# Hamilton description of plasmas and other models of matter: structure and applications II

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Survey Hamiltonian systems that describe matter: particles, fluids, plasmas, e.g., magnetofluids, kinetic theories, . . . .

"Hamiltonian systems .... are the basis of physics." M. Gutzwiller

**Infinite-dimensional systems**

$\longleftrightarrow$

**Field Theories**

# Functionals

Functions: number $\longmapsto$ number or $f : \mathbb{R}^n \to \mathbb{R}$

example

Generalized Coordinate Path: $q(t) = A\cos(\omega t + \phi)$ e.g. SHO

Functionals: function $\longmapsto$ number or $F : \mathcal{F} \to \mathbb{R}$

examples

General: $F[u] = \int_D \mathcal{F}(u, u_x, u_{xx}, \dots)\, dx$ $u : D \to \mathbb{R}$

Hamiltonians Principle: $S[q] = \int_{t_0}^{t_1} L(q, \dot{q})\, dt\,.$

Vlasov Energy: $H[f] = m\int f v^2/2\, dxdv + \int E^2/2\, dx.$

# Functional Differentiation

First variation of function:

$$\delta f(z;\delta z) = \sum_{i=1}^{n} \frac{\partial f(z)}{\partial z_i}\,\delta z_i =: \nabla f \cdot \delta z\,, \qquad f(z) = f(z_1, z_2, \ldots, z_n)\,.$$

First variation of functional:

$$\delta F[u;\delta u] = \frac{d}{d\epsilon} F[u + \epsilon\,\delta u]\Big|_{\epsilon=0} = \int_{x_0}^{x_1} \delta u\, \frac{\delta F}{\delta u(x)}\, dx =: \left\langle \frac{\delta F}{\delta u}, \delta u \right\rangle\,.$$

| | | | | |
|---|---|---|---|---|
| dot product | $\cdot$ | $\Longleftrightarrow$ | scalar product | $<,>$ |
| index | $i$ | $\Longleftrightarrow$ | integration variable | $x$ |
| gradient | $\frac{\partial f(z)}{\partial z_i}$ | $\Longleftrightarrow$ | functional derivative | $\frac{\delta F[u]}{\delta u(x)}$ |

**Vary and Isolate $\longrightarrow$ Functional Derivative**

Examples: $\frac{\partial z_i}{\partial z_j} = \delta_{ij} \quad \Longleftrightarrow \quad \frac{\delta u(x)}{\delta u(x')} = \delta(x - x')$

# Infinite-Dimensional Hamiltonian Structure

Field Variables: $\psi(\mu, t)$ e.g. label $\mu = x$, $\mu = (x, v)$, . . .

Poisson Bracket:

$$\{F, G\} = \int \frac{\delta F}{\delta \psi} \mathcal{J}(\psi) \frac{\delta G}{\delta \psi} \, d\mu = \left\langle \frac{\delta F}{\delta \psi}, \mathcal{J} \frac{\delta G}{\delta \psi} \right\rangle$$

Lie-Poisson Bracket:

$$\{F, G\} = \left\langle \psi, \left[ \frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \psi} \right] \right\rangle$$

Poisson Operator:

$$\mathcal{J} \, \cdot \; \sim [\psi, \; \cdot \; ]$$

where [ , ] is a finite-dimensional Lie bracket on functions.

Unifying form for <u>Eulerian theories</u>: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, XMHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

# Simple Examples

Canonical Poisson Bracket:

$$\{F,G\} = \int d^3x \left( \frac{\delta F}{\delta \psi} \frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi} \frac{\delta G}{\delta \psi} \right) = \int d^3x \, \frac{\delta F}{\delta \chi} \, \mathcal{J}_c \frac{\delta G}{\delta \chi} \,,$$

where $\chi(x,t) := (\psi, \pi)$ with $\pi$ a momentum density.

Examples: wave equation, nonlinear Schroedinger, $\phi^4$ field theory, classical versions of quantum fields, etc.

Gardner KdV Bracket:

$$\{F,G\} = \int_{\mathbb{R}} dx \, \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \frac{\delta G}{\delta u}$$

where $u(x,t)$, i.e., a 1+1 theory. Here $\mathcal{J} = \partial / \partial x$.

