Deformation of Lie-Poisson algebras and chiral non-Hamiltonian dynamics*

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Investigation of strange behavior caused by singularities of Poisson manifolds.

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Usual Symplectic Geometry

Dynamics takes place in phase space, $\mathcal{Z}$ (needn't be T^*Q), a differential manifold endowed with a closed, nondegenerate 2-form ω. A patch has canonical coordinates $z = (q, p)$.

Hamiltonian dynamics $\Leftrightarrow$ flow on symplectic manifold: $i_X\omega = dH$

Poisson tensor (J_c) is Hamiltonian bivector inverse of symplectic 2-form (ω), defining the Poisson bracket

$$\{f, g\} = \langle df, J_c(dg)\rangle = \omega(X_f, X_g) = \frac{\partial f}{\partial z^\alpha} J_c^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,, \quad \alpha, \beta = 1, 2, \ldots 2N$$

Flows generated by Hamiltonian vector fields $Z_H = J_c dH$, H a 0-form, dH a 1-form. Poisson bracket = commutator of Hamiltonian vector fields etc.

Early refs.: Jost, Mackey, Souriau, Arnold, Abraham &Marsden

Noncanonical Hamiltonian Definition

A phase space $\mathcal{P}$ diff. manifold with binary bracket operation on $C^\infty(\mathcal{P})$ functions $f, g\colon \mathcal{P} \to \mathbb{R}$, s.t. $\{\cdot\,,\cdot\}\colon C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ satisfies

- Bilinear: $\{f + \lambda g, h\} = \{f, h\} + \lambda\{g, h\}\,, \quad \forall f, g, h$ and $\lambda \in \mathbb{R}$
- Antisymmetric: $\{f, g\} = -\{g, f\}\,, \qquad \forall f, g$
- Jacobi: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{F, g\}\} \equiv 0\,, \qquad \forall f, g, h$
- Leibniz: $\{fg, h\} = f\{g, h\} + \{f, h\}g\,, \qquad \forall f, g, h.$

Above is a Lie algebra realization on functions. Take fg to be pointwise multiplication.

Eqs. Motion: $\frac{\partial \Psi}{\partial t} = \{\Psi, H\}$, Ψ an observable & H a Hamiltonian.

Example: flows on Poisson manifolds, e.g. Weinstein 1983 ....

Noncanonical Hamiltonian Dynamics

Sophus Lie (1890)

Noncanonical Coordinates:

$$\frac{dz^\alpha}{dt} = J^{\alpha\beta}\frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\}\,, \qquad \{f,g\} = \frac{\partial f}{\partial z^\alpha}J^{\alpha\beta}(z)\frac{\partial g}{\partial z^\beta}\,, \qquad \alpha,\beta = 1, 2, \ldots M$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{f,g\} = -\{g,f\}\,,$

Jacobi identity $\longrightarrow$ $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

$$J \to J_d = \begin{pmatrix} 0_N & I_N & 0 \\ -I_N & 0_N & 0 \\ 0 & 0 & 0_{M-2N} \end{pmatrix}\,.$$

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{M}$ is differentiable manifold with bracket $\{\,,\,\}: C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ st $C^\infty(\mathcal{M})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $Z_H = JdH$.

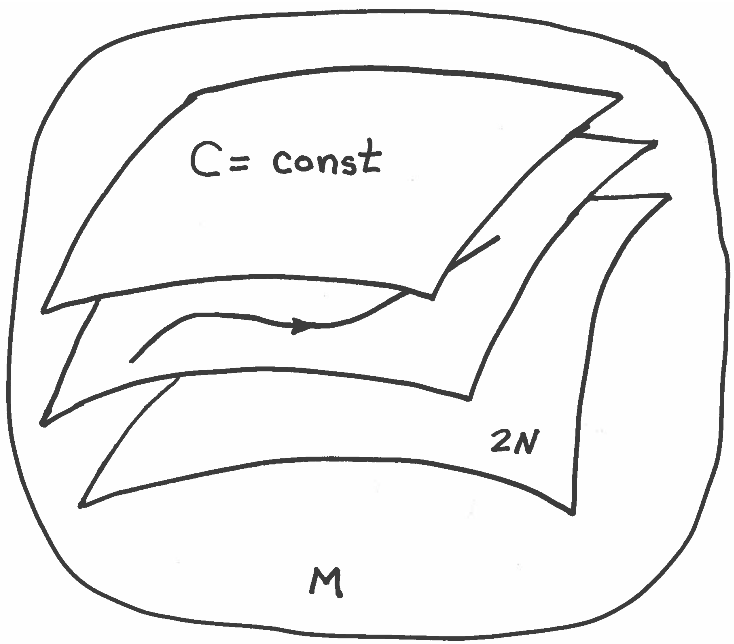
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{M})$. Called Casimir invariants (Lie's distinguished functions.)

