On an inclusive curvature-like framework for describing dissipation: metriplectic 4-bracket dynamics

Philip J. Morrison
Department of Physics
Institute for Fusion Studies, and ODEN Institute
The University of Texas at Austin
morrison@physics.utexas.edu
http://www.ph.utexas.edu/~morrison/

IWoSP, Chile (online)
December 15, 2023


Geometry of metriplectic 4-brackets: with Michael Updike


→ Theory of thermodynamically consistent theories!
Thermodynamic Consistency – Examples

Navier-Stokes is inconsistent:

\[ \partial_t v = -v \cdot \nabla v - \frac{1}{\rho_0} \nabla p + \nu \nabla^2 v, \quad \nabla \cdot v = 0 \Rightarrow p[v] \]

\[ H = \int_\Omega \rho_0 |v|^2/2 \quad \text{and} \quad \dot{H} \leq 0 \]

Thermodynamic Navier-Stokes is consistent (Eckart 1940):

\[ \partial_t v = -v \cdot \nabla v - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot T \quad \leftarrow T \text{ viscous stress tensor} \]

\[ \partial_t \rho = -\nabla \cdot (\rho v) \]

\[ \partial_t s = -v \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot q + \frac{1}{\rho T} T : \nabla v, \quad q = -\kappa \nabla T \quad \leftarrow T \text{ heat flux} \]

\[ H = \int_\Omega \rho |v|^2/2 + \rho U(\rho, s), \quad \dot{H} = 0 \quad \text{and} \quad S = \int_\Omega \rho s \quad \rightarrow \quad \dot{S} \geq 0 \]
Theories & Models → Dynamics

Goal:
Predict the future or explain the past ⇒

\[ \dot{z} = V(z), \quad z \in \mathcal{Z}, \text{ Phase Space} \]

Ultimately a dynamical system. Vector fields on manifold. Maps, ODEs, PDEs, etc.

Whence vector field \( V \)?
- **Fundamental** parent theory (microscopic, \( N \) interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, rigorous asymptotics
  \[ \rightarrow \]
  **Reduced Computable Model** \( V \).
- **Phenomena** based modeling using known properties, constraints, etc. used to intuit
  \[ \rightarrow \]
  **Reduced Computable Model** \( V \). \( \leftarrow \) structure can be useful.
Types of Vector Fields, $V(z)$ (cont)

Only (?) Natural Split:

$$V(z) = V_H + V_D$$

- Hamiltonian vector fields, $V_H$: conservative, properties, etc.
- Dissipative vector fields, $V_D$: not conservative of something, relaxation/asymptotic stability, etc.

**General Hamiltonian Form:**

$$V_H = J \frac{\partial H}{\partial z} \quad \text{or} \quad V_H = J \frac{\delta H}{\delta \psi} \quad \leftarrow \infty \text{ dim}$$

where $J(z)$ is Poisson tensor/operator and $H$ is the Hamiltonian. Basic product decomposition.

**General Dissipation:**

$$V_D =?... \quad \rightarrow \quad V_D = G \frac{\partial F}{\partial z}$$

Codifying Dissipation – Some History

Is there a framework for dissipation akin to the Hamiltonian formulation for nondissipative systems?

Rayleigh (1873): \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\nu} \right) - \left( \frac{\partial L}{\partial q^\nu} \right) + \left( \frac{\partial F}{\partial \dot{q}^\nu} \right) = 0 \]
Linear dissipation e.g. of sound waves. *Theory of Sound.*

Lagrangian variational formulation with constraints.

Cahn-Hilliard (1958): \[ \frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 \left( n^3 - n - \nabla^2 n \right) \]
Phase separation, nonlinear diffusive dissipation, binary fluid ..

Other Gradient Flows: \[ \frac{\partial \psi}{\partial t} = G \frac{\delta F}{\delta \psi} \]
Otto, Ricci Flows, Poincarè conjecture on $S^3$, Perelman (2002)…
Metriplectic Dynamics
(Metric ∪ Symplectic Flows)

- Formalism for natural split of vector fields

- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.