Example: $H = \int_{\mathbb{R}} dx \, (u_x^2/2 - u^3/6)$

# Vlasov-Poisson System

Phase space density ($1 + 1 + 1$ field theory):

$$f(x, v, t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi[x, t; f]}{\partial x}\frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\phi_{xx} = -4\pi \left[ e \int_{\mathbb{R}} f(x, v, t)\, dv + \rho_B \right]$$

Energy:

$$H = \frac{m}{2} \int_{\mathbb{T}} \int_{\mathbb{R}} v^2 f \, dx dv + \frac{1}{8\pi} \int_{\mathbb{T}} (\phi_x)^2 \, dx$$

# Two Hamiltonian Systems

**Vlasov Poisson:** phase space density $f(x,v,t)$

$$\frac{\partial f}{\partial t} + [\mathcal{E}, f] = 0\,, \quad \text{where } [\mathcal{E}, f] = \mathcal{E}_x f_v - \mathcal{E}_v f_x\,,\ \mathcal{E} := mv^2/2 + e\phi.$$

**2D Euler/QG:** vorticity/pv $\omega(x,y,t)$

$$\frac{\partial \omega}{\partial t} + [\psi, \omega] = 0\,, \qquad \psi = \triangle^{-1}[\omega]$$

**Kinematic Commonality:**

energy, momentum, Casimir conservation; dynamics is re-arrangement; continuous spectra; . . . $\longrightarrow$

**Noncanonical Hamiltonian Structure:**

Organizing principle. Do one do all!

# Vlasov/2D Euler Fluid Hamiltonian Structure

How? Energy is quadratic $\Rightarrow$ SHO? Vlasov/Euler are quadratically nonlinear. Canonically conjugate variables?

Noncanonical Poisson Bracket (pjm 1980):

$$\{F,G\} = \int_{\mathbb{R}} \int_{\mathbb{T}} f \left[ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right] dx dv$$

Lie-Poisson Operator (group of canonical transformations):

$$\mathcal{J} \,\cdot = \frac{1}{m} \left( \frac{\partial f}{\partial x} \frac{\partial \;\cdot}{\partial v} - \frac{\partial \;\cdot}{\partial x} \frac{\partial f}{\partial v} \right)$$

Vlasov:

$$\frac{\partial f}{\partial t} = \{f, H\} = [f, \mathcal{E}].$$

Hamiltonian:

$$H[f] = \tfrac{m}{2} \int_{\mathbb{R}} \int_{\mathbb{T}} f\, v^2 \, dx dv + \tfrac{1}{2} \int_{\mathbb{T}} E^2 \, dx \,, \qquad \mathcal{E} = mv^2/2 + e\phi = \frac{\delta H}{\delta f}$$

**MHD= magnetohydrodynamics**

# MHD

Equations of Motion:

$$
\begin{array}{lrcl}
\text{Force} & \rho\dfrac{\partial \boldsymbol{v}}{\partial t} & = & -\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p + \dfrac{1}{c}\boldsymbol{J}\times\boldsymbol{B} \\
\text{Density} & \dfrac{\partial \rho}{\partial t} & = & -\nabla\cdot(\rho\boldsymbol{v}) \\
\text{Entropy} & \dfrac{\partial s}{\partial t} & = & -\boldsymbol{v}\cdot\nabla s \\
\text{Ohm's Law} & \boldsymbol{E} & + & \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{J} \approx 0 \\
\text{Magnetic Field} & \dfrac{\partial \boldsymbol{B}}{\partial t} & = & -\nabla\times\boldsymbol{E} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})
\end{array}
$$

Energy:

$$
H = \int_D d^3x \left(\frac{1}{2}\rho|\boldsymbol{v}|^2 + \rho U(\rho, s) + \frac{1}{2}|\boldsymbol{B}|^2\right)
$$

Thermodynamics:

$$
p = \rho^2\frac{\partial U}{\partial \rho} \qquad\qquad s = \frac{\partial U}{\partial s}
$$

Noncanonical Lie-Poisson Bracket (pjm & Greene 1980):

$$
\begin{aligned}
\{F,G\} \;=\; & -\int_D d^3x \left[ M_i \left( \frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \right. \\
& + \rho \left( \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left( \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
& + \boldsymbol{B} \cdot \left[ \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{B}} \right] \\
& \left. + \boldsymbol{B} \cdot \left[ \nabla \left( \frac{\delta F}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} - \nabla \left( \frac{\delta G}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta F}{\delta \boldsymbol{B}} \right] \right],
\end{aligned}
$$

Dynamics:

$\frac{\partial \rho}{\partial t} = \{\rho, H\}$, $\quad \frac{\partial s}{\partial t} = \{s, H\}$, $\quad \frac{\partial \boldsymbol{v}}{\partial t} = \{\boldsymbol{v}, H\}$, and $\frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}$.

