Hamiltonian and Metriplectic Descriptions of Matter

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Overview

I. Hamiltonian Dynamics

II. Metriplectic Dynamics

I. Hamiltonian Dynamics

Classical Field Theory for Classical Purposes

Dynamics of matter described by

- **Fluid models**
  - Euler's equations, Navier-Stokes, ...
- **Magnetofluid models**
  - MHD, XMHD (Hall, electron mass physics), 2-fluid, ...
- **Kinetic theories**
  - Vlasov-Maxwell, Landau-Lenard-Balescu, gyrokinetics, ...
- **Fluid-Kinetic hybrids**
  - MHD + hot particle kinetics, gyrokinetics, ...

Applications:

atmospheres, oceans, fluidics, natural and laboratory plasmas

Classical Field Theories for Classical Purposes Have Common Structure

Two Dichotomies:

- **Lagrangian vs. Eulerian variables**
  - particle or material vs. spatial or observable
- **Lagrangian vs. Hamiltonian formalisms**
  - Action principle vs. Poisson bracket

Basic procedure of **reduction**:

action principle $\rightarrow$ Hamiltonian $\rightarrow$ noncanonical Poisson bracket

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Plasma Parent Model as Example

Relativistic N-Particle Action

Dynamical Variables: $\quad q_i(t), \phi(x,t), A(x,t)$

$$S[q,\phi,A] \;=\; -\sum_{i=1}^{N}\int dt\;\, mc^2\sqrt{1-\frac{|\dot{q}_i^2|}{c^2}} \qquad \longleftarrow \text{ ptle kinetic energy}$$

$$\text{coupling} \longrightarrow \qquad -e\int dt\sum_{i=1}^{N}\int d^3x\left[\phi(x,t)+\frac{\dot{q}_i}{c}\cdot A(x,t)\right]\delta\left(x-q_i(t)\right)$$

$$\text{field `energy'} \longrightarrow \qquad +\frac{1}{8\pi}\int dt\int d^3x\left[|E|^2(x,t)-|B|^2(x,t)\right].$$

Variation:

$$\frac{\delta S}{\delta q^i(t)} = 0 \quad \Longrightarrow \quad \text{Newton's 2nd \& Fields},$$

$$\frac{\delta S}{\delta \phi(x,t)} = 0\,, \quad \frac{\delta S}{\delta A(x,t)} = 0 \quad \Longrightarrow \quad \text{Maxwell eqs. \& Sources}$$

Too Much Information

Reductions, Approximations, Mutilations, . . . :

$\Rightarrow$ Constraints (explicit or implicit) $\Rightarrow$ Interesting!

Finite Systems

B-lines, ptle orbits, self-consistent models, . . .

Infinite Systems

kinetic theories, fluid models, mixed . . .

Usually Eulerian (spacial) variable field theories

Continuum Action – Particle to Field Theory

Dynamical Variables: $\quad q(z_0,t), \phi(x,t), A(x,t)$

Particles to Fields: $\quad i \to z_0, \quad q_i \to q(z_0,t),$ and $\sum_{i=1}^N \to \int dz_0$

$$S[q,\phi,A] = \int dt \int d^6 z_0 \, f_0(z_0) \, \frac{m}{2} |\dot{q}|^2(z_0,t)$$
$$-e \int dt \int d^6 z_0 \, f_0(z_0) \int d^3x \, \left[\phi(x,t) + \frac{\dot{q}}{c} \cdot A(x,t) \right] \delta\Big(x - q(z_0,t)\Big)$$
$$+ \frac{1}{8\pi} \int dt \int d^3x \, \Big(|E|^2(x,t) - |B|^2(x,t) \Big) \, .$$

Continuum Low-Like Actions: Kinetic Theories, Guiding Center/Gyro Kinetic Theories, Fluid Theories, . . .

Canonical Hamiltonian Field Theory

Legendre Transform: $\{(q, \pi), (E, A)\}$ $\leftarrow$ canonical conjugates

Canonical Poisson Bracket:

$$\{F, G\} = \int d^6 z_0 \left(\frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \pi} \right) + \int d^3 x \left(\frac{\delta F}{\delta E} \cdot \frac{\delta G}{\delta A} - \frac{\delta G}{\delta E} \cdot \frac{\delta F}{\delta A} \right)$$

Equations of Motion:

$$\frac{\partial q}{\partial t} = \{q, H\} = \frac{\delta H}{\delta \pi} \quad \text{and} \quad \frac{\partial \pi}{\partial t} = \{\pi, H\} = -\frac{\delta H}{\delta q}$$

$$\frac{\partial E}{\partial t} = \{E, H\} = \frac{\delta H}{\delta A} \quad \text{and} \quad \frac{\partial A}{\partial t} = \{\pi, H\} = -\frac{\delta H}{\delta E}$$

Here H the Hamiltonian functional, $\delta H / \delta q$ the functional derivative, z_0 the particle label, x the electromagnetic field label, ...

Reduction Field Theory Example

We will see the map

$$\{q, \pi, E, A\} \ \rightarrow \{f(x, v, t), E(x, t), B(x, t)\}$$

gives a gauge-free field theory Hamiltonian theory in terms of noncanonical Poisson bracket.

But, first consider how it works in general in finite dimensions.

Hamiltonian Reduction:
Canonical to Noncanonical Poisson Brackets

Hamiltonian reduction is a way to reduce the dimension of a system. The process may take canonical to noncanonical or noncanonical to a smaller noncanonical.

For matter models, one can first construct underlying canonical 'particle-like' (Lagrangian variable) description. Then effect <u>Hamiltonian reduction.</u> (Souriau's momentum map).

Hamiltonian Reduction

Bracket Reduction:

Reduced set of variables $(q,p) \mapsto w(q,p)$ $\leftarrow$ noninertible

Bracket Closure:

$$\{w,w\} = c(w) \qquad\qquad f(q,p) = \hat{f} \circ w = \hat{f}(w(q,p))$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket

Hamiltonian Closure:

$$H(q,p) = \widehat{H}(w)$$

Note $\exists$ symmetry, consequently a group theory interpretation ...

Reduced dynamics: $\dot{w} = \{w, \widehat{H}\}$

Angular Momentum Example

Simple particle with canonical coordinates: $(\boldsymbol{r}, \boldsymbol{p})$

Equations of motion:

$$\dot{\boldsymbol{r}} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \text{and} \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{r}}$$

Angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

Reduction:

$$\{L_x, L_y\} = L_z$$

Casimir:

$$\{|\boldsymbol{L}|^2, f\} = 0 \qquad \forall\, f$$

If $H(\boldsymbol{L}) \Rightarrow$ closure, i.e. reduction of system to three dimensions!

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980)....

Noncanonical Coordinates:

$$\dot{w}^i = J^{ij}\frac{\partial H}{\partial w^j} = \{w^j, H\}\,, \qquad\qquad \{A,B\} = \frac{\partial A}{\partial w^i}J^{ij}(w)\frac{\partial B}{\partial w^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{A,B\} = -\{B,A\}\,,$

Jacobi identity $\longrightarrow$ $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus <u>Casimirs</u>

Matter models in Eulerian variables: $J^{ij} = c_k^{ij}w^k \leftarrow \mathrm{Lie-Poisson\ Brackets}$

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\,,\,\}: C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

i) bilinear,
ii) antisymmetric,
iii) Jacobi, and
iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

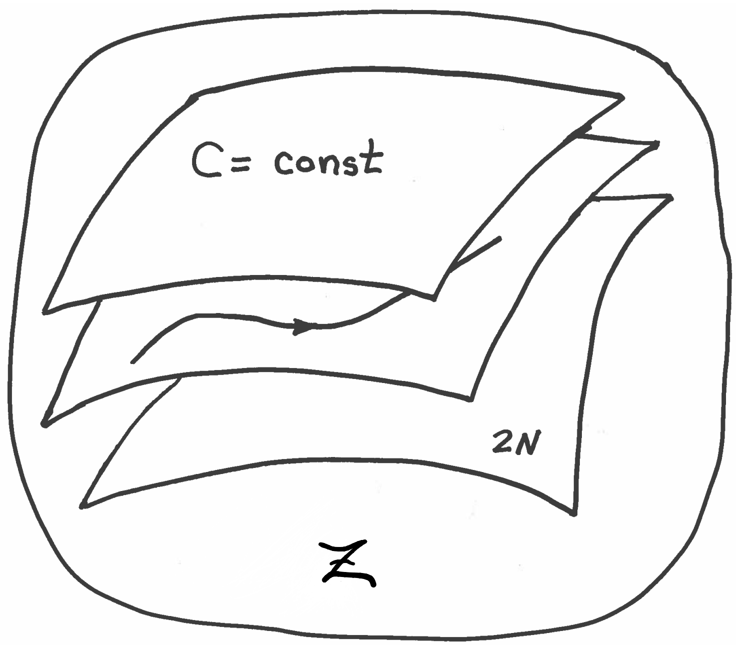
Because of degeneracy, $\exists$ functions C st $\{A, C\} = 0$ for all $A \in C^\infty(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{A, C\} = 0 \quad \forall\ A : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Lie Poisson Flows

