Hamilton description of plasmas and other models of matter: structure and applications I

P. J. Morrison

Department of Physics and Institute for Fusion Studies
The University of Texas at Austin

`morrison@physics.utexas.edu`

`http://www.ph.utexas.edu/~morrison/`

MSRI August 20, 2018

Survey Hamiltonian systems that describe matter: particles, fluids, plasmas, e.g., magnetofluids, kinetic theories, ...

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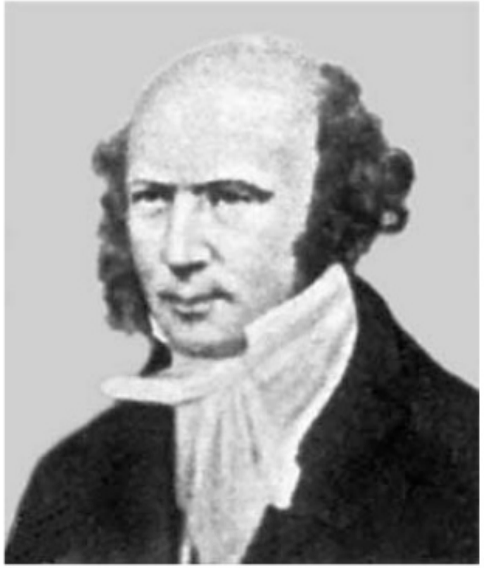
"Hamiltonian systems .... are the basis of physics." M. Gutzwiller

Coarse Outline



Hamilton in middle age

William Rowan Hamilton (August 4, 1805 - September 2, 1865)

I. **Today:** Finite-dimensional systems. Particles etc. ODEs

II. **Tomorrow:** Infinite-dimensional systems. Hamiltonian field theories. PDEs

Why Hamiltonian?

- Beauty, Teleology, . . . : Still a good reason!
- 20th Century framework for physics: Fluids, Plasmas, etc. too.
- Symmetries and Conservation Laws: energy-momentum . . . .
- Generality: do one problem $\Rightarrow$ do all.
- Approximation: perturbation theory, averaging, . . . 1 function.
- Stability: built-in principle, Lagrange-Dirichlet, δW, ...
- Beacon: $\exists$ ∞-dim KAM theorem? Krein with Cont. Spec.?
- Numerical Methods: structure preserving algorithms: symplectic, conservative, Poisson integrators, . . . e.g. GEMPIC.
- Statistical Mechanics: energy, measure . . . e.g. absolute equil.

Today

- Natural Hamiltonian systems
- “Unnatural” Hamiltonian systems
- Noncanonical Hamiltonian systems

Action Principle

Hero of Alexandria (60 AD) $\longrightarrow$ Fermat (1600's) $\longrightarrow$

$\boxed{\text{Hamilton's Principle (1800's)}}$

The Procedure:

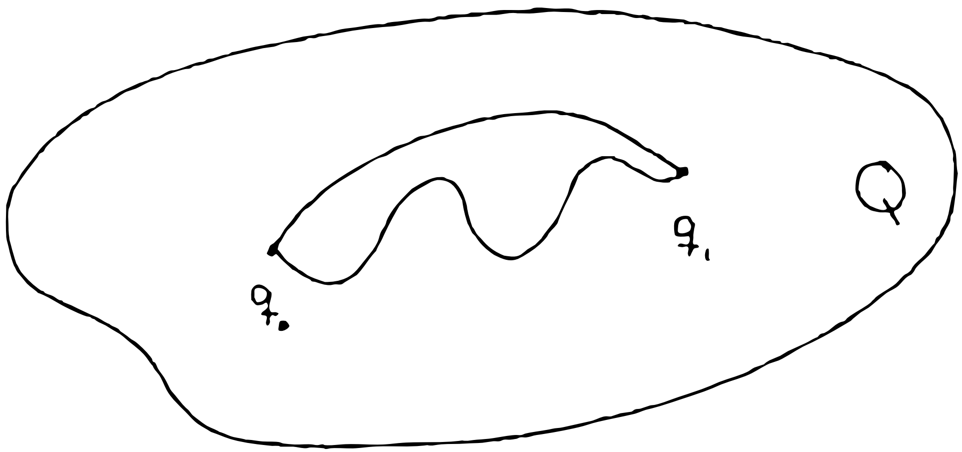
- Configuration Space/Manifold Q: $q^i(t)\,,\ i = 1, 2, \ldots, N \leftarrow$ #DOF
- Lagrangian (Kinetic Potential): $L = T - V \quad \leftarrow L : TQ \to \mathbb{R}$
- Action Functional:

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q})\, dt\,, \qquad \delta q(t_0) = \delta q(t_1) = 0$$

$\boxed{\text{Extremal path} \Rightarrow \text{Lagrange's equations}}$

Variation Over Paths

$S[q_{\text{path}}] =$ number



First Variation (Fréchet derivative):

$$\delta S[q;\delta q] = DS\cdot\delta q = \frac{d}{d\epsilon}S[q+\epsilon\,\delta q]\Big|_{\epsilon=0} \equiv 0 \qquad \forall\ \delta q(t) \qquad \Rightarrow$$

Lagrange's Equations:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = 0\,.$$

Hamilton's Equations

Canonical Momentum: $p_i = \frac{\partial L}{\partial \dot{q}^i}$ $\leftarrow$ inverse function theorem

Legendre Transform: $H(q,p) = p_i \dot{q}^i - L(\dot{q}, q)$

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,}$$

Phase Space Coordinates: $z = (q,p)\,, \quad \alpha, \beta = 1, 2, \ldots 2N$

$$\dot{z}^\alpha = J_c^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}\,, \qquad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

symplectic 2-form = (cosymplectic form)$^{-1}$: $\omega^c_{\alpha\beta} J_c^{\beta\gamma} = \delta^\gamma_\alpha$,

Natural Hamiltonian Systems

Natural Hamiltonian Systems

Natural Hamiltonian:

$$H(q,p) = T(q,p) + V(q)$$

Kinetic Energy:

$$T(q,p) = \frac{1}{2} \sum_{i,j} m_{ij}^{-1}(q)\, p_i\, p_j$$

where m_{ij} pos. def. mass matrix (metric tensor).