Poisson Manifold $\mathcal{M}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{M} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Leaf vector fields, $Z_H = \{z, H\} = JdH$ are tangent to leaves.

Lie-Poisson Brackets

Coordinates:

$$J^{\alpha\beta} = c_{\gamma}^{\alpha\beta} z^{\gamma}$$

where $c_{\gamma}^{\alpha\beta}$ are the structure constants for some Lie algebra.

Examples:

• The 3-dimensional Bianchi algebras for the free rigid body, the Kida vortex, & other?

• Many infinite-dimensional theories - matter models: Ideal fluid flow, MHD, shearflow, extended MHD, Vlasov-Maxwell, BBGKY, plethora of other plasma models, etc.

Lie-Poisson Geometry

Lie Algebra: $\mathfrak{g}$, a vector space with

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\,, \qquad \mathsf{ad}_{\boldsymbol{v}} \cdot = [\boldsymbol{v}, \cdot]$$

antisymmetric, bilinear, satisfies Jacobi identity

Pairing:

$$\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}\,, \qquad \mathsf{ad}^*_{\boldsymbol{v}} \cdot = [\boldsymbol{v}, \cdot]^*$$

with $\mathfrak{g}^*$ vector space dual to $\mathfrak{g}$

Lie-Poisson Bracket:

$$\{f, g\} = \left\langle z, \left[\frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right]\right\rangle\,, \qquad z \in \mathfrak{g}^*,\ \frac{\partial f}{\partial z} \in \mathfrak{g}$$

Dynamics:

$$\frac{dz}{dt} = \mathsf{ad}^*_{\boldsymbol{v}}\, z = \left[\frac{\partial H}{\partial z}, z\right]^*\,, \qquad \boldsymbol{v} = \frac{\partial H}{\partial z} \in \mathfrak{g}$$

Example: Rattleback

Tokieda Moffat system is Hamiltonian,

$$\frac{d}{dt}\begin{pmatrix} P \\ R \\ S \end{pmatrix} = \begin{pmatrix} \alpha PS \\ -RS \\ R^2 - \alpha P^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha P \\ 0 & 0 & -R \\ -\alpha P & R & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial P \\ \partial H/\partial R \\ \partial H/\partial S \end{pmatrix}$$

$z = (P, R, S)$ with P pitch, R roll, and S spin.

$$H = \frac{1}{2}(P^2 + R^2 + S^2)\,, \qquad C = PR^\alpha$$

where paramter α is aspect ratio.

Pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ yields the Lie-Poisson bracket:

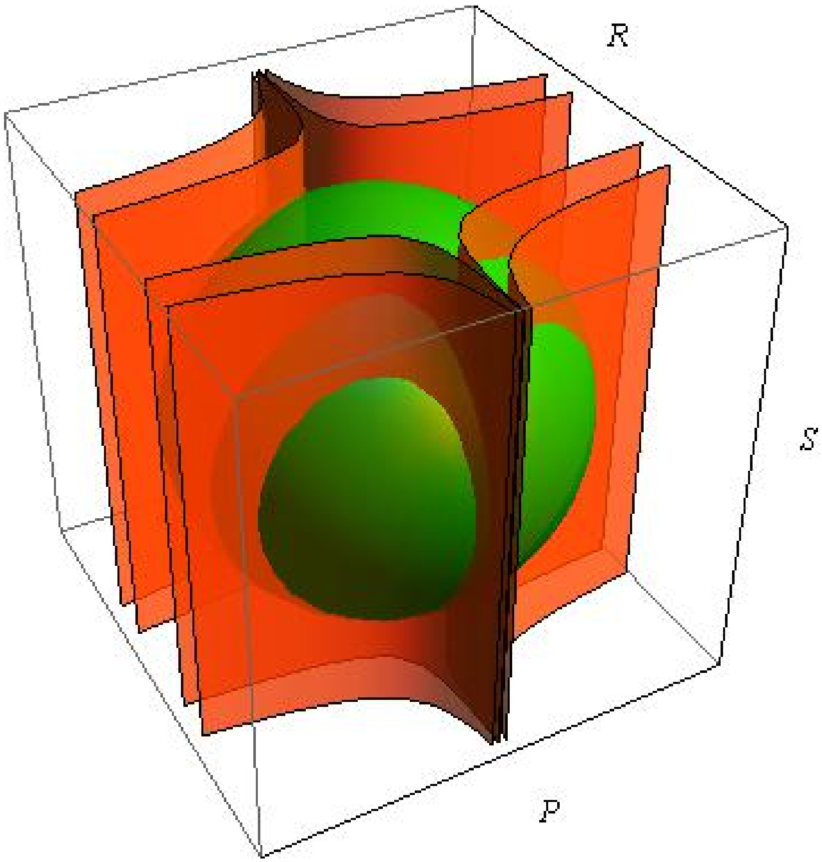
$$\{f, g\} = c_\alpha^{\beta\gamma}\, z^\alpha\, \frac{\partial f}{\partial z^\beta}\frac{\partial g}{\partial z^\gamma}\,,$$