Densities:

$$\boldsymbol{M} := \rho \boldsymbol{v} \qquad\qquad \sigma := \rho s$$

# Casimir Invariants

Helicities are Casimir Invariants:

$$\{F, C\}^{MHD} = 0 \qquad \forall \text{ functionals } F.$$

Casimirs Invariants (helicities):

$$C_B = \int d^3x\, \boldsymbol{B} \cdot \boldsymbol{A}\,, \qquad C_V = \int d^3x\, \boldsymbol{B} \cdot \boldsymbol{v}$$

Topological content, linking etc.

# XMHD= Extended MHD

Fortschritte der Physik 7, 503—558 (1959)

# Über die Ausbreitung von Wellen in einem Plasma

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## Zusammenfassung

Einleitend wird kurz auf die Bedeutung und die Anwendungsmöglichkeiten hingewiesen, die die Wellen in einem Plasma für verschiedene Gebiete der Physik und der Astrophysik haben können (Abschnitt 1). Die Grundgleichungen für ein nicht quasineutrales Plasma werden in Abschnitt 2 abgeleitet, wobei

Die Differenz der Gln. (2.3) und (2.4) ergibt nach Division durch $m_i$ bzw. $m_e$ und nach einigen weiteren Umformungen die „Diffusionsgleichung", die auch als das verallgemeinerte OHMsche Gesetz bezeichnet wird:

$$\frac{1}{e^2}\,\frac{m_i m_e}{n_i m_e + n_e m_i}\left\{\frac{d_2 \boldsymbol{j}}{dt} + \boldsymbol{N}_2\right\} + \frac{1}{\sigma}(\boldsymbol{j} - \lambda \boldsymbol{v}) = \boldsymbol{E} + \frac{\varrho}{n_i m_e + n_e m_i}\,\frac{1}{c}\,[\boldsymbol{v}\boldsymbol{B}] + $$
$$+ \frac{1}{e(n_i m_e + n_e m_i)}\left\{(m_e - m_i)\,\frac{1}{c}\,[\boldsymbol{j}\boldsymbol{B}] - m_e \operatorname{grad} p_i + m_i \operatorname{grad} p_e\right\}. \qquad (2.11)$$



nicht mehr auftritt [2]. Weiterhin ist auf der linken Seite $\lambda \boldsymbol{v}$ der Konvektionsstrom infolge der elektrischen Raumladungsdichte. Schließlich ist

$$\frac{d_2 \boldsymbol{j}}{dt} \equiv \frac{\partial \boldsymbol{j}}{\partial t} + \frac{\varrho}{m^2}\left(\frac{m_e}{n_i} + \frac{m_i}{n_e}\right)\{(\boldsymbol{v} \operatorname{grad})\,\boldsymbol{j} + (\boldsymbol{j} \operatorname{grad})\,\boldsymbol{v}\} + $$
$$+ \frac{1}{e m^2}\left(\frac{m_e^2}{n_i} - \frac{m_i^2}{n_e}\right)(\boldsymbol{j} \operatorname{grad})\,\boldsymbol{j} + \frac{1}{e m^2}\left\{\boldsymbol{j} \operatorname{grad}\left(\frac{m_e^2}{n_i} - \frac{m_i^2}{n_e}\right)\right\}\boldsymbol{j} + $$
$$+ \frac{1}{m^2}\,\varrho\left\{\boldsymbol{v} \operatorname{grad}\left(\frac{m_e}{n_i} + \frac{m_i}{n_e}\right)\right\}\boldsymbol{j} + $$
$$+ \frac{1}{m}\left\{m_e n_i \frac{\partial}{\partial t}\left(\frac{1}{n_i}\right) + m_i n_e \frac{\partial}{\partial t}\left(\frac{1}{n_e}\right)\right\} \qquad (2.13)$$

und

$$\boldsymbol{N}_2 \equiv \frac{e}{m}\left\{m_e n_i \frac{\partial}{\partial t}\left(\frac{n_e}{n_i}\right) - m_i n_e \frac{\partial}{\partial t}\left(\frac{n_i}{n_e}\right)\right\}\boldsymbol{v} + \frac{e\varrho^2}{m^2}\left(\frac{1}{n_i} - \frac{1}{n_e}\right)(\boldsymbol{v} \operatorname{grad})\,\boldsymbol{v} + $$
$$+ \frac{e\varrho}{m}\left\{\boldsymbol{v} \operatorname{grad}\left(m_e \frac{n_e}{n_i} - m_i \frac{n_i}{n_e}\right)\right\}\boldsymbol{v} + \frac{1}{m^2}\left\{\boldsymbol{j} \operatorname{grad}\left(m_e^2 \frac{n_e}{n_i} + m_i^2 \frac{n_i}{n_e}\right)\right\}\boldsymbol{v}. \qquad (2.14)$$