Potential energy:

$$V(q) = V(q_1, q_2, \ldots, q_N)$$

Equations of motion:

$$\dot{q}_i = \sum_j m_{ij}^{-1}\, p_j \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

for m_{ij} constant.

Natural Hamiltonian Examples

- Mass spring systems, pendula, particle in potential well, etc.

- N-Body problem $\boldsymbol{q}_i = (q_{xi}, q_{yi}, q_{zi}) \in Q \subset \mathbb{R}^3,\ i = 1, 2, \ldots N$

$$H = \sum_{i=1}^{N} \frac{||\boldsymbol{p}_i||^2}{2m_i} + \sum_{i,j=1}^{N} \frac{c_{ij}}{||\boldsymbol{q}_i - \boldsymbol{q}_j||}$$

where depending on sign c_{ij} it represents attracting gravitational interaction (satellites, planets, stars, ...), repelling electrostatic interaction (electrons), attracting electrons and ions (protons), collection of both in plasmas.

“Unnatural” Hamiltonian Systems

"Unnatural" := $\neg$ Natural Hamiltonian Systems

- Charged particle in given electromagnetic fields:

$$m\ddot{\boldsymbol{q}} = e\boldsymbol{E}(\boldsymbol{q},t) + \frac{e}{c}\dot{\boldsymbol{q}} \times \boldsymbol{B}(\boldsymbol{q},t)$$

where $\boldsymbol{E}, \boldsymbol{B}$ electric and magnetic fields, respectively, e charge, m mass, c speed of light.

Potentials:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

Hamiltonian:

$$H(\boldsymbol{q},\boldsymbol{p},t) = \frac{1}{2m}\left\|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{q},t)\right\|^2 + e\phi(\boldsymbol{q},t)$$

Other Unnatural Hamiltonian Systems

- Interaction of point vortices in the plane

$$H = c \sum_{ij=1}^{N} \kappa_i \kappa_j \ln\left((x_i - x_j)^2 + (y_i - y_j)^2\right)$$

- Chaotic advection in two dimensions

$$H = \psi(x, y, t)\,, \qquad \nabla \cdot \boldsymbol{v} = 0 \to \boldsymbol{v}(x, y, t) = (\partial\psi/\partial y, -\partial\psi/\partial x)\,,$$

neutrally buoyant particle or dye moves with fluid. Stream function ψ.

- Magnetic field line flow (integral curves of $\boldsymbol{B}(\boldsymbol{x})$)

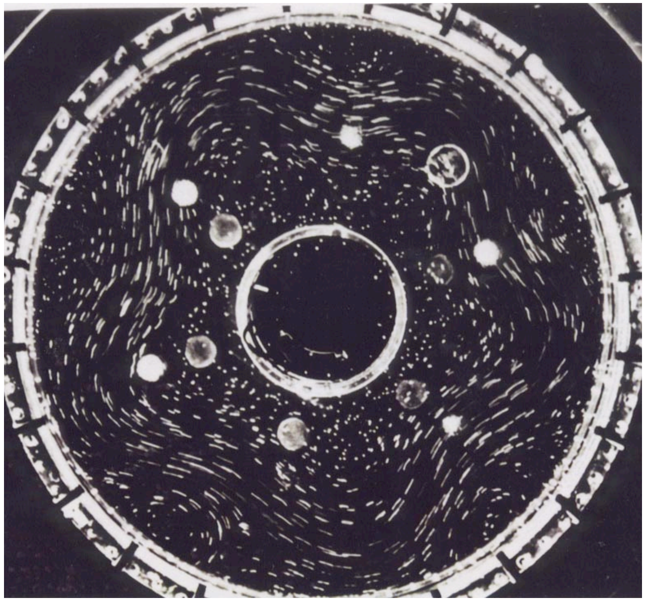
- Other: preditor-prey, etc.

Chaotic Advection

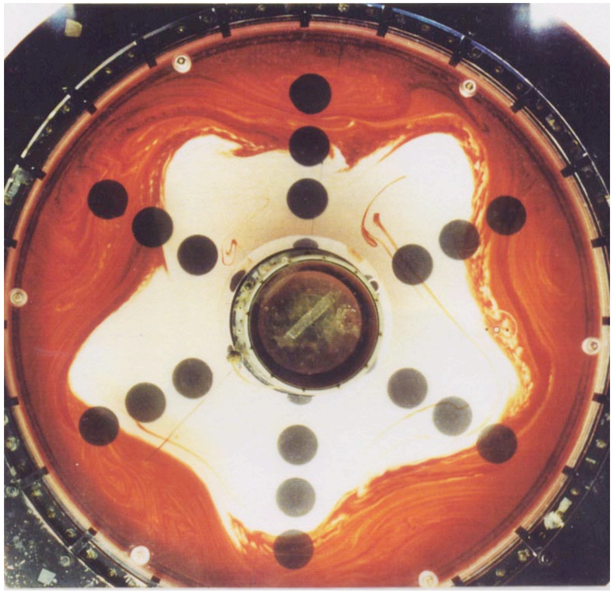
H. Swinney Lab circa 1989. Nontwist! del-Castillo-Negrete et al.