- Other invariants? E.g., collision operators preserve, mass, momentum, .... There exists some theory for building in, but won't discuss today.

- **Encompassing 4-bracket theory**: “curvature” as dissipation

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in pjm (1984). Metriplectic in pjm (1986).
Hamiltonian Review

Poisson Bracket: \( \{f, g\} \)
Hamilton’s Canonical Equations

Phase Space with Canonical Coordinates: \((q, p)\)

Hamiltonian function: \(H(q, p) \leftarrow \text{the energy}\)

Equations of Motion:

\[
\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}, \quad \dot{q}^\alpha = \frac{\partial H}{\partial p_i}, \quad \alpha = 1, 2, \ldots N
\]

Phase Space Coordinate Rewrite:

\(z = (q, p), \quad i, j = 1, 2, \ldots 2N\)

\[
\dot{z}^i = J^i_j \frac{\partial H}{\partial z^j} = \{z^i, H\}_c, \quad (J^i_j) = \begin{pmatrix}
I_N & 0_N \\
-I_N & 0_N
\end{pmatrix},
\]

\(J_c := \text{Poisson tensor}, \) Hamiltonian bi-vector, cosymplectic form
Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\rightarrow$ PJM (1980) $\rightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

\[ z^i = \{z^i, H\} = J^{ij}(z) \frac{\partial H}{\partial z^j} \]

Noncanonical Poisson Bracket:

\[ \{f, g\} = \frac{\partial f}{\partial z^i} J^{ij}(z) \frac{\partial g}{\partial z^j} \]

Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f, g\} = -\{g, f\}$

Jacobi identity $\rightarrow$ $\{f, \{g, h\}\} + \{b, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Leibniz $\rightarrow$ $\{fh, g\} = f\{h, g\} + \{h, g\}f$

G. Darboux: $det J \neq 0 \implies J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \implies$ Canonical Coordinates plus Casimirs
(Lie's distinguished functions!)
Poisson Brackets – Flows on Poisson Manifolds

**Definition.** A Poisson manifold $\mathcal{Z}$ has bracket

$$\{,\} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

s.t. $C^\infty(\mathcal{Z})$ with $\{,\}$ is a Lie algebra realization, i.e., is

- bilinear,
- antisymmetric,
- Jacobi, and
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $d$ exterior derivative.

$$\{f,g\} = \langle df, Jdg \rangle = J(df, dg).$$

$J$ the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $JdH$, i.e.,

$$\dot{z}^i = J^{ij}(z) \frac{\partial H(z)}{\partial z^j}, \quad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \ldots, z^N)$$

Because of degeneracy, $\exists$ functions $C$ s.t. $\{f,C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$. Casimir invariants (Lie's distinguished functions!).
Poisson Manifold (phase space) \( \mathcal{Z} \) Cartoon

Degeneracy in \( J \) ⇒ Casimirs:

\[
\{ f, C \} = 0 \quad \forall \ f : \mathcal{Z} \to \mathbb{R}
\]

Lie-Darboux Foliation by Casimir (symplectic) leaves:
Metriplectic 4-Bracket: \((f, k; g, n)\)
Why a 4-Bracket?

- Two slots for two fundamental functions: Hamiltonian, $H$, and Entropy (Casimir), $S$.

- There remains two slots for bilinear bracket: one for observable one for generator ($F$?) s.t. $\dot{H} = 0$ and $\dot{S} \geq 0$.

- Provides natural reductions to other bilinear & binary brackets.

- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.
The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\cdot, \cdot; \cdot, \cdot) : \Lambda^0(\mathbb{Z}) \times \Lambda^0(\mathbb{Z}) \times \Lambda^0(\mathbb{Z}) \times \Lambda^0(\mathbb{Z}) \to \Lambda^0(\mathbb{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathbb{Z})$

$$(f, k; g, n) := R(df, dk, dg, dn),$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f, k; g, n) = R^{ijkl}(z) \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l}. \quad \leftarrow \text{quadravector?}$$

• A blend of my previous ideas: Two important functions $H$ and $S$, symmetries, curvature idea, multilinear brackets.