Auch hier ist die Aufspaltung zwischen $\frac{d_2 \boldsymbol{j}}{dt}$ und $\boldsymbol{N}_2$ wieder so vorgenommen worden, daß $\boldsymbol{N}_2$ verschwindet, wenn man Quasineutralität annimmt. $\boldsymbol{N}_2$ ist

# Extended MHD

Ohm's Law:

$$\boldsymbol{E} + \frac{\boldsymbol{V} \times \boldsymbol{B}}{c} = \frac{m_e}{e^2 n} \left( \frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V}) \right) - \frac{m_e}{e^2 n} (\boldsymbol{J} \cdot \nabla) \left( \frac{\boldsymbol{J}}{n} \right) + \frac{(\boldsymbol{J} \times \boldsymbol{B})}{enc} - \frac{\nabla p_e}{en} .$$

Momentum:

$$nm \left( \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} \right) = -\nabla p + \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \frac{m_e}{e^2} (\boldsymbol{J} \cdot \nabla) \left( \frac{\boldsymbol{J}}{n} \right) .$$

# Extended MHD

Ohm's Law:

$$\boldsymbol{E} + \frac{\boldsymbol{V} \times \boldsymbol{B}}{c} = \frac{m_e}{e^2 n} \left( \frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V}) \right) - \frac{m_e}{e^2 n} (\boldsymbol{J} \cdot \nabla) \left( \frac{\boldsymbol{J}}{n} \right) + \frac{(\boldsymbol{J} \times \boldsymbol{B})}{enc} - \frac{\nabla p_e}{en} .$$

Momentum:

$$nm \left( \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} \right) = -\nabla p + \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \frac{m_e}{e^2} (\boldsymbol{J} \cdot \nabla) \left( \frac{\boldsymbol{J}}{n} \right) .$$

# Extended MHD

Ohm's Law:

$$\boldsymbol{E} + \frac{\boldsymbol{V} \times \boldsymbol{B}}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V})\right) - \frac{m_e}{e^2 n}(\boldsymbol{J} \cdot \nabla)\left(\frac{\boldsymbol{J}}{n}\right) + \frac{(\boldsymbol{J} \times \boldsymbol{B})}{enc} - \frac{\nabla p_e}{en}.$$

Momentum:

$$nm\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right) = -\nabla p + \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \frac{m_e}{e^2}(\boldsymbol{J} \cdot \nabla)\left(\frac{\boldsymbol{J}}{n}\right).$$

# Extended MHD (XMHD) Scaled

Ohm's Law:

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \frac{d_e^2}{\rho}\left(\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot \left(\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V} - \frac{d_i}{\rho}\boldsymbol{J}\boldsymbol{J}\right)\right)$$
$$+ \frac{d_i}{\rho}\left(\boldsymbol{J} \times \boldsymbol{B} - \nabla p_e\right).$$

Momentum:

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}$$
$$-d_e^2\, \boldsymbol{J} \cdot \nabla \left(\frac{\boldsymbol{J}}{\rho}\right).$$

Two parameters, $d_e = \frac{c}{\omega_{p_e} L}$ measures electron inertia and $d_i = \frac{c}{\omega_{p_i} L}$ accounts for current carried by electrons mostly ... .

# Energy Conservation

Candidate Hamiltonian:

$$H = \int d^3x \left[ \rho \frac{|\boldsymbol{V}|^2}{2} + \rho U(\rho) + \frac{|\boldsymbol{B}|^2}{2} + d_e^2 \, \frac{|\boldsymbol{J}|^2}{2\rho} \right]$$

Kimura and pjm 2014

$H$ is conserved. Pressure, $p = \rho^2 \partial U / \partial \rho$.

What is the Poisson bracket? Casimirs? Helicities?