Cyclonic (eastward) jet

particle streaks



dye

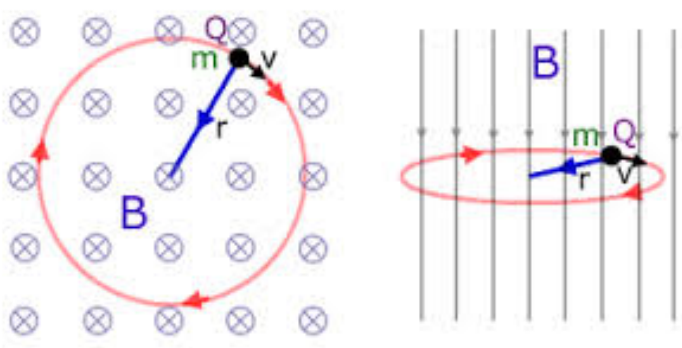


Particle in B-Field

- Equation of motion:

$$m\ddot{\boldsymbol{q}} = \frac{e}{c}\dot{\boldsymbol{q}} \times \boldsymbol{B}(\boldsymbol{q}, t)$$

- Solution for $\boldsymbol{B}$ uniform:



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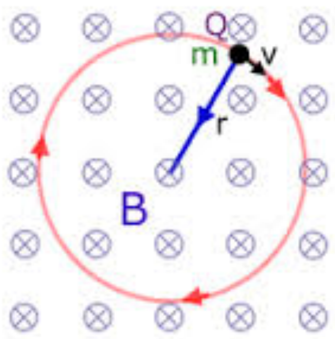

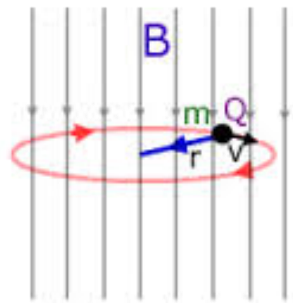

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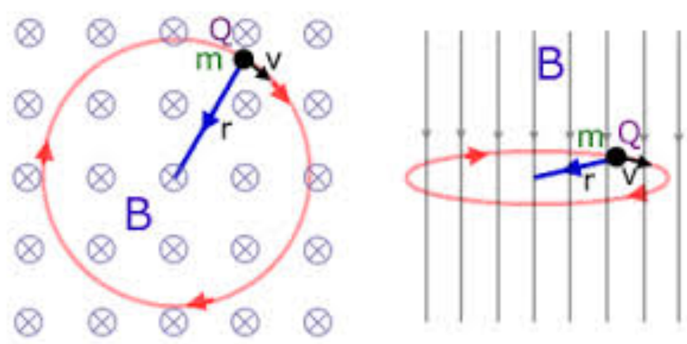
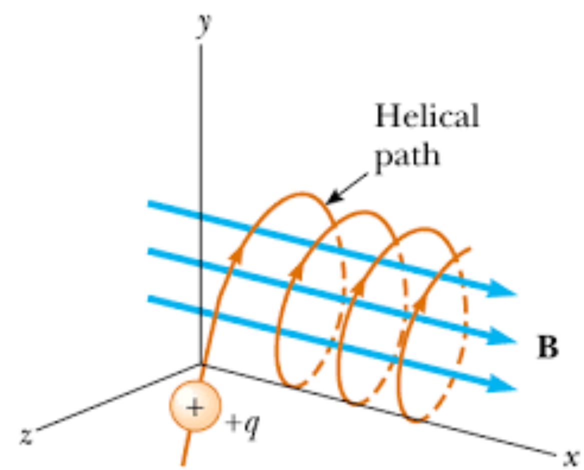




gyroradius: $\rho_g = mcv_\perp/(eB)$. gyrofrequency: $\Omega_g = eB/(mc)$

B-lines as Hamiltonian system

If particles are tied to field lines $\Rightarrow$ natural to look at field line flow. If interested in confinement, then if field lines escape particle will.

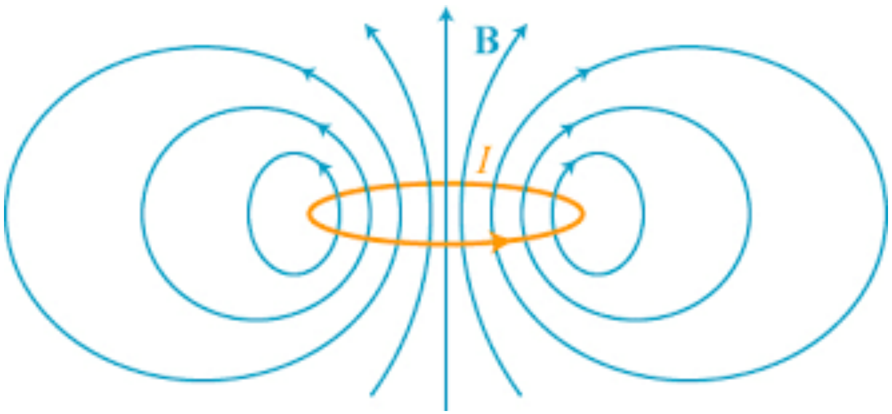
Use fields for confinement: Stellerator, Tokamak, etc.

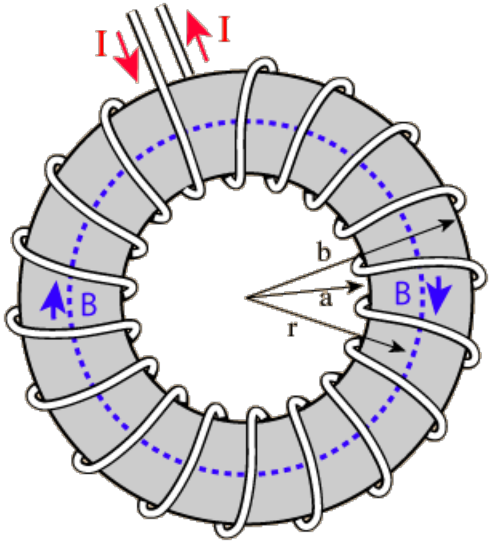
Also related are particle accelerators.

Field line flow is Hamiltonian: Kruskal, Kerst, I. M. Gelfand, Morozov & Solov'ev, ... Cary and Littlejohn 1983

Tokamak Fields

Superposition of dipole + toroid ≈ Tokamak





B-line Hamitonian for Straight Torus

For simplicity remove metrical components and consider topological torus with cylindrical coords (r, θ, z) with z 'toroidal' angle (long way aorund) θ poloidal angle (short way around):

$$\boldsymbol{B} = B_T \hat{z} + \hat{z} \times \nabla \psi(r, \theta, z) \qquad B_T = \text{const} >> B_\perp \Rightarrow 1 \text{ way}$$

assures $\nabla \cdot \boldsymbol{B} = 0$.