• Manifolds with both Poisson tensor, $J^{ij}$, and compatible quadravector $R^{ijkl}$, where $S$ and $H$ come from Hamiltonian part.
**Metriplectic 4-Bracket Properties**

(i) $\mathbb{R}$-linearity in all arguments, e.g.,

$$ (f + h, k; g, n) = (f, k; g, n) + (h, k; g, n) $$

(ii) algebraic identities/symmetries

$$ (f, k; g, n) = -(k, f; g, n) $$
$$ (f, k; g, n) = -(f, k; n, g) $$
$$ (f, k; g, n) = (g, n; f, k) $$

(iii) derivation in all arguments, e.g.,

$$ (fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h $$

which is manifest in coordinates. Here, as usual, $fh$ denotes pointwise multiplication. Symmetries of algebraic curvature without cyclic identity. Often see $R_{ijk}^l$ or $R_{lijk}$ but not $R_{lijk}$!

**Minimal Metriplectic** if above and

(iv) positive semidefiniteness

$$ (f, g; f, g) \geq 0 $$
**Metriplectic 4-Bracket Properties**

(i) \( \mathbb{R} \)-linearity in all arguments, e.g.,

\[
(f + h, k; g, n) = (f, k; g, n) + (h, k; g, n)
\]

(ii) algebraic identities/symmetries

\[
\begin{align*}
(f, k; g, n) &= -(k, f; g, n) \\
(f, k; g, n) &= -(f, k; n, g) \\
(f, k; g, n) &= (g, n; f, k) \\
(f, k; g, n) + (f, g; n, k) + (f, n; k, g) &= 0
\end{align*}
\]

\( \leftarrow \) not needed

(iii) derivation in all arguments, e.g.,

\[
(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h
\]

which is manifest in coordinates. Here, as usual, \( fh \) denotes pointwise multiplication. Symmetries of algebraic curvature without cyclic identity. Often see \( R^{l}_{ijk} \) or \( R_{lijk} \) but not \( R^{lijk} \)!

**Minimal Metriplectic** if above and

(iv) positive semindefiniteness

\[
(f, g; f, g) \geq 0
\]
1980s Binary 2-Brackets and Dissipation

Ingredients:
Binary Brackets (Poisson and Dissipative) + Generators

\[ \dot{z} = \{z, H\} + ((z, F)) \]

If \(((\cdot,\cdot))\) Leibniz & bilinear

\[ \dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} + G^{ij} \frac{\partial F}{\partial z^j} \]

where

\[ (((,))) : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z}) \]

What is \(F\) and what are the algebraic properties of \(((,)))\)?
(f, g) symmetric, bilinear, appropriately degenerate

Casimirs of noncanonical PB \( \{, \} \) are ‘candidate’ entropies. Election of particular \( S \in \{ \text{Casimirs} \} \Rightarrow \) thermodynamic equilibrium (relaxed) state.

Generator: \( \mathcal{F} = H + S \) ← “Free Energy”

1st Law: identify energy with Hamiltonian, \( H \), then

\[
\dot{H} = \{H, \mathcal{F}\} + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0
\]

Foliate \( \mathcal{Z} \) by level sets of \( H \), with \( (H, f) = 0 \) \( \forall f \in C^\infty(\mathcal{Z}) \).