$\mathfrak{g}$ Lie algebra; basis $\{E_1, E_2, \ldots, E_n\}$; structure constants c_{ij}^k, i.e., $[E_i, E_j] = c_{ij}^k E_k$;

Dual $\mathfrak{g}^*$; dual basis $\{E_*^1, E_*^2, \ldots, E_*^n\}$; $\left\langle E_*^i, E_j \right\rangle = \delta_j^i$; standard pairing $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

Smooth $A \colon \mathfrak{g}^* \to \mathbb{R}$ has derivative $DA(\mu) \in \mathfrak{g}$ at $\mu \in \mathfrak{g}^*$ for any $\delta\mu \in \mathfrak{g}^*$,

$$\langle \delta\mu, DA(\mu) \rangle = \frac{d}{ds} A(\mu + s\delta\mu)\Big|_{s=0} \qquad \Rightarrow \qquad DA(\mu) = \frac{\partial A}{\partial \mu_i}(\mu)\, E_i.$$

Lie-Poisson bracket on $\mathfrak{g}^*$, for all $A, B \colon \mathfrak{g}^* \to \mathbb{R}$,

$$\{A, B\}_{LP} := \langle \mu, [DA, DB] \rangle = \mu_k c_{ij}^k \frac{\partial A}{\partial \mu_i} \frac{\partial B}{\partial \mu_j}.$$

Dynamics with Hamiltonian $H \colon \mathfrak{g}^* \to \mathbb{R}$

$$\dot{\mu}_i = \{\mu_i, H\}_{LP} = \mu_k c_{ij}^k \frac{\partial H}{\partial \mu_j} \qquad \Leftrightarrow \qquad \dot{\mu} = -\,\mathrm{ad}_{DH}^* \mu$$

Maxwell-Vlasov Reduction

Under the map

$$\{q, \pi, E, B\} \;\rightarrow \{f(x, v, t), E(x, t), B(x, t)\}$$

the canonical Poisson bracket

$$\{F, G\} = \int d^6 z_0 \left(\frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \pi} \right) + \int d^3 x \left(\frac{\delta F}{\delta E} \cdot \frac{\delta G}{\delta A} - \frac{\delta G}{\delta E} \cdot \frac{\delta F}{\delta A} \right)$$

gives $\rightarrow$

Maxwell-Vlasov Poisson Bracket

Hamiltonian:

$$H = \sum_s \frac{m_s}{2}\int |\boldsymbol{v}|^2 f_s\, d^3x\, d^3v + \frac{1}{8\pi}\int (|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)\, d^3x\,,$$

Bracket:

$$\begin{aligned}
\{F,G\} \;=\; & \sum_s \int \Big(\frac{1}{m_s} f_s \left(\nabla F_{f_s}\cdot\partial_{\boldsymbol{v}} G_{f_s} - \nabla G_{f_s}\cdot\partial_{\boldsymbol{v}} F_{f_s}\right) \\
& + \frac{e_s}{m_s^2 c} f_s\, \boldsymbol{B}\cdot\left(\partial_{\boldsymbol{v}} F_{f_s}\times\partial_{\boldsymbol{v}} G_{f_s}\right) \\
+ \;& \frac{4\pi e_s}{m_s} f_s \left(G_{\boldsymbol{E}}\cdot\partial_{\boldsymbol{v}} F_{f_s} - F_{\boldsymbol{E}}\cdot\partial_{\boldsymbol{v}} G_{f_s}\right)\Big) d^3x\, d^3v \\
+ \;& 4\pi c \int \left(F_{\boldsymbol{E}}\cdot\nabla\times G_{\boldsymbol{B}} - G_{\boldsymbol{E}}\cdot\nabla\times F_{\boldsymbol{B}}\right) d^3x\,,
\end{aligned}$$

where $\partial_{\boldsymbol{v}} := \partial/\partial\boldsymbol{v}$, F_{f_s} means functional derivative of F with respect to f_s etc.

pjm 1980,1982; Marsden and Weinstein 1982

Maxwell-Vlasov Equations and Casimirs

Equations of Motion:

$$\frac{\partial f_s}{\partial t} = \{f_s, H\}\,, \quad \frac{\partial \boldsymbol{E}}{\partial t} = \{\boldsymbol{E}, H\}\,, \quad \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}\,.$$

Casimirs invariants:

$$\begin{aligned}
\mathcal{C}^f_s[f_s] &= \int \mathcal{C}_s(f_s)\, d^3x d^3v \\
\mathcal{C}^E[\boldsymbol{E}, f_s] &= \int h^{\boldsymbol{E}}(x) \left(\nabla \cdot \boldsymbol{E} - 4\pi \sum_s e_s \int f_s\, d^3v \right) d^3x\,, \\
\mathcal{C}^B[\boldsymbol{B}] &= \int h^{\boldsymbol{B}}(x)\, \nabla \cdot \boldsymbol{B}\, d^3x\,,
\end{aligned}$$

where $\mathcal{C}_s$, $h^{\boldsymbol{E}}$ and $h^{\boldsymbol{B}}$ are arbitrary functions of their arguments. These satisfy the degeneracy conditions

$$\{F, C\} = 0 \quad \forall F\,.$$

Main Conclusion

Equations for the dynamics of (dissipation free) matter are naturally given in terms of noncanonical Poisson brackets of Lie-Poisson form. When coupled to a gauge field like electromagnetism, there will also be a canonical component.

All good models have this form!

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II. Metriplectic Dynamics

An encompassing formulation that combines Hamiltonian dynamics with dissipation, consistent with thermodynamical Laws.

pjm 1984; Kaufman almost: Grmela renamed Generic

Overview

1. Other attempts
   (a) Rayleigh Dissipation Function
   (b) Cahn-Hilliard Equation
2. Metriplectic Dynamics
   (a) gradient flows
   (b) Hamiltonian flows
   (c) metriplectic flows
3. Geometrical Aspects

Other Attempts

Rayleigh Dissipation Function

Introduced for study of vibrations, stable linear oscillations, in 1873 (see e.g. Rayleigh, Theory of Sound, Chap. IV §81)

Linear friction law for n-bodies, $\mathbf{F}_i = -b_i(\mathbf{r}_i)\mathbf{v}_i$, with $\mathbf{r}_i \in \mathbb{R}^3$. Rayleigh was interested in linear vibrations, $\mathcal{F} = \sum_i b_i \, ||\mathbf{v}_i||^2/2$.

Coordinates $\mathbf{r}_i \to q_\nu$ etc. $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$$

Ad hoc, phenomenological, yet is generalizable, geometrizable (e.g. Bloch et al.,...)

Cahn-Hilliard Equation

Models phase separation, nonlinear diffusive dissipation, in binary fluid with 'concentrations' n, $n = 1$ one kind $n = -1$ the other

$$\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 \left(n^3 - n - \nabla^2 n \right)$$

Lyapunov Functional

$$F[n] = \int d^3x \left[\frac{1}{4} \left(n^2 - 1 \right)^2 + \frac{1}{2} |\nabla n|^2 \right]$$

$$\frac{dF}{dt} = \int d^3x \frac{\delta F}{\delta n} \frac{\partial n}{\partial t} = \int d^3x \frac{\delta F}{\delta n} \nabla^2 \frac{\delta F}{\delta n} = -\int d^3x \left| \nabla \frac{\delta F}{\delta n} \right|^2 \leq 0$$

For example in 1D

$$\lim_{t \to \infty} n(x,t) = \tanh(x/\sqrt{2})$$

Ad hoc, phenomenological, yet generalizable and very important (Otto, Ricci Flows, Poincarè conjecture on S^3, ...)

Metriplectic Dynamics

A dynamical model of thermodynamics that ‘captures’:.

- First Law: conservation of energy

- Second Law: entropy production

Example – Transport Equation

$$\frac{\partial f}{\partial t} = -v \cdot \nabla f - a \cdot \nabla_v f + \left.\frac{\partial f}{\partial t}\right)_c$$

where

$$\text{Collision term} \quad \rightarrow \quad \left.\frac{\partial f}{\partial t}\right)_c$$

could be Boltzmann, Landau, etc.

Conserves, mass, momentum, energy,

$$\frac{dH}{dt} = \frac{d}{dt}\int \frac{1}{2}mv^2 f + \text{interaction} = 0$$

and makes entropy

$$\frac{dS}{dt} = -\frac{d}{dt}\int f \ln(f) \geq 0$$

Vlasov Kinetic Theory

Noncanonical Poisson Brackets:

$$\{F, G\} = \int dxdv\, f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] = -\int dxdv\, \frac{\delta F}{\delta f} [f, \cdot] \frac{\delta G}{\delta f}$$

f = distribution fn, $\mathcal{E} = v^2/2 - \phi(f; x) = \delta H/\delta f$ = particle energy

$$[f, g] = f_x g_v - f_v g_x$$

Hamiltonian:

$$H[f] = \frac{1}{2}\int dxdv\, v^2 + \frac{1}{2}\int dx\, |\nabla\phi|^2$$

Equation of Motion:

$$f_t = \{f, H\}$$

PJM (1980)

Metripletic Flows

- Casimirs of $\{\,,\,\}$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator: $\mathcal{F} = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = \{H, \mathcal{F}\} + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0$$

  Foliate P by level sets of H i.e. $(H, F) = 0 \; \forall \; F \in C^{\infty}(P)$.

- 2nd Law: entropy production

$$\dot{S} = \{S, \mathcal{F}\} + (S, \mathcal{F}) = (S, S) \geq 0$$

  Lyapunov relaxation to equilbrium: i.e., dynamics effects the variational principle: $\delta \mathcal{F} = 0$.

Examples

- Finite dimensional theories, rigid body, etc.

- Kinetic theories: Boltzmann equation, Landau-Lenard-Balescu equation, ...

- Fluid flows: various nonideal fluids, etc.

- Magnetofluid flows, MHD, XMHD, gyrofluids, etc.

Collision Operator

Two counting dichotomies:

- Exclusion vs. Nonexclusion
- Distinguishability vs. Indistinguishability

$\Rightarrow$ 4 possibilities

$$\begin{aligned}
\text{Indistinguishable} + \text{Exclusion} &\rightarrow \text{Fermi} - \text{Dirac} \\
\text{Indistinguishable} + \text{Nonexclusion} &\rightarrow \text{Bose} - \text{Einstein} \\
\text{Distinguishable} + \text{Nonexclusion} &\rightarrow \text{Maxwell} - \text{Boltzmann} \\
\text{Distinguishable} + \text{Exclusion} &\rightarrow \text{Lynden} - \text{Bell}^*
\end{aligned}$$

* Lynden-Bell (1967) proposed this for stars which are distinguishable.

Collision Operator

Kadomstev and Pogutse (1970) collision operator
with formal H-theorm to F-D ?

Metriplectic formalism $\rightarrow$ can do for *any* monotonic distribution

$$
\begin{aligned}
(A,B) \;=\; &\int dz \int dz' \left[\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta A}{\delta f(z')}\right] T_{ij}(z,z') \\
&\times \left[\frac{\partial}{\partial v_j}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta B}{\delta f(z')}\right]
\end{aligned}
$$

$$T_{ij}(z,z') = w_{ij}(z,z')M(f(z))M(f(z')/2$$

Conservation and Lyapunov:

$w_{ij}(z,z') = w_{ji}(z,z') \quad w_{ij}(z,z') = w_{ij}(z',z) \quad g_i w_{ij} = 0$ with $g_i = v_i - v_i'$

'Entropy' Compatibility:

$$S[f] = \int dz\, s(f) \qquad \Rightarrow \qquad M\frac{d^2 s}{df^2} = 1$$

Collision Operator Examples

Landau kernel:

$$w_{ij}^{(L)} = (\delta_{ij} - g_i g_j / g^2)\delta(\mathbf{x} - \mathbf{x}')/g$$

Landau Entropy Compatibility

$$S[f] = \int dz\, f \ln f \qquad \Rightarrow \qquad M \frac{d^2 s}{df^2} = 1 \Rightarrow M = f$$

Lynden-Bell Entropy Compatibility

$$S[f] = \int dz\, s(f) \qquad \Rightarrow \qquad M \frac{d^2 s}{df^2} = 1 \Rightarrow M = f(1 - f)$$

General Form

$$(F, G) = \int d^n z \int d^n z' \, \mathcal{L}' \left(\frac{\delta F}{\delta \chi} \right) \cdot g(z, z'; \chi) \, \cdot \mathcal{L} \left(\frac{\delta G}{\delta \chi} \right)$$

$\mathcal{L}$ a formally self-adjoint pseudo-differential operator, g a symmetric operator, $z = (z^1, \ldots, z^n)$, and $\chi = \chi^1, \ldots \chi^m)$.

Degeneracies can appear from kernel of $\mathcal{L}$ and g

Geometrical Aspects

Bloch, PJM, Ratiu 2013

Geometical Definition

A *metriplectic system* consists of a smooth manifold P, two smooth vector bundle maps $\pi, \gamma : T^*P \to TP$ covering the identity, and two functions $H, S \in C^\infty(P)$, the *Hamiltonian* and the *entropy* of the system, such that

(i) $\{F, G\} := \langle \mathbf{d}F, \pi(\mathbf{d}G)\rangle$ is a Poisson bracket; $\pi^* = -\pi$;

(ii) $(F, G) := \langle \mathbf{d}F, \gamma(\mathbf{d}G)\rangle$ is a positive semidefinite symmetric bracket, i.e., $(\,,\,)$ is $\mathbb{R}$-bilinear and symmetric, so $\gamma^* = \gamma$, and $(F, F) \geq 0$ for every $F \in C^\infty(P)$;

(iii) $\{S, F\} = 0$ and $(H, F) = 0$ for all $F \in C^\infty(P)$
$$\iff \pi(\mathbf{d}S) = \gamma(\mathbf{d}H) = 0.$$

The Flow

The *metriplectic dynamics* of the system is given in terms of the two brackets by

$$\begin{aligned} \frac{dF}{dt} &= \{F, H+S\} + (F, H+S) \qquad (1) \\ &= \{F, H\} + (F, S), \qquad \forall\ F \in C^{\infty}(P), \end{aligned}$$

or, equivalently, as an ordinary differential equation, by

$$\frac{dz(t)}{dt} = \pi(z(t))\mathbf{d}H(z(t)) + \gamma(z(t))\mathbf{d}S(z(t)). \qquad (2)$$

The Hamiltonian vector field $X_H := \pi(\mathbf{d}H) \in \mathfrak{X}(P)$ represents the *Hamiltonian part*, whereas $Y_S := \gamma(\mathbf{d}S) \in \mathfrak{X}(P)$ the *dissipative part* of the full metriplectic dynamics (1) or (2).

General Construction

Suppose manifold P has both Riemannian and Symplectic structure: Given two vector fields $Z_{1,2} \in \mathfrak{X}(P)$ the following is defined:

$$\mathbf{g}(Z_1, Z_2) : \mathfrak{X}(P) \times \mathfrak{X}(P) \to \mathbb{R}$$

If the two vector fields are Hamiltonian, say Z_F, Z_G, then we have the bracket

$$(F, G) = \mathbf{g}(Z_F, Z_G)$$

which produces a 'relaxing' gradient flow. Such flows exist for Kähler manifolds. If P is a Poisson manifold with Casimir C, then $(F, C) \equiv 0 \;\; \forall\, F$.

Summary

• The noncanonical Lie-Poisson bracket description is natural for describing classical field theories intended for classical purposes.

• Metriplectic dynamics serves as a normal form for dissipation, one that gives a dynamical version of the first and second laws of thermodynamics