Integral curves of $\boldsymbol{B}(\boldsymbol{x})$:

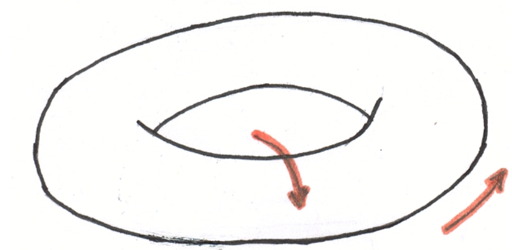
$$\frac{d\boldsymbol{R}}{d\sigma} = \boldsymbol{B}$$

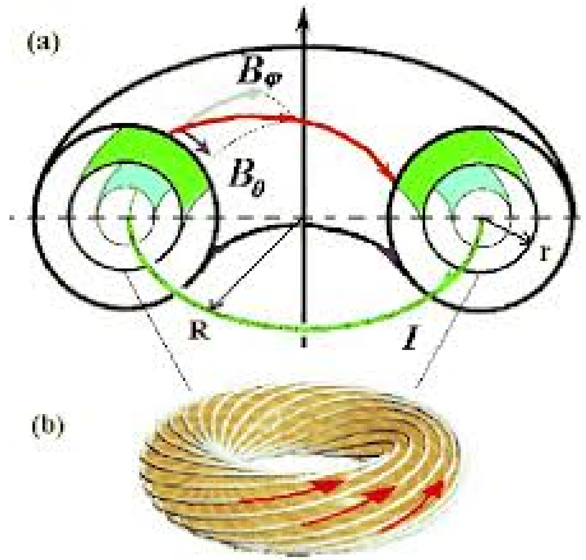
parametrize by $z \Rightarrow$

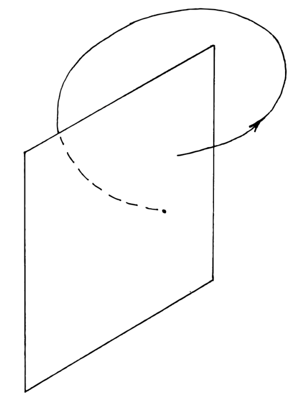
$$\frac{dX}{dz} = B_x = -\frac{\partial \psi}{\partial Y} \quad \text{and} \quad \frac{dY}{dz} = B_y = \frac{\partial \psi}{\partial X}$$

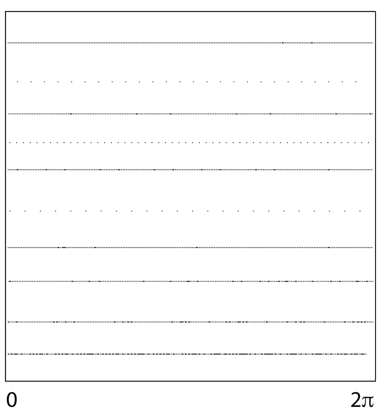
$\psi(X, Y, z)$ is the Hamiltonian. General system is 1.5 dof but integrable if $\partial \psi / \partial z \equiv 0$ (becomes 1 dof) desired equilibrium state.

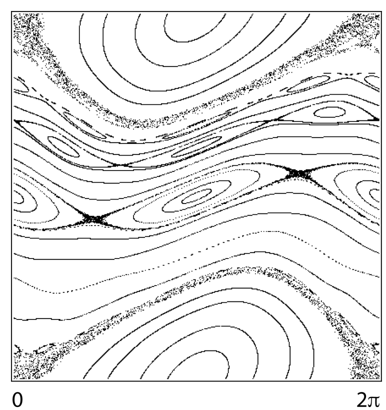
Surface of Section

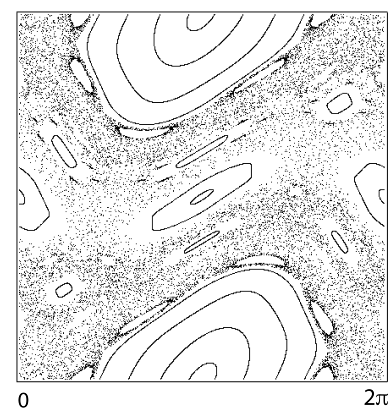












Symmetry breaking $\Leftrightarrow$ $k \uparrow$

Early Symplectic Map

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PROJECT MATTERHORN

Forrestal Research Center
Princeton University
Princeton, New Jersey

SOME PROPERTIES OF ROTATIONAL TRANSFORMS

February 18, 1952

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READING ROOM.

Report written by:

Martin D. Kruskal

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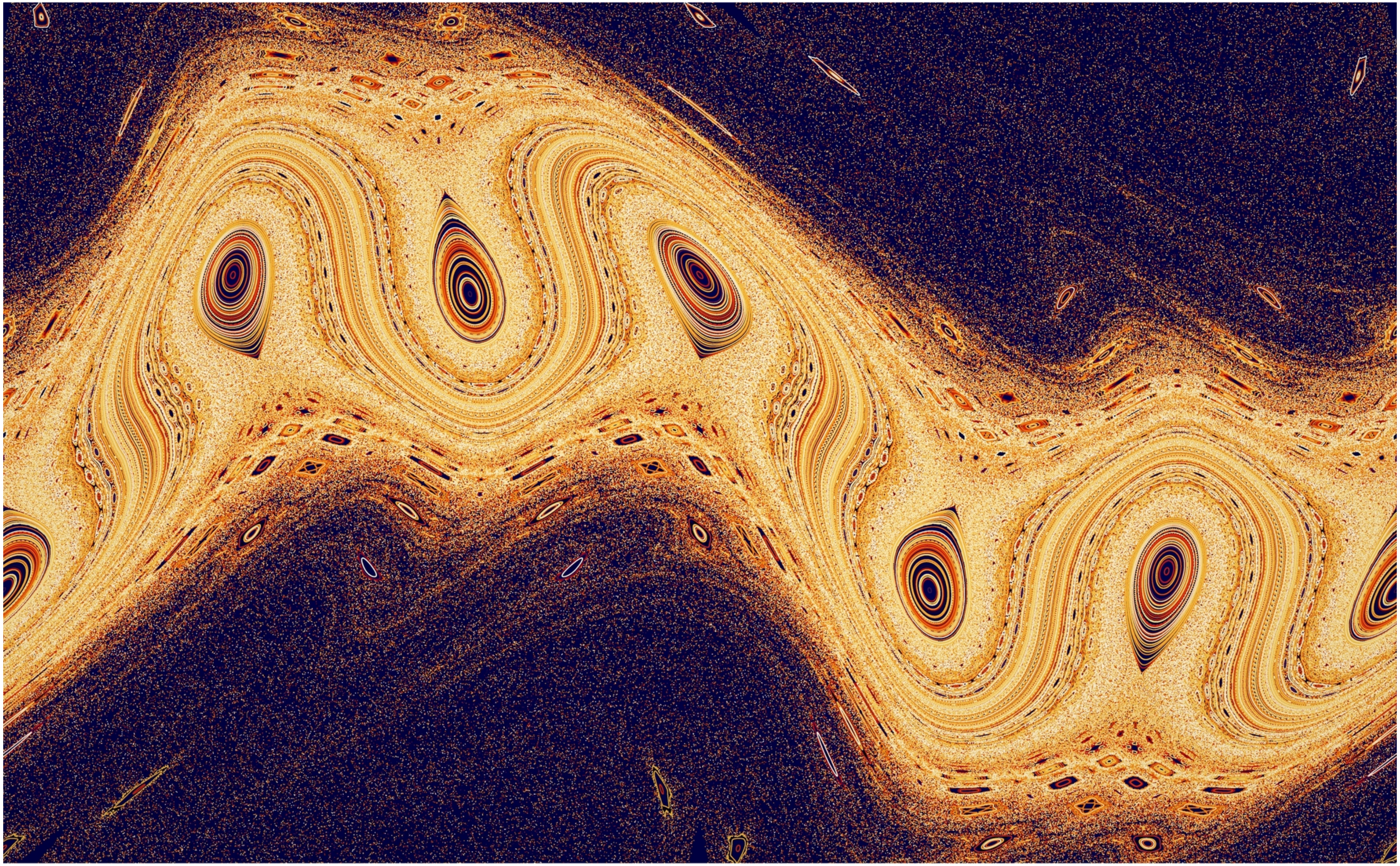
Early Symplectic Map (cont)

whence again from (4) it follows that X_n is similarly periodic.

Part III

Inasmuch as it appears difficult to obtain formulas for the deviation from periodicity, computations for particular cases were carried out by Miss Edith Guertler. In the first case (for system (1)) the transformation from $\mathbf{X}_n$ to $\mathbf{X}_{n+1}$ was taken to be

$$\begin{aligned} x_{n+1} &= x_n - \frac{1}{3} g\, y_n (2 + y_n), \\ y_{n+1} &= y_n + g \log_e (1 + \frac{1}{2} x_{n+1}) \end{aligned} \tag{11}$$



George Miloshevich ← $250

Universal Symplectic Maps of Dimension Two

Standard (Twist) Map:

$$\begin{aligned} x' &= x + y' \\ y' &= y - \frac{k}{2\pi}\sin(2\pi x) \end{aligned}$$

Standard Nontwist Map:

$$\begin{aligned} x' &= x + a(1 - y'^2) \\ y' &= y - b\sin(2\pi x) \end{aligned}$$

Parameters:

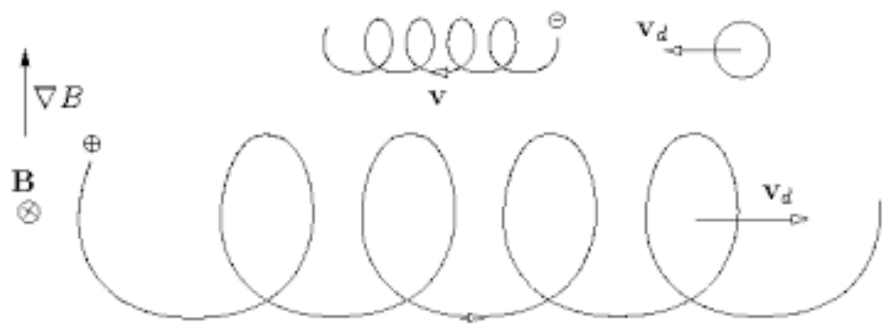
a measures shear, while b and k measure ripple

Drifts: $\exists$ E and B Not Uniform

Even for large $\boldsymbol{B}$, particles don't follow field lines because of drifts.

Drift Types: $\boldsymbol{E} \times \boldsymbol{B}$, $\nabla \boldsymbol{B}$, curvature, polarization.

Example $\nabla \boldsymbol{B}$ (recall $\rho_g \sim 1/B$):