2nd Law: entropy production

\[
\dot{S} = \{S, \mathcal{F}\} + (S, \mathcal{F}) = (S, S) \geq 0
\]

Lyapunov relaxation to the equilibrium state. Dynamics solves the equilibrium variational principle: \( \delta \mathcal{F} = \delta (H + S) = 0 \).
**Metriplectic 4-Bracket Reduction to 2-Bracket**

**Symmetric 2-bracket:**

\[(f, g)_H = (f, H; g, H) = (g, f)_H\]

**Dissipative dynamics:**

\[\dot{z} = (z, S)_H = (z, H; S, H)\]

**Energy conservation:**

\[(g, H)_H = (H, g)_H = 0 \quad \forall g.\]

**Entropy dynamics:**

\[\dot{S} = (S, S)_H = (S, H; S, H) \geq 0\]

**Metriplectic 4-brackets → metriplectic 2-brackets of 1984, 1986!**
Metriplectic 4-Bracket: Encompassing Definition of Dissipation

- Lots of geometry on Poisson manifolds with metric or connection. Emerges naturally.

- If Riemannian, entropy production rate is positive contravariant sectional curvature. For $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by
  \[ \dot{S} = K(\sigma, \eta) := (S, H; S, H), \]
  where the second equality follows if $\sigma = dS$ and $\eta = dH$. 
Binary Brackets for Dissipation circa 1980 →


- Antisymmetric Bracket (possibly degenerate) (Kaufman and pjm 1982)


- Double Brackets (Vallis, Carnevale, Young, Shepherd; Brockett, Bloch . . . 1989)

- GENERIC (Grmela 1984, with Oettinger 1997, . . .) Binary but **not** Symmetric and **not** Bilinear ⇔ Metriplectic Dynamics!
4-Bracket Reduction to K-M Brackets
(Kaufman and Morrison 1982)

K-M done for plasma quasilinear theory.

Dynamics:
\[ \dot{z} = [z, H]_S = (z, H; S, H) \]

Bracket Properties:
\[ [f, g]_S = (f, g; S, H) \]
- bilinear
- antisymmetric, possibly degenerate
- energy conservation and entropy production

\[ \dot{H} = [H, H]_S = 0 \quad \text{and} \quad \dot{S} = [S, H]_S \geq 0 \quad \Rightarrow \quad z \mapsto z_{eq} \]
Interchanging the role of $H$ with a Casimir $S$:

$$(f,g)_S = (f,S; g,S)$$

Can show with assumptions

$$(C,g)_S = (C,S; g,S) = 0$$

for any Casimir $C$. Therefore $\dot{C} = 0$.

Practical tool for equilibria computation $\rightarrow$ Beautiful geometry with Fernandes-Koszul connection!
4-Bracket Reduction to 2-Brackets $\equiv$ GENERIC \\
(Grmela 1984, with Öttinger 1997)

- Grmela 1984 bracket for Boltzmann not bilinear and not symmetric, unlike metriplectic 2-bracket.

**GENERIC Vector Field in terms of dissipation function $\Xi(z, z^*)$:**

$$\dot{z}^i = Y_S^i = \frac{\partial \Xi(z, z^*)}{\partial z^i} \bigg|_{z^* = \partial S/\partial z}.$$  

**Special Case:**

$$\Xi(z, z^*) = \frac{1}{2} \frac{\partial S}{\partial z^i} G^{ij}(z) \frac{\partial S}{\partial z^j} \Rightarrow \dot{z}^i = Y_S^i = G^{ij}(z) \frac{\partial S}{\partial z^j},$$

- **General Case:** there exists a bracket and procedure (pjm & Updike) for linearizing and symmetrizing $\Rightarrow$

Existence – General Constructions

• For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

• Methods of construction? We describe two: Kulkarni-Nomizu and Lie algebra based. Goal is to develop intuition like building Lagrangians.
Construction via Kulkarni-Nomizu Product

Given $\sigma$ and $\mu$, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $df, dk$ and $dg, dn$, the K-N product is

$$
\sigma \bowtie \mu (df, dk, dg, dn) = \sigma(df, dg) \mu(dk, dn) - \sigma(df, dn) \mu(dk, dg) + \mu(df, dg) \sigma(dk, dn) - \mu(df, dn) \sigma(dk, dg).
$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \bowtie \mu (df, dk, dg, dn).$$