# XMHD Hamiltonian Structure

Yoshida, Abdelhamid, Kawazura, pjm, Lingam, Miloshevich

Poisson Bracket:

$$\begin{aligned}\{F,G\}^{XMHD} &= \{F,G\}^{MHD} \\ &+ d_e^2 \int_D d^3x \left[\frac{\nabla\times\mathbf{V}}{\rho}\cdot\Big((\nabla\times F_{\mathbf{B}^\star})\times(\nabla\times G_{\mathbf{B}^\star})\Big)\right] \\ &+ d_i \int_D d^3x\, \frac{\mathbf{B}^\star}{\rho}\cdot\left[(\nabla\times F^\star_{\mathbf{B}})\times(\nabla\times G^\star_{\mathbf{B}})\right]\end{aligned}$$

where we introduce the 'inertial' magnetic field

$$\mathbf{B}^\star = \mathbf{B} + d_e^2\,\nabla\times\left(\frac{\nabla\times\mathbf{B}}{\rho}\right),$$

Hamiltonian:

$$H = \int_D d^3x\left[\frac{\rho|\mathbf{V}|^2}{2} + \rho U(\rho) + \frac{\mathbf{B}\cdot\mathbf{B}^\star}{2}\right].$$

# XMHD Hamiltonian Structure (cont)

Casimirs;

$$C^{\pm}_{XMHD} = \int_D d^3x \; (\mathbf{V} + \lambda_\pm \mathbf{A}^\star) \cdot (\nabla \times \mathbf{V} + \lambda_\pm \mathbf{B}^\star) \; ,$$

where

$$\lambda_\pm = \frac{-d_i \pm \sqrt{d_i^2 + 4d_e^2}}{2d_e^2} \, .$$

Jacobi Identity:

Directly Abdelhamid et al.; remarkable transformations Lingam et al. which lead to **normal fields**.

# Normal Fields

Normal Fields:

$$\boldsymbol{\mathcal{B}}_{\pm} := \mathbf{B} + d_e^2\, \nabla \times \left[\frac{\nabla \times \mathbf{B}}{\rho}\right] + \lambda_{\pm} \nabla \times \mathbf{V}$$

XMHD remarkably yields:

$$\frac{\partial \boldsymbol{\mathcal{B}}_{\pm}}{\partial t} + \pounds_{\boldsymbol{V}_{\pm}} \boldsymbol{\mathcal{B}}_{\pm} = 0 \qquad \leftarrow \qquad \text{Lie dragging of two } 2-\text{forms}$$

Dragging velocities:

$$\mathbf{V}_{\pm} = \mathbf{V} - \lambda_{\mp} \nabla \times \mathbf{B}/\rho$$

Helicities:

$$K_{\pm} = \int \mathsf{A}_{\pm} \wedge d\mathsf{A}_{\pm}\,, \qquad \boldsymbol{\mathcal{B}}_{\pm} = \nabla \times \mathbf{A}_{\pm} \sim d\mathsf{A}_{\pm}$$

# Extended MHD

Ohm's Law:

$$\boldsymbol{E} + \frac{\boldsymbol{V}\times\boldsymbol{B}}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial \boldsymbol{J}}{\partial t} + \nabla\cdot(\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V})\right) - \frac{m_e}{e^2 n}(\boldsymbol{J}\cdot\nabla)\left(\frac{\boldsymbol{J}}{n}\right) + \frac{(\boldsymbol{J}\times\boldsymbol{B})}{enc} - \frac{\nabla p_e}{en}\,.$$

Momentum:

$$nm\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right) = -\nabla p + \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} - \frac{m_e}{e^2}(\boldsymbol{J}\cdot\nabla)\left(\frac{\boldsymbol{J}}{n}\right)\,.$$

# Ohm's Laws

Normal Potentials:

$$\mathbf{E}_{\pm} = -\nabla\phi_{\pm} - \frac{1}{c}\frac{\partial \mathbf{A}_{\pm}}{\partial t}$$

Ohms Laws:

$$\mathbf{E}_{\pm} + \frac{\mathbf{v}_{\pm} \times \mathbf{B}_{\pm}}{c} = \nabla\Upsilon_{\pm}$$

# Maxwell-Vlasov

# Maxwell Part

$$\frac{\partial \mathbf{B}}{\partial t} = -c\,\nabla \times \mathbf{E} \qquad \frac{\partial \mathbf{E}}{\partial t} = c\,\nabla \times \mathbf{B} - 4\pi \boldsymbol{J}_e$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 4\pi \rho_e$$

$$\mathbf{E}(\boldsymbol{x}, t),\ \mathbf{B}(\boldsymbol{x}, t),\ \rho_e(\boldsymbol{x}, t),\ \boldsymbol{J}_e(\boldsymbol{x}, t)$$