where $c_\alpha^{\beta\gamma}$ are the structure constants for Bianchi Type $\mathrm{VI}_{h<-1}$.

Equilibrium S_e has non-Hamiltonian spectrum: $(0, \alpha S_e, -S_e)$

Rattleback Orbits (All real 3D Lie-Poisson systems)

Orbits lie on intersection of Casimir leaves and energy surface. Singular equilibrium is at $(R = P = 0, S \neq 0)$.

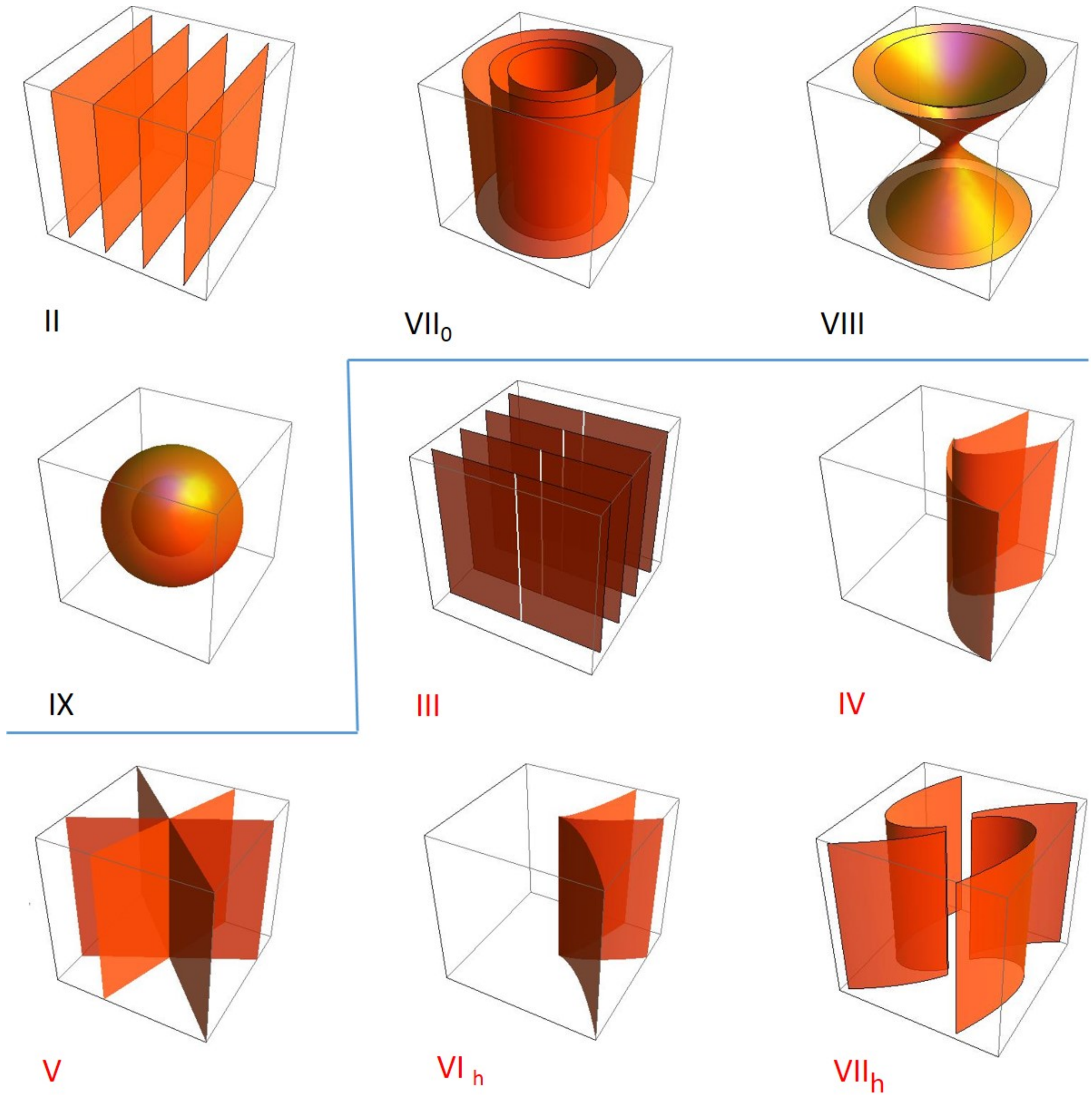


All Real 3D Lie-Poisson Structures

Bianchi classification (cf. Jacobson) of real Lie algebras

$$c^{\alpha}_{\beta\gamma} = \epsilon_{\beta\gamma\delta}\, m^{\delta\alpha} + \delta^{\alpha}_{k}\, a_{\beta} - \delta^{\alpha}_{\beta}\, a_{\gamma}\,, \qquad \alpha, \beta, \gamma = 1, 2, 3$$

Class	Type	m	a_α
A	I	0	0
A	II	diag$(1, 0, 0)$	0
A	VI_{-1}	$-\alpha$	0
A	VII_0	diag$(-1, -1, 0)$	0
A	VIII	diag$(-1, 1, 1)$	0
A	IX	diag$(1, 1, 1)$	0
B	III	$-\frac{1}{2}\alpha$	$-\frac{1}{2}\delta^{\alpha}_{3}$
B	IV	diag$(1, 0, 0)$	$-\delta^{\alpha}_{3}$
B	V	0	$-\delta^{\alpha}_{3}$
B	$\mathrm{VI}_{h\neq-1}$	$\frac{1}{2}(h-1)\alpha$	$-\frac{1}{2}(h+1)\delta^{\alpha}_{3}$
B	$\mathrm{VII}_{h=0}$	diag$(-1, -1, 0) + \frac{1}{2}h\alpha$	$-\frac{1}{2}h\delta^{\alpha}_{3}$


II
VII0
VIII
IX
III
IV
V
VI h
VIIh

Division of Real 3D Lie-Poisson Structures

Class A:

Type IX – Free rigid body, spin, ...

Type II – Heisenberg algebra

Type $VIII$ – Kida vortex of fluid mechanics

Class B: ?

Type $\mathrm{VI}_{h<-1}$ – Rattleback

Other B – ?

Casimir surfaces may not be algebraic varieties

Properties of 3D Lie-Poisson Structures

• Type $\mathrm{VI}_{h<-1}$ governs rattleback system of Moffat and Tokieda.

• Chirality comes from equilibria that live on the singular set.

• Such equilibria need not have Hamiltonian spectra.

Yoshida, Tokieda and pjm, Phys. Lett. A **381**, 2772 (2017)

• Rank changing is responsible for the Casimir deficit problem.

Relationship to b-symplectic and presymplectic systems.

Regular and Singular Equilibria

Let $z = z_e + \tilde{z}$, $F = H + C$, and expand

$$\frac{d\tilde{z}}{dt} = J(z_e)F''(z_e)\tilde{z} + J(\tilde{z})h(z_e)$$

where

$$h(z_e) := \left.\frac{\partial F}{\partial z}\right|_{z=z_e} \in \mathfrak{g}\,, \qquad \left(F''(z_e)_{jk}\right) := \left.\frac{\partial^2 F}{\partial z^k \partial z^j}\right|_{z=z_e} \in \mathrm{Hom}(\mathfrak{g}*, \mathfrak{g})$$

Regular Equlibria $h(z_e) = 0$:

$$\frac{dz}{dt} = J(z_e)\partial_z H_L \qquad H_L = \frac{1}{2}\langle F''(z_e)\tilde{z}, \tilde{z}\rangle$$