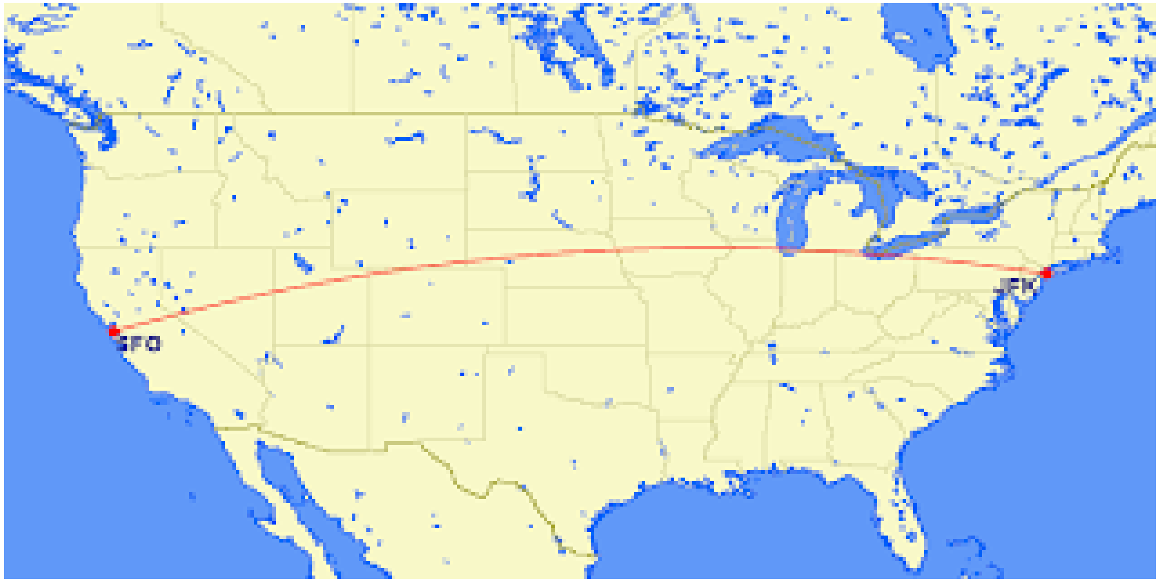


Reductions base on magnetic moment $\mu = m|v_\perp|^2/(2B)$ being an adiabatic invariant.

Drifts: $\exists$ E and B Not Uniform (cont)

Drifts: $\exists$ E and B Not Uniform (cont)

Drifts: $\exists$ E and B Not Uniform (cont^2)



Drifts - $\exists$ $\boldsymbol{E}$ and $\boldsymbol{B}$ Not Uniform (cont^2)

Guiding center equations for transport. Hannes Alfvén.

$\longrightarrow$ Hamiltonian system in noncanonical coordinates

$\subset$ Noncanonical Hamiltonian systems.

Kinetic theories on guiding centers etc.

$\longrightarrow$ Drift kinetics and Gyrokinetics.

Noncanonical Hamiltonian Systems

Usual Geometry

Dynamics takes place in phase space, $\mathcal{Z}$ (needn't be T^*Q), a differential manifold endowed with a closed, nondegenerate 2-form ω. A patch has canonical coordinates $z = (q, p)$.

Hamiltonian dynamics $\Leftrightarrow$ flow on symplectic manifold: $i_X\omega = dH$

Poisson tensor (J_c) is bivector inverse of ω, defining the Poisson bracket

$$\{f, g\} = \langle df, J_c(dg)\rangle = \omega(X_f, X_g) = \frac{\partial f}{\partial z^\alpha} J_c^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,, \qquad \alpha, \beta = 1, 2, \ldots 2N$$

Flows generated by Hamiltonian vector fields $Z_H = JdH$, H a 0-form, dH a 1-form. Poisson bracket = commutator of Hamiltonian vector fields etc.

Early refs.: Jost, Mackey, Souriau, Arnold, Abraham &Marsden

Noncanonical Hamiltonian Definition

A phase space $\mathcal{P}$ diff. manifold with binary bracket operation on $C^\infty(\mathcal{P})$ functions $f, g\colon \mathcal{P} \to \mathbb{R}$, s.t. $\{\cdot\,,\cdot\}\colon C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ satisfies

- Bilinear: $\{f + \lambda g, h\} = \{f, h\} + \lambda\{g, h\}\,, \quad \forall f, g, h$ and $\lambda \in \mathbb{R}$
- Antisymmetric: $\{f, g\} = -\{g, f\}\,, \qquad \forall f, g$
- Jacobi: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{F, g\}\} \equiv 0\,, \qquad \forall f, g, h$
- Leibniz: $\{fg, h\} = f\{g, h\} + \{f, h\}g\,, \qquad \forall f, g, h$.

Above is a Lie algebra realization on functions. Take fg to be pointwise multiplication.

Eqs. Motion: $\frac{\partial \Psi}{\partial t} = \{\Psi, H\}$, Ψ an observable & H a Hamiltonian.

Example: flows on Poisson manifolds, e.g. Weinstein 1983 ....

Noncanonical Hamiltonian Dynamics

Sophus Lie (1890)

Noncanonical Coordinates:

$$\dot z^\alpha = J^{\alpha\beta}\frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}\,, \qquad \{f,g\} = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta}(z)\frac{\partial g}{\partial z^\beta}\,, \qquad \alpha,\beta = 1,2,\ldots M$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{f,g\} = -\{g,f\}\,,$

Jacobi identity $\longrightarrow$ $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

$$J \rightarrow J_d = \begin{pmatrix} 0_N & I_N & 0 \\ -I_N & 0_N & 0 \\ 0 & 0 & 0_{M-2N} \end{pmatrix}.$$

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{M}$ is differentiable manifold with bracket $\{\,,\,\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ st $C^\infty(\mathcal{M})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $Z_H = JdH$.

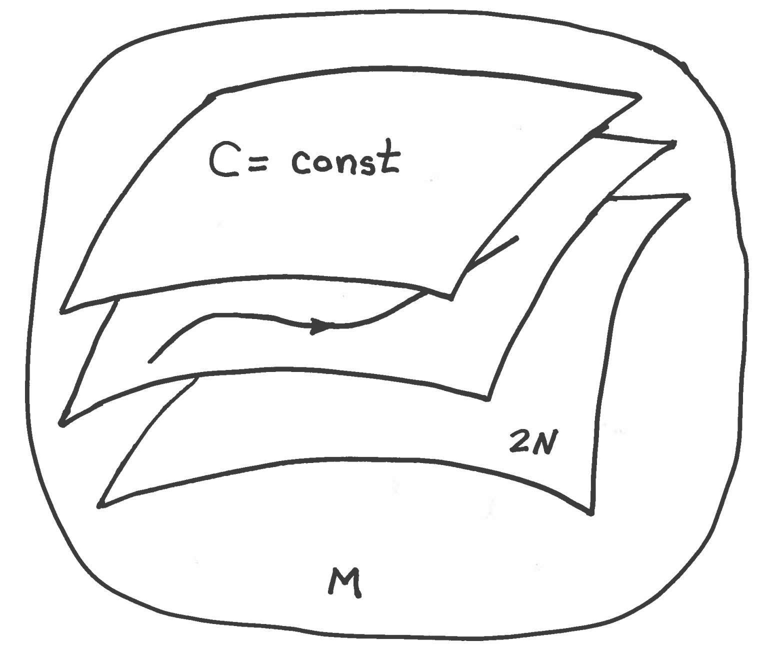
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{M})$. Called Casimir invariants (Lie's distinguished functions.)

