \{f, g\} = -\{g, f\},\]

Jacobi identity \[\rightarrow \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0\]

G. Darboux: \[det J \neq 0 \implies J \rightarrow J_c \quad \text{Canonical Coordinates}\]

Sophus Lie: \[det J = 0 \implies \text{Canonical Coordinates plus Casimirs}\]

\[
J \rightarrow J_d = \begin{pmatrix}
0_N & I_N & 0 \\
-I_N & 0_N & 0 \\
0 & 0 & 0_{M-2N}
\end{pmatrix}.
\]
Flow on Poisson Manifold

**Definition.** A Poisson manifold $\mathcal{M}$ is a differentiable manifold with bracket $\{\cdot,\cdot\}:\mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ such that $\mathcal{C}^\infty(\mathcal{M})$ with $\{\cdot,\cdot\}$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $Z_H = JdH$.

Because of degeneracy, $\exists$ functions $C$ st $\{f,C\} = 0$ for all $f \in \mathcal{C}^\infty(\mathcal{M})$. Called Casimir invariants (Lie's distinguished functions.)
Degeneracy in $J \Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall f : \mathcal{M} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:

Leaf vector fields, $Z_f = \{z, f\} = J df$ are tangent to leaves.
Lie-Poisson Brackets

Coordinates:

\[ J^{\alpha \beta} = c_{\gamma}^{\alpha \beta} z^{\gamma} \]

where \( c_{\gamma}^{\alpha \beta} \) are the structure constants for some Lie algebra.

Examples:

- 3-dimensional Bianchi algebras for free rigid body, Kida vortex, & other?
- Infinite-dimensional theories - matter models: Ideal fluid flow, MHD, shearflow, extended MHD, Vlasov-Maxwell, BBGKY, etc.
**Lie-Poisson Geometry**

**Lie Algebra:** $\mathfrak{g}$, a vector space with

\[ [ , ] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} , \]

antisymmetric, bilinear, satisfies Jacobi identity

**Pairing:**

\[ \langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R} \]

with $\mathfrak{g}^*$ vector space dual to $\mathfrak{g}$

**Lie-Poisson Bracket:**

\[ \{f, g\} = \langle z, \begin{bmatrix} \frac{\partial f}{\partial z} \end{bmatrix} , \begin{bmatrix} \frac{\partial g}{\partial z} \end{bmatrix} \rangle , \quad z \in \mathfrak{g}^*, \frac{\partial f}{\partial z} \in \mathfrak{g} \]
Example $\mathfrak{so}(3)$

Lie Algebra is antisymmetric matrices, or $s = (s_1, s_2, s_3)$, a vector space with

$$[f, g] = \frac{\partial f}{\partial s} \times \frac{\partial g}{\partial s}$$

where $\times$ is vector cross product.

Pairing between $s \in \mathfrak{so}(3)^*$ and $\partial f / \partial s \in \mathfrak{g}$ yields the Lie-Poisson bracket:

$$\{f, g\} = s \cdot \frac{\partial f}{\partial s} \times \frac{\partial g}{\partial s} = \epsilon_{\alpha\beta\gamma} s_\alpha \frac{\partial f}{\partial s_\beta} \frac{\partial g}{\partial s_\gamma} ,$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita (permutation) symbol, which denotes the structure constants for $\mathfrak{so}(3)$.

Casimirs (nested spheres $S^2$ foliation):

$$C = s_1^2 + s_2^2 + s_3^2$$

Examples: spin system, free rigid body with Euler’s equations
All Real 3D Lie-Poisson Structures

Bianchi classification (cf. Jacobson) of real Lie algebras

\[ c^\alpha_{\beta \gamma} = \epsilon_{\beta \gamma \delta} m^{\delta \alpha} + \delta^\alpha_k a_{\beta} - \delta^\alpha_\beta a_{\gamma}, \quad \alpha, \beta, \gamma = 1, 2, 3 \]

<table>
<thead>
<tr>
<th>Class</th>
<th>Type</th>
<th>( m )</th>
<th>( a_\alpha )</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>I</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>II</td>
<td>diag(1, 0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>VI(_{-1})</td>
<td>(-\alpha)</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>VII(_0)</td>
<td>diag(-1, -1, 0)</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>VIII</td>
<td>diag(-1, 1, 1)</td>
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</tr>
<tr>
<td>A</td>
<td>IX</td>
<td>diag(1, 1, 1)</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>III</td>
<td>(-\frac{1}{2}\alpha)</td>
<td>(-\frac{1}{2}\delta_3^\alpha)</td>
</tr>
<tr>
<td>B</td>
<td>IV</td>
<td>diag(1, 0, 0)</td>
<td>(-\delta_3^\alpha)</td>
</tr>
<tr>
<td>B</td>
<td>V</td>
<td>0</td>
<td>(-\delta_3^\alpha)</td>
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<tr>
<td>B</td>
<td>VI(_{h \neq -1})</td>
<td>(\frac{1}{2}(h - 1)\alpha)</td>
<td>(-\frac{1}{2}(h + 1)\delta_3^\alpha)</td>
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<tr>
<td>B</td>
<td>VII(_{h=0})</td>
<td>diag(-1, -1, 0) + (\frac{1}{2}h\alpha)</td>
<td>(-\frac{1}{2}h\delta_3^\alpha)</td>
</tr>
</tbody>
</table>
All Real 3D Lie-Poisson Structures (cont²)

Class A:
Type $IX$ – Free rigid body, spin, ...
Type $II$ – Heisenberg algebra
Type $VIII$ – Kida vortex of fluid mechanics

Class B: ?
All Real 3D Lie-Poisson Structures (cont$^3$)

Orbits lie on intersection of Casimir leaves and energy surface. Singular equilibrium is at ($R = P = 0, S \neq 0$).
All Real 3D Lie-Poisson Structures (cont^4)

• Type VI_{h<-1} governs rattleback system of Moffat and Tokieda.

Chirality comes from equilibria that live on the singular set.

Such equilibria need not have Hamiltonian spectra.

Yoshida, Tokieda, pjm (2017)

Rank changing is responsible for the Casimir deficit problem.

Relationship to b-symplectic and presymplectic systems.
End Part I