Singular Equlibria $J(z_e)$:

$$\frac{dz}{dt} = J(\tilde{z})h(z_e) = [h(z_e), \tilde{z}]^*$$

Question: When is $[h(z_e), \cdot\,]^*$ a Hamiltonian matrix? $J\mathcal{H}$

Lie Algebra Deformation

Modified Observables:

$$\{G,H\}_M = \langle [\partial_z G, \partial_z H], Mz\rangle = \langle \partial_z G, [\partial_z, Mz]^*\rangle\,, \qquad M \in \mathsf{End}(\mathfrak{g}^*)$$

Modified Bivector and Bracket:

$$J_M(z) = J(Mz)$$

$$\{G,H\}_M = \langle M^T[\partial_z G, \partial_z H], z\rangle = \langle [\partial_z G, \partial_z H]_M, z\rangle\,, \qquad M \in \mathsf{End}(\mathfrak{g})$$

Central Question: Is the Jacobi Identity satisfied?

Main Theorem for 3D

Theorem 1 (deformation of $\mathfrak{so}(3)$) *Every 3-dimensional real Lie bracket can be written as* $[\ ,\ ]_M = M^{\mathsf{T}}[\ ,\ ]_{\mathsf{XI}}$ *with* $M \in \mathrm{End}(\mathbb{R}^3)$ *which is chosen from the following two classes:*

1. *class A:* M *is an arbitrary symmetric* 3×3 *matrix.*
2. *class B:* $M = N \oplus 0$ *(*N *is an arbitrary asymmetric* 2×2 *matrix).*

Accordingly, we have a unified representation of all 3-dimensional Lie-Poisson brackets:

$$\{G, H\}_M = \langle [\partial_{\boldsymbol{\xi}} G, \partial_{\boldsymbol{\xi}} H]_M, \boldsymbol{\xi} \rangle = \langle [\partial_{\boldsymbol{\xi}} G, \partial_{\boldsymbol{\xi}} H]_{\mathrm{IX}}, M\boldsymbol{\xi} \rangle. \qquad (1)$$

The corresponding Poisson operator is

$$J_M(\boldsymbol{\xi}) \cdot = J_{\mathrm{IX}}(M\boldsymbol{\xi}) \cdot = [\,\cdot\,, M\boldsymbol{\xi}]^*_{\mathrm{IX}} = -(M\boldsymbol{\xi}) \times \cdot\,. \qquad (2)$$

The singularity (where the rank of the Poisson operator becomes zero) is

$$\sigma = \mathrm{Ker}\, M.$$

Higher Dimensions

Definition 1 (classification into A, B and C) *Let $\mathfrak{g}$ be an n-dimensional real Lie algebra.*

- *If $\mathfrak{g}$ is fully antisymmetric (i.e. the Lie bracket is given by fully antisymmetric structure constants), or it is the deformation of some fully antisymmetric Lie algebra by a symmetric matrix $M \in \mathrm{End}(\mathbb{R}^n)$, we say that $\mathfrak{g}$ is class A.*

- *If $\mathfrak{g}$ is the deformation of some fully antisymmetric Lie algebra by an asymmetric matrix $M \in \mathrm{End}(\mathbb{R}^n)$, we say that $\mathfrak{g}$ is class B.*

- *If $\mathfrak{g}$ is neither class A nor class B, we say that $\mathfrak{g}$ is class C.*

Higher Dimensions

Theorem 2 (Hamiltonian spectral symmetry) *Suppose that* $\mathfrak{g}_M$ *is a real* n*-dimensional class-A Lie algebra endowed with a Lie bracket* $[\ ,\ ]_M = M[\ ,\ ]_{\mathsf{AS}}$*, where* $[\ ,\ ]_{\mathsf{AS}}$ *is a fully antisymmetric Lie bracket, and* $M \in \mathrm{End}(\mathbb{R}^n)$ *is a symmetric matrix. Then, the linearized generator*

$$\mathcal{A} = -[\boldsymbol{h}, M\cdot]^*_{\mathsf{AS}} \quad (\boldsymbol{h} \in \mathfrak{g}_M)$$

has Hamiltonian symmetric spectra. On the other hand, the linearization of a class-B or class-C system has chiral (non-Hamiltonian) spectra.

Other Material

- 4d Lie Algebras: 24 Real Lie algebras, 10 non-composites, none semi-simple/compact. We examined all of them.

- Infinite Dimensions: Working on fluid and plasma field theories.

Yoshida and pjm: arXiv:2001.03744v1 [math-ph] 11 Jan 2020