In coordinates:

$$R^{ijkl} = \sigma^{ik} \mu^{jl} - \sigma^{il} \mu^{jk} + \mu^{ik} \sigma^{jl} - \mu^{il} \sigma^{jk}.$$
Lie Algebras: Denoted \( \mathfrak{g} \), is a vector space (over \( \mathbb{R}, \mathbb{C} \), for us \( \mathbb{R} \)) with binary, bilinear product \( [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \). In basis \( \{e_i\} \), \( [e_i, e_j] = c_{ij}^k e_k \). Structure constants \( c_{ij}^k \). For example \( \mathfrak{so}(3) \), which has \( A \times (B \times C) + B \times (C \times A) + C \times (A \times B) \equiv 0 \).

Lie-Poisson Brackets: special noncanonical Poisson brackets associated with any Lie algebra, \( \mathfrak{g} \).

Natural phase space \( \mathfrak{g}^* \). For \( f, g \in C^\infty(\mathfrak{g}^*) \) and \( z \in \mathfrak{g}^* \).

Lie-Poisson bracket has the form

\[
\{f, g\} = \langle z, [\nabla f, \nabla g] \rangle = \frac{\partial f}{\partial z^i} c_{ij}^k z_k \frac{\partial g}{\partial z^j}, \quad i, j, k = 1, 2, \ldots, \text{dim } \mathfrak{g}
\]

Pairing \( <,> : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R} \), \( z^i \) coordinates for \( \mathfrak{g}^* \), and \( c_{ij}^k \) structure constants of \( \mathfrak{g} \). Note \( J^{ij}_k = c_{ij}^k z_k \).
Lie Algebra Based Metriplectic 4-Brackets

- For structure constants $c_{kl}^{s}$:
  \[
  (f, k; g, n) = c_{ij}^{r} c_{kl}^{s} g_{rs} \frac{\partial f}{\partial z^{i}} \frac{\partial k}{\partial z^{j}} \frac{\partial g}{\partial z^{k}} \frac{\partial n}{\partial z^{l}}.
  \]
  Lacks cyclic symmetry, but $\exists$ procedure to remove torsion (Bianchi identity) for any symmetric ‘metric’ $g^{rs}$. Dynamics does not see torsion, but manifold does.

- For $g_{CK}^{rs} = c_{rl}^{k} c_{sk}^{l}$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra.

- Covariant connection $\nabla: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$. A contravariant connection $D: \Lambda^{1}(\mathbb{Z}) \times \Lambda^{1}(\mathbb{Z}) \to \Lambda^{1}(\mathbb{Z})$ satisfying Koszul identities, but Leibniz becomes $D_{\alpha}(f \gamma) = f D_{\alpha} \gamma + J(\alpha)[f] \gamma$ where $J(\alpha)[f] = \alpha_{i} J^{ij} \frac{\partial f}{\partial z^{j}}$ is a 0-form that replaces the term $X(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^{1}(\mathbb{Z})$, $f \in \Lambda^{0}(\mathbb{Z})$. Add a metric, build 4-bracket like curvature from connection.
Examples

- finite-dimensional
- $1+1$ fluid theory
- $3+1$ fluid theory
- kinetic theory
Free Rigid Body (pjm Physica D 1986)

Angular momenta \((L^1, L^2, L^3)\), Lie-Poisson bracket with Lie algebra \(so(3)\), 
\(c_{ik}^{\ j} = -\epsilon_{ijk}\).

Hamiltonian:
\[
H = \frac{(L^1)^2}{2I_1} + \frac{(L^2)^2}{2I_2} + \frac{(L^3)^2}{2I_3}
\]

principal moments of inertia, \(I_i\) Casimir
\[
C = ||L||^2 = (L^1)^2 + (L^3)^2 + (L^3)^2 = S,
\]

Euler’s equations:
\[
\dot{L}^i = \{L^i, H\}
\]

“Thermodynamics” → design a system s.t. \(\dot{H} = 0\) and \(\dot{S} \geq 0\) (or \(\dot{S} \leq 0\)).
Growing entropy sphere $\dot{S} \geq 0$ at fixed energy ellipsoid $\dot{H} = 0$
“Thermodynamical” Free Rigid Body (cont)$^2$

Use K-N product. Choose $\sigma^{ij} = \mu^{ij} = g^{ij} \Rightarrow$

$$R^{ijkl} = K \left( g^{ik} g^{jl} - g^{il} g^{jk} \right),$$

Riemannian space form with constant sectional curvature $K$.