# Coupling to Vlasov

$$\frac{\partial f_s}{\partial t} = -\boldsymbol{v}\cdot\nabla f_s - \frac{e_s}{m_s}\left(\mathbf{E} + \frac{\boldsymbol{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}$$

$$\rho_e(\boldsymbol{x},t) = \sum_s e_s \int f_s(\boldsymbol{x},\boldsymbol{v},t)\, d^3v\,, \qquad \boldsymbol{J}_e(\boldsymbol{x},t) = \sum_s e_s \int \boldsymbol{v}\, f_s(\boldsymbol{x},\boldsymbol{v},t)\, d^3v$$

$f_s(\boldsymbol{x},\boldsymbol{v},t)$ is a phase space density for particles of species $s$ with charge and mass, $e_s, m_s$.

An inclusive matter field theory that includes point particles and fluids as exact reductions.

# Maxwell-Vlasov Structure

Hamiltonian:

$$H = \sum_s \frac{m_s}{2}\int |\boldsymbol{v}|^2 f_s\, d^3x\, d^3v + \frac{1}{8\pi}\int (|\mathbf{E}|^2 + |\mathbf{B}|^2)\, d^3x\,,$$

Bracket:

$$\begin{aligned}
\{F,G\} \;=\; & \sum_s \int \Bigg(\frac{1}{m_s} f_s \left(\nabla F_{f_s}\cdot \partial_{\boldsymbol{v}} G_{f_s} - \nabla G_{f_s}\cdot \partial_{\boldsymbol{v}} F_{f_s}\right) \\
& + \frac{e_s}{m_s^2 c} f_s\, \mathbf{B}\cdot \left(\partial_{\boldsymbol{v}} F_{f_s} \times \partial_{\boldsymbol{v}} G_{f_s}\right) \\
+ \;\; & \frac{4\pi e_s}{m_s} f_s \left(G_{\mathbf{E}}\cdot \partial_{\boldsymbol{v}} F_{f_s} - F_{\mathbf{E}}\cdot \partial_{\boldsymbol{v}} G_{f_s}\right)\Bigg) d^3x\, d^3v \\
+ \;\; & 4\pi c \int (F_{\mathbf{E}}\cdot \nabla\times G_{\mathbf{B}} - G_{\mathbf{E}}\cdot \nabla\times F_{\mathbf{B}})\, d^3x\,,
\end{aligned}$$

where $\partial_{\boldsymbol{v}} := \partial/\partial\boldsymbol{v}$, $F_{f_s}$ means functional derivative of $F$ with respect to $f_s$ etc.

pjm**5,9** 1980,1982; Marsden and Weinstein 1982

# Maxwell-Vlasov Structure (cont)

Equations of Motion:

$$\frac{\partial f_s}{\partial t} = \{f_s, H\}\,, \quad \frac{\partial \mathbf{E}}{\partial t} = \{\mathbf{E}, H\}\,, \quad \frac{\partial \mathbf{B}}{\partial t} = \{\mathbf{B}, H\}\,.$$

Casimirs invariants:

$$\begin{aligned}
\mathcal{C}^f_s[f_s] &= \int \mathcal{C}_s(f_s)\, d^3x d^3v \\
\mathcal{C}^E[\mathbf{E}, f_s] &= \int h^{\mathbf{E}}(x) \left( \nabla \cdot \mathbf{E} - 4\pi \sum_s e_s \int f_s\, d^3v \right) d^3x\,, \\
\mathcal{C}^B[\mathbf{B}] &= \int h^{\mathbf{B}}(x)\, \nabla \cdot \mathbf{B}\, d^3x\,,
\end{aligned}$$

where $\mathcal{C}_s$, $h^{\mathbf{E}}$ and $h^{\mathbf{B}}$ are arbitrary functions of their arguments. These satisfy the degeneracy conditions

$$\{F, C\} = 0 \quad \forall F\,.$$

# Reduction

# Whence Noncanonical?

Hamiltonian reduction is a way to reduce the dimension of a system. The process may take canonical to noncanonical or noncanonical to a smaller noncanonical.

For matter models, one can first construct underlying canonical 'particle-like' (Lagrangian variable) description. Then effect Hamiltonian reduction. (Souriau's momentum map).

# Hamiltonian Reduction

**Bracket Reduction:**

Reduced set of variables $(q,p) \mapsto w(q,p)$ $\leftarrow$ noninertible

Bracket Closure:

$$\{w,w\} = c(w) \qquad\qquad f(q,p) = \hat{f} \circ w = \hat{f}(w(q,p))$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket

**Hamiltonian Closure:**

$$H(q,p) = \hat{H}(w)$$

**Reduced dynamics:** $\dot{w} = \{w, \hat{H}\}$

Note $\exists$ symmetry, consequently a group theory interpretation; e.g. Marsden-Weinstein reduction $T^*G/G \cong \mathfrak{g}^*$

# Reduction Example

Simple particle with canonical coordinates: $(\boldsymbol{r}, \boldsymbol{p}) \in \mathbb{R}^6$

Equations of motion:

$$\dot{\boldsymbol{r}} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \text{and} \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{r}}$$

Angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

Reduction:

$$\{L_x, L_y\} = L_z$$

Casimir:

$$\{|\boldsymbol{L}|^2, f\} = 0 \qquad \forall\, f$$

If $H(\boldsymbol{L}) \Rightarrow$ closure, i.e. reduction of system to three dimensions!

# Reduction Examples

- 2D Euler's fluid equations to Kirchhoff point vortices
- 2D Euler's fluid equations to Contour Dynamics. Flierl.
- 2D Euler's fluid equations to Kirchhoff's elliptical vortex patch and Kida's reduction. Infinite to 3-dimensional ODE.
- Riemann self-gravitating ellipsoids. Quasigeostrophy ellipsoids. Lebovitz, Meacham, Flierl, et al.
- Vlasov-Poisson to waterbag fluid model. Perin, Chandre, Tassi, et al.
- ... examples abound

# Canonical Hamiltonian Description of Idea MHD

Lagrangian fluid variable description is a continuum version of particle mechanics and, consequently, canonical, while Eulerian fluid variables are noncanonical variables. Lagrange to Euler is a reduction.

# Usual Eulerian MHD

Momentum:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)=-\nabla p+\boldsymbol{J}\times\boldsymbol{B} \qquad \leftarrow \qquad \boldsymbol{J}=\nabla\times\boldsymbol{B}$$

Mass:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0$$

Ohm's Law:

$$\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}=\eta\boldsymbol{J}\approx 0 \qquad \leftarrow \qquad \text{ideal MHD}$$

Faraday's Law:

$$\frac{\partial \boldsymbol{B}}{\partial t}=-\nabla\times\boldsymbol{E}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

Thermo:

$$\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\nabla s=0 \qquad \text{or barotropic} \rightarrow \quad p=p(\rho)=\kappa\rho^{\gamma}$$

# Eulerian Variable Description

# Eulerian Variable Description

Observables $\{s(r,t), \rho(r,t), \boldsymbol{v}(r,t), \boldsymbol{B}(r,t)\}$ where $r \in D \subset \mathbb{R}^3$.

# Eulerian Variable Description

Observables $\{s(r,t)\rho(r,t), \boldsymbol{v}(r,t), \boldsymbol{B}(r,t)\}$ are fields.

## Lagrangian Variable Description

Assume a continuum of fluid particles or fluid elements and follow them around.

# Lagrangian Variable Description

# Lagrangian Variable Description



Dynamical canonical variables $\{q(a,t), \pi(a,t)\}$.

# Fluid Kinematics

Lagrange *Mécanique Analytique* (1788) $\rightarrow$ Newcomb (1962) MHD

Lagrangian Variables:

Fluid occupies domain $\quad D \subset \mathbb{R}^3$

Fluid particle position $\quad q(a,t), \quad q_t : D \rightarrow D \quad$ bijection, etc.

Particles label: $\quad a \quad$ e.g. $q(a,0) = a$.

Deformation: $\quad \frac{\partial q^i}{\partial a^j} = q^i_{,j}$

Determinant: $\quad \mathcal{J} = \det(q^i_{,j}) \neq 0 \Rightarrow a = q^{-1}(r,t)$

Identity: $\quad q^i_{,k} a^k_{,j} = \delta^i_j$

Volume: $d^3q = \mathcal{J} d^3a$

Area: $(d^2q)_i = \mathcal{J} a^j_{,i} (d^2a)_j$

Line: $(dq)_i = q^i_{,j} (da)_j$

Eulerian Variables:

Observation point: $r$

Velocity field: $\boldsymbol{v}(r,t) =?$ Probe sees $\dot{q}(a,t)$ for some $a$.