Poisson Manifold $\mathcal{M}$ Cartoon

Degeneracy in $J \;\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\; f : \mathcal{M} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Leaf vector fields, $Z_f = \{z, f\} = Jdf$ are tangent to leaves.

Lie-Poisson Brackets

Coordinates:

$$J^{\alpha\beta} = c_{\gamma}^{\alpha\beta} z^{\gamma}$$

where $c_{\gamma}^{\alpha\beta}$ are the structure constants for some Lie algebra.

<u>Examples</u>:

• 3-dimensional Bianchi algebras for free rigid body, Kida vortex, & other ?

• Infinite-dimensional theories - matter models: Ideal fluid flow, MHD, shearflow, extended MHD, Vlasov-Maxwell, BBGKY, etc.

Lie-Poisson Geometry

Lie Algebra: $\mathfrak{g}$, a vector space with

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\,,$$

antisymmetric, bilinear, satisfies Jacobi identity

Pairing:

$$\langle\ \ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$$

with $\mathfrak{g}^*$ vector space dual to $\mathfrak{g}$

Lie-Poisson Bracket:

$$\{f, g\} = \left\langle z, \left[\frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right]\right\rangle\,, \qquad z \in \mathfrak{g}^*,\ \frac{\partial f}{\partial z} \in \mathfrak{g}$$

Example $\mathfrak{so}(3)$

Lie Algebra is antisymmetric matrices, or $\mathbf{s} = (s_1, s_2, s_3)$, a vector space with

$$[f, g] = \frac{\partial f}{\partial \mathbf{s}} \times \frac{\partial g}{\partial \mathbf{s}}$$

where $\times$ is vector cross product.

Pairing between $\mathbf{s} \in \mathfrak{so}(3)^*$ and $\partial f / \partial \mathbf{s} \in \mathfrak{g}$ yields the Lie-Poisson bracket:

$$\{f, g\} = \mathbf{s} \cdot \frac{\partial f}{\partial \mathbf{s}} \times \frac{\partial g}{\partial \mathbf{s}} = \epsilon_{\alpha\beta\gamma}\, s_\alpha \frac{\partial f}{\partial s_\beta} \frac{\partial g}{\partial s_\gamma}\,,$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita (permutation) symbol, which denotes the structure constants for $\mathfrak{so}(3)$.

Casimirs (nested spheres S^2 foliation):

$$C = s_1^2 + s_2^2 + s_3^2$$

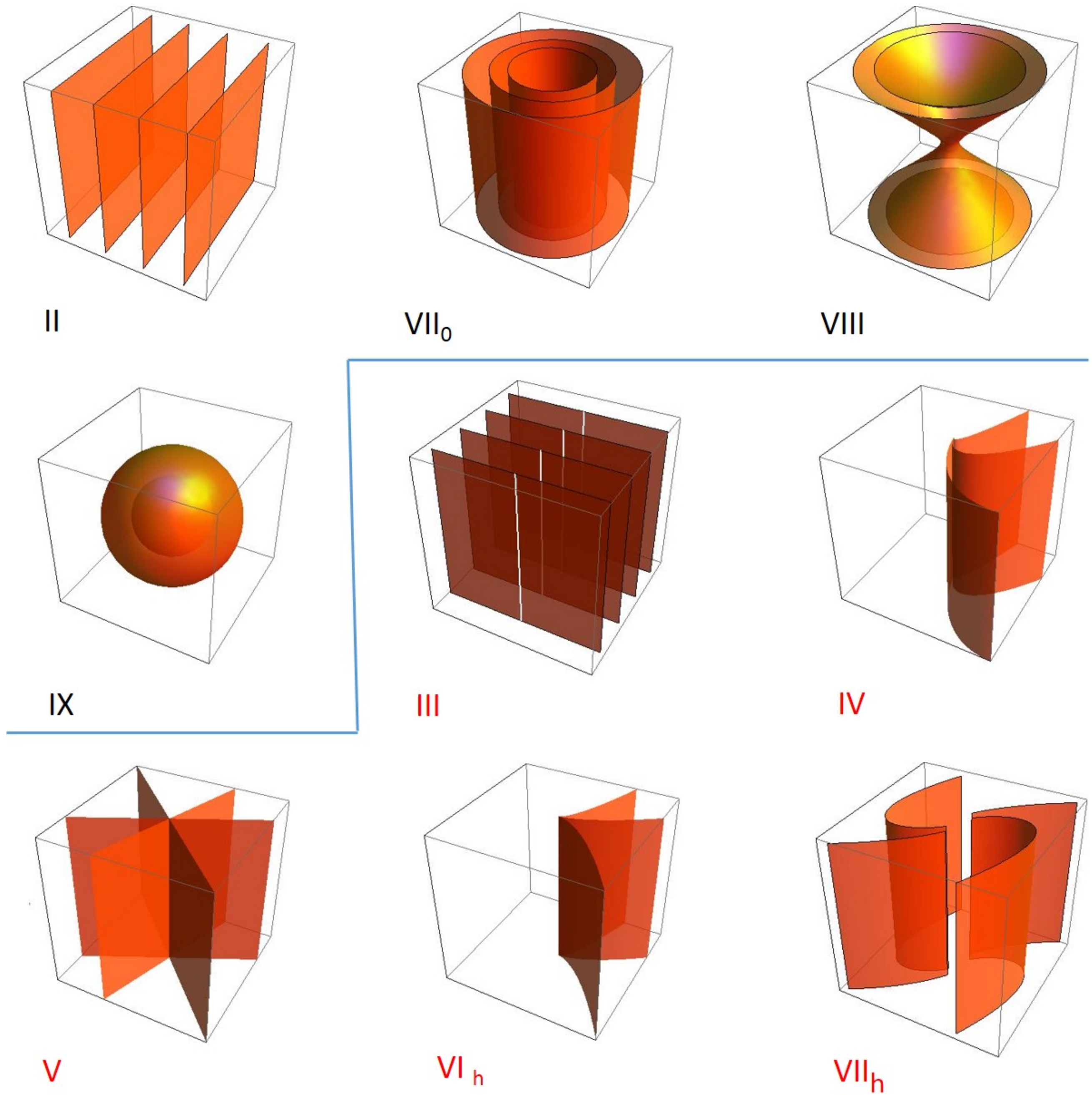
Examples: spin system, free rigid body with Euler's equations

All Real 3D Lie-Poisson Structures

Bianchi classification (cf. Jacobson) of real Lie algebras

$$c^{\alpha}_{\beta\gamma} = \epsilon_{\beta\gamma\delta}\, m^{\delta\alpha} + \delta^{\alpha}_{k}\, a_{\beta} - \delta^{\alpha}_{\beta}\, a_{\gamma}\,, \qquad \alpha, \beta, \gamma = 1, 2, 3$$

Class	Type	m	a_α
A	I	0	0
A	II	diag$(1,0,0)$	0
A	VI_{-1}	$-\alpha$	0
A	VII_0	diag$(-1,-1,0)$	0
A	VIII	diag$(-1,1,1)$	0
A	IX	diag$(1,1,1)$	0
B	III	$-\frac{1}{2}\alpha$	$-\frac{1}{2}\delta^{\alpha}_{3}$
B	IV	diag$(1,0,0)$	$-\delta^{\alpha}_{3}$
B	V	0	$-\delta^{\alpha}_{3}$
B	$\mathrm{VI}_{h\neq -1}$	$\frac{1}{2}(h-1)\alpha$	$-\frac{1}{2}(h+1)\delta^{\alpha}_{3}$
B	$\mathrm{VII}_{h=0}$	diag$(-1,-1,0) + \frac{1}{2}h\alpha$	$-\frac{1}{2}h\delta^{\alpha}_{3}$


II
VII0
VIII
IX
III
IV
V
VI h
VIIh

All Real 3D Lie-Poisson Structures (cont^2)

Class A:

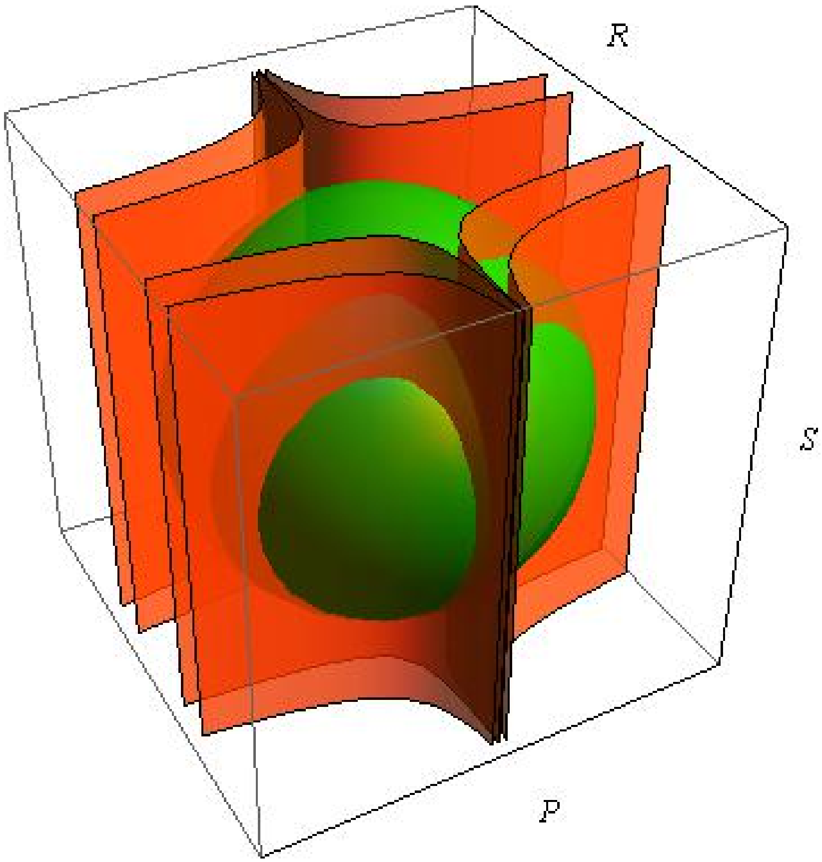
Type IX	$-$	Free rigid body, spin, ...
Type II	$-$	Heisenberg algebra
Type $VIII$	$-$	Kida vortex of fluid mechanics

Class B: ?

Showtime

All Real 3D Lie-Poisson Structures (cont^3)

Orbits lie on intersection of Casimir leaves and energy surface. Singular equilibrium is at $(R = P = 0, S \neq 0)$.



All Real 3D Lie-Poisson Structures (cont^4)

- Type $\mathrm{VI}_{h<-1}$ governs rattleback system of Moffat and Tokieda.

- Chirality comes from equilibria that live on the singular set.

- Such equilibria need not have Hamiltonian spectra.

Yoshida, Tokieda, pjm (2017)

- Rank changing is responsible for the Casimir deficit problem.

Relationship to b-symplectic and presymplectic systems.

End Part I