Assume Euclidean gives metriplectic 4-bracket:

$$(f, k; g, n) = K \left( \delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk} \right) \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l},$$

Metriplectic 2-bracket:

$$(f, g)_H = (f, H; g, H)$$

Precisely bracket and dynamics of pjm 1986!

$$\dot{L}^i = \{L^i, H\} + (L^i, S)_H = \{L^i, H\} + (L^i, H; S, H)$$
Infinite Dimensions — Field Theories $\partial f/\partial z \rightarrow \delta F/\delta \chi$

Multi-component fields:

$$\chi(z, t) = (\chi^1(z, t), \chi^2(z, t), \ldots, \chi^M(z, t)),$$

$z \in D$

Metrplectic 4-bracket:

$$(F, G; K, N) = \int d^Nz \int d^Nz' \int d^Nz'' \int d^Nz''' \hat{R}^{ijkl}(z, z', z'', z''') \times \frac{\delta F}{\delta \chi^i(z)} \frac{\delta G}{\delta \chi^j(z')} \frac{\delta K}{\delta \chi^k(z'')} \frac{\delta N}{\delta \chi^l(z''')}$$

Fréchet derivative:

$$\delta F[\chi; \eta] = \frac{d}{d\epsilon} F[\chi + \epsilon \eta] \bigg|_{\epsilon=0} = \int_D d^Nz \frac{\delta F[\chi]}{\delta \chi^i} \eta^i$$

$\delta F/\delta \chi$ the functional (variational) derivative (a gradient)

$\hat{R}^{ijkl}(z, z', z'', z''')$ defined as distribution, an operator (e.g. a pseudo-differential ...) acting on the functional derivatives.
1D fluid \( u(x,t) \): 1+ 1 + (1) Field Theory

Again use K-N product with operators \( \Sigma \) and \( M \)

\[
(F, K; G, N) = \int_{\mathbb{R}} dx \, W \left( \Sigma(F_u, G_u) M(K_u, N_u) 
- \Sigma(F_u, N_u) M(K_u, G_u) + M(F_u, G_u) \Sigma(K_u, N_u) 
- M(F_u, N_u) \Sigma(K_u, G_u) \right),
\]

\( W \) a constant and \( F_u = \delta F/\delta u \), etc.

Choose

\[
M(F_u, G_u) = F_u G_u
\]

\[
\Sigma(F_u, G_u)(x) = \partial F_u(x) \mathcal{H}[G_u](x) + \partial G_u(x) \mathcal{H}[F_u](x),
\]

\( \partial = \partial/\partial x \) and \( \mathcal{H} \) the Hilbert transform \( \Rightarrow \)

\[
(F, G)_H = (F, H; G, H) = \int_{\mathbb{R}} dx \, W \left( \partial F_u \mathcal{H}[G_u] + \partial G_u \mathcal{H}[F_u] \right).
\]

\[
 u_t = ... (u, S)_H = -2W \mathcal{H}[\partial u].
\]

Ott & Sudan 1969 fluid model of electron Landau damping (Hammett-Perkins 1990). \( \mathcal{H} \rightarrow \partial \Rightarrow \) viscous dissipation
**Thermodynamic Navier-Stokes** (Eckart, 1940) \( \chi = \{ \rho, \sigma = \rho s, M = \rho v \} \)