What is $a$ ? $r = q(a,t) \quad \Rightarrow \quad a = q^{-1}(r,t)$

$$\boldsymbol{v}(r,t) = \dot{q} \circ q^{-1} = \dot{q}(a,t)\big|_{a=q^{-1}(r,t)}$$

# IDEAL MHD

Attributes and the **Lagrange to Euler Map**:

Velocity (vector field):

$$\boldsymbol{v}(r,t) = \dot{q} \circ q^{-1} = \dot{q}(a,t)\big|_{a=q^{-1}(r,t)}$$

Entropy (1-form):

$$s(r,t) = s_0\big|_{a=q^{-1}(r,t)} \,,$$

Mass (3-form):

$$\rho d^3x = \rho_0 d^3a \quad \Rightarrow \quad \rho(r,t) = \frac{\rho_0}{\mathcal{J}}\bigg|_{a=q^{-1}(r,t)} \,.$$

$B$-Flux (2-form):

$$B \cdot d^2x = B_0 \cdot d^2a \quad \Rightarrow \quad B^i(r,t) = \frac{q^i_{,j} B^j_0}{\mathcal{J}}\bigg|_{a=q^{-1}(r,t)} \,.$$

# Hamiltonian

Kinetic Energy:

$$T[q] = \frac{1}{2}\int_D d^3a\, \rho_0 |\dot{q}|^2 = \frac{1}{2}\int_D d^3a\, |\pi|^2/\rho_0 = \frac{1}{2}\int_D d^3x\, \rho\, |\boldsymbol{v}|^2$$

Potential Energy:

$$\begin{aligned} V[q] &= \int_D d^3a \left(\rho_0 \mathcal{U}(\rho_0/\mathcal{J}, s_0) + \frac{1}{2}\frac{|q^i_{,j} B^j_0|^2}{\mathcal{J}^2}\right) \\ &= \int_D d^3x \left(\rho U(\rho, s) + \frac{1}{2}|\mathbf{B}|^2\right) \end{aligned}$$

Hamiltonian (energy):

$$H[q] = T + V\,,$$

# Hamiltonian Structure – an early field theory

Fréchet Derivative → Variational Derivative:

$$\delta F = \frac{d}{d\epsilon} F[q + \epsilon \delta q]\Big|_{\epsilon=0} = DF \cdot \delta q \qquad \rightarrow \qquad \frac{\delta F}{\delta q}$$

Poisson Bracket:

$$\{F, G\} = \int_D d^3a \left( \frac{\delta F}{\delta q^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta \pi_i} \right)$$

EOM:

$$\dot{q} = \{q, H\} \qquad \dot{\pi} = \{\pi, H\}$$

# Eulerian Reduction

An example of Souriau's momentum map.

$$F[q,\pi] = \widehat{F}[\rho, s, \boldsymbol{v}, \boldsymbol{B}]$$

Chain Rule $\Rightarrow$ noncanonical Eulerian Poisson Bracket:

$$\begin{aligned} \{F,G\}^{MHD} \;=\; & -\int_D d^3x \Bigg\{ [F_\rho \nabla \cdot G_{\boldsymbol{v}} + F_{\boldsymbol{v}} \cdot \nabla G_\rho] \\ & - \left[ \frac{(\nabla \times \boldsymbol{v})}{\rho} \cdot (F_{\boldsymbol{v}} \times G_{\boldsymbol{v}}) \right] \\ & - \left[ \frac{\mathbf{B}}{\rho} \cdot \Big( F_{\boldsymbol{v}} \times (\nabla \times G_{\mathbf{B}}) - G_{\boldsymbol{v}} \times (\nabla \times F_{\mathbf{B}}) \Big) \right] \Bigg\} \end{aligned}$$

where $F_{\boldsymbol{v}} := \delta F / \delta \boldsymbol{v}$ etc. With Hamiltonian

$$H = \int_D d^3x \Big( \rho |\boldsymbol{v}|^2/2 + \rho U(\rho, s) + |\boldsymbol{B}|^2/2 \Big)$$

gives MHD in Eulerian form $\Psi_t = \{\Psi, H\}$.

# Energy-Casimir Stability

An application that originated in plasma physics that I don't have time to cover.

# End Part II