K-N again:

\[
M(F_\chi, G_\chi) = F_\sigma G_\sigma
\]

\[
\Sigma(F_\chi, G_\chi) = \tilde{\Lambda}_{ijkl} \partial_j F_{Mi} \partial_k G_{Ml} + a \nabla F_\sigma \cdot \nabla G_\sigma
\]

\( \partial_i := \partial/\partial x^i \) with general isotropic Cartesian tensor of order 4

\[
\tilde{\Lambda}_{ikst} = \alpha \delta_{ik} \delta_{st} + \beta (\delta_{is} \delta_{kt} + \delta_{it} \delta_{ks}) + \gamma (\delta_{is} \delta_{kt} - \delta_{it} \delta_{ks})
\]

Construct

\[
(F, G)_H = (F, H; G, H) \rightarrow \chi_t = \{ \chi, H \} + (\chi, S)_H \Rightarrow
\]

using \( S = \int d^3 x \rho s \) and \( H = \int d^3 x \left( \rho |v|^2 / 2 + \rho U(\rho, s) \right) \)

\[
\partial_t v = -v \cdot \nabla v - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot T \quad \leftarrow \quad T \text{ viscous stress}
\]

\[
\partial_t \rho = -\nabla \cdot (\rho v)
\]

\[
\partial_t s = -v \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot q + \frac{1}{\rho T} T : \nabla v, \quad q = -\kappa \nabla T
\]

Reproduces pjm 1984!
Kinetic Theory Collision Operator

Phase space $z = (x, v)$, density $f(z, t)$

Define operator on $w : \mathbb{R}^6 \to \mathbb{R}$ (at fixed time)

$$P[w]_i = \frac{\partial w(z)}{\partial v_i} - \frac{\partial w(z')}{\partial v'_i}$$

$$(F, K; G, N) = \int d^6 z \int d^6 z' G(z, z')$$

$$\times (\delta \otimes \delta)_{ijkl} P[Ff]_i P[Kf]_j P[Gf]_k P[Nf]_l ,$$

where simplest K-N

$$(\delta \otimes \delta)_{ijkl} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

with $S = -\int d^2 z f \ln f$

$$(f, H; SH) = ??$$

Landau-Lenard-Balescu collision operator!

Metrplectic 2-bracket $(f, g)_H$ in pjm 1984 again!
Final Comments


• Useful for thermodynamically consistent model building, e.g., multiphase flow (Navier-Stokes-Cahn-Hilliard) with many constitutive relation effects (with A. Zaidni) and inhomogeneous collision operators for plasma (with N. Sato).

• Given that double brackets and metriplectic brackets have been used for computation of equilibria, metriplectic 4-bracket can be a new tool for equilibria. □ Easy to maintain $R^{ijkl}$ symmetries in semi-discrete projection.

• New kind of structure to preserve: Symplectic, Poisson, FEEC, .... metriplectic 2-bracket, metriplectic 4-bracket?
Existing Computational Uses


Dynamical extremization with constraints:

- Simulated Annealing: Double brackets for equilibria

- Metriplectic relaxation
Double Bracket for Vortex States 1989

Good Idea:

Vallis, Carnevale, and Young, Shepherd (1989,1990)

\[
\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, H\} + ((\mathcal{F}, H)) = ((\mathcal{F}, \mathcal{F})) \geq 0
\]

where

\[
((F, G)) = \int d^3x \frac{\delta F}{\delta \chi} J^2 \frac{\delta G}{\delta \chi}
\]

Lyapunov function, \(\mathcal{F}\), yields asymptotic stability to rearranged equilibrium.

- Maximizing energy at fixed Casimir: Except only works sometimes, e.g., limited to circular vortex states ....
Simulated Annealing

Use various bracket dynamics to effect extremization.

Many relaxation methods exist: gradient descent, etc.

Simulated annealing: an artificial dynamics that solves a variational principle with constraints for equilibria states.

Coordinates:

\[ \dot{z}^i = ((z^i, H)) = J^{ik} g_{kl} J^{jl} \frac{\partial H}{\partial z^j} \]

symmetric, definite, and kernel of \( J \).

\[ \dot{C} = 0 \quad \text{with} \quad \dot{H} \leq 0 \]
Simulated Annealing with Generalized (Noncanonical) Dirac Brackets

Dirac Bracket:

\[
\{F, G\}_D = \{F, G\} + \frac{\{F, C_1\}\{C_2, G\}}{\{C_1, C_2\}} - \frac{\{F, C_2\}\{C_1, G\}}{\{C_1, C_2\}}
\]

Preserves any two incipient constraints \(C_1\) and \(C_2\).

Our New Idea:

Do simulated Annealing with Generalized Dirac Bracket

\[
((F, G))_D = \int dx dx' \{F, \zeta(x)\}_D \mathcal{G}(x, x') \{\zeta(x'), G\}_D
\]

Preserves any Casimirs of \(\{F, G\}\) and Dirac constraints \(C_{1,2}\)

For implementation with contour dynamics see PJM (with Flierl) Phys. Plasmas \textbf{12} 058102 (2005).
Vorticity contours. The three-fold symmetric initial condition finds tri-polar state using Dirac bracket Simulated Annealing.
Double Bracket SA for Reduced MHD


High-beta reduced MHD (Strauss, 1977) given by

\[
\frac{\partial U}{\partial t} = [U, \varphi] + [\psi, J] - \epsilon \frac{\partial J}{\partial \zeta} + [P, h]
\]

\[
\frac{\partial \psi}{\partial t} = [\psi, \varphi] - \epsilon \frac{\partial \varphi}{\partial \zeta}
\]

\[
\frac{\partial P}{\partial t} = [P, \varphi]
\]

Extremization

\[
\mathcal{F} = H + \sum_i C_i + \lambda^i P_i \to \text{equilibria, maybe with flow}
\]

Cs Casimirs and Ps dynamical invariants.
Nested Tori are level sets of \( \psi \); \( q \) gives pitch of helical \( B \)-lines.
Double Bracket SA for Stability


Since SA searches for an energy extremum, it can also be used for stability analysis when initiated from a state where a perturbation is added to an equilibrium. Three steps:

1) choose **any** equilibrium of unknown stability

2) perturb the equilibrium with dynamically accessible (leaf) perturbation

3) perform double bracket SA

If it finds the equilibrium, then it is an energy extremum and must be stable
Sample Double Bracket SA unstable equilibria

FIG. 11: Radial profiles of the $(m, n) = (2, 1)$ components are plotted at several times during the SA evolution. The perturbation amplitudes decreased in time.

FIG. 12: Poloidal rotation velocity $v_p$ profile.
FIG. 16: Radial profiles of the $(m, n) = (2, 1)$ components are plotted at several times during the SA evolution. The perturbation amplitudes grew in time.
Metriplectic Simulated Annealing.

Camilla Bressen Ph.D.
TUM & Max Planck, Garching, Germany

Vortex states and MHD equilibria
Figure 6.7: Relaxed state for the test case euler-ilgr. The same as in Figure 6.2, but for the collision-like operator. The relaxed state is presented in Figure 6.9: from both the color plot (a) and scatter plot (b) we see that the initial condition has relaxed to a solution in accordance with the variational principle. For this test case, the results strongly suggest that the relaxation to the state of constrained minimum entropy, which corresponds to the solution of the variational principle, appears to be a feature of the collision-like operator, rather than being caused by numerical dissipation, as we observed for the diffusion-like operator.

Figure 6.29: Relaxed state for the gs-imgc test case. The same as in Figure 6.23, but for the collision-like operator and the case of the Czarny domain discussed in Section A.4.2. With respect to Figure 6.27(b) for the diffusion-like operator, we see from (b) that the agreement between the relaxed state and the prediction of the variational principle is better.
Computation Summary

- Poisson Integrators

- Dirac Double Bracket Simulated Annealing for Equilibria and Stability

- Metriplectic Simulated Annealing for Equilibria
References: