Variational Principles for Equilibria & Dynamical Relaxation

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Unfinished work with Bob Dewar on **weakening the frozen-in flux constraint** to allow for islands & alternative Hamiltonian approach by removing Casimir constraints.

- Dewar & Qu, J. Plasma Phys. **88**, 835880101 (2022).

Two Variational Principles: Lagrangian and Hamiltonian

Lagrangian and Hamiltonian Dynamics:

$$\delta A = \delta \int dt\, \mathcal{L} = \delta \int dt\, (T - V) = 0 \qquad \Rightarrow \text{dynamics via Lagrange's eqs.}$$

$$\dot{z} = J_c \nabla H = 0 \qquad \Rightarrow \text{equilibrium eqs. } \nabla H = 0 \,. \qquad \text{Note here } z = (q, p)$$

MHD (field theory) Dynamics via Hamilton's Principle?

$$A_{MHD} = \int dt \int d^3x\, (T - V) = \int dt \int d^3x \left(\frac{\rho}{2}\, |\boldsymbol{v}|^2 - \rho U(\rho, s) - \frac{|\boldsymbol{B}|^2}{2} \right)$$

$$H_{MHD} = \int d^3x \left(\frac{\rho}{2} |\boldsymbol{v}|^2 + \rho U(\rho, s) + \frac{|\boldsymbol{B}|^2}{2} \right)$$

$$\delta A_{MHD} = 0 \qquad \Rightarrow \boldsymbol{v} = \boldsymbol{B} \equiv 0, \qquad \rho = \text{constant}, \ldots \rightarrow \qquad \textbf{no dynamics!}$$

$$\frac{\delta H_{MHD}}{\delta \boldsymbol{v}} = \rho \boldsymbol{v} = 0\,, \qquad \frac{\delta H_{MHD}}{\delta \boldsymbol{B}} = \boldsymbol{B} = 0 \ \ldots \rightarrow \qquad \textbf{trivial equilibrium!}$$

Lagrange (1788) and Newcomb (1962)

Lagrange (1788): Lagrangian variables, Lagrangian for the ideal fluid (compressible and incompressible) the latter by method of Lagrange multipliers. $\leftarrow$ holonomic constraint.

$$A_L = \int dt \int d^3a \left(\frac{\rho_0}{2} |\dot{q}|^2 - \rho_0 U(\rho_0/\mathcal{J}) \right) ,$$

where $q(a,t)$ fluid element position, $q(a,0) = a$, ρ_0 fluid element attribute, $\mathcal{J} = \det(\partial q/\partial a)$

$$\delta A_L = 0 \quad \Rightarrow \quad \rho_0 \ddot{q} = ... \quad \leftarrow \text{ideal fluid EoM in Lagrangian variables}$$

Newcomb (1962):

$$A_N = \int dt \int d^3a \left(\frac{\rho_0}{2} |\dot{q}|^2 - \rho_0 U(\rho_0/\mathcal{J}, s_0) - \frac{\left| B_0^j \partial q/\partial a^j \right|^2}{2\mathcal{J}^2} \right) ,$$

New term is frozen flux.

$$\delta A_N = 0 \quad \Rightarrow \quad \rho_0 \ddot{q} = ... \quad \leftarrow \text{ideal MHD EoM in Lagrangian variables}$$

Constraints: Hamel, Poincare (1904) and Newcomb (1962)

Lagrangian induce Eulerian variations:

$$
\begin{aligned}
\delta \boldsymbol{v} &= \partial_t \xi + \boldsymbol{v} \cdot \nabla \xi - \xi \cdot \nabla \boldsymbol{v} \\
\delta \rho &= -\nabla \cdot (\rho \xi) \\
\delta p &= -\gamma p \nabla \cdot \xi - \xi \cdot \nabla p \\
\delta \boldsymbol{B} &= \nabla \times (\xi \times \boldsymbol{B})
\end{aligned}
$$

Here $\xi = \delta q$. With the above constrained variations

$$\delta A_{MHD} = 0 \quad \Rightarrow \quad \text{ideal MHD dynamics!}$$

$$\frac{\delta H_{MHD}}{\delta \xi} = 0 \quad \Rightarrow \quad \text{ideal MHD equilibrium equations!}$$

The Idea of Dewar and Qu (2022)

Goal $\rightarrow$ **Weaken frozen in flux** to allow for islands. Then relaxation?

Phase Space Lagrangian:

$$\mathcal{L} = \int d^3x \left(\rho\, \boldsymbol{u} \cdot \boldsymbol{v} - \frac{\rho |\boldsymbol{u}|^2}{2} - \frac{p}{\gamma - 1} - \frac{|\boldsymbol{B}|^2}{2} \right)$$

New Local Constraint:

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0$$

Global Constraints:

$$K_{\boldsymbol{A}\cdot\boldsymbol{B}} = \frac{1}{2} \int d^3x\, \boldsymbol{A} \cdot \boldsymbol{B} \quad \text{and} \quad K_{\boldsymbol{u}\cdot\boldsymbol{B}} = \int d^3x\, \boldsymbol{u} \cdot \boldsymbol{B}$$

Action:

$$\mathcal{L}_D = \mathcal{L}_{MHD} + \int d^3x\, \lambda \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \mu K_{\boldsymbol{A}\cdot\boldsymbol{B}} + \nu K_{\boldsymbol{u}\cdot\boldsymbol{B}}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -\nabla\Phi - \partial_t \boldsymbol{A}$$

Mixed variations: $\delta\boldsymbol{A}, \delta\Phi$ with $\delta\boldsymbol{v}$ via $\delta\xi$ $\Rightarrow$ equations of motion. Identify multipliers.

Relaxation & Variational Principle for Equilibrium

• Aristotle's principle that "nature does nothing in vain" ancient teleological approach to nature.

Taylor-Woltjer-Beltrami states:

$$\delta\left(\int d^3x\ |\boldsymbol{B}|^2 + \mu \int d^3x\, \boldsymbol{A}\cdot\boldsymbol{B}\right) = 0$$

Nature minimizes energy at constant helicity or vice verse. Selective decay hypothesis in GFD.

Procedure: Find some invariants, minimize one at constant other, make medieval argument!

Questions:

• Why does this even yield an equilibrium state in general? Observed after the fact.

• What if you had a VP for any equilibrium? Then, why Beltrami?

• Lagrangian and Hamiltonian variational principles don't relax? Whence relaxation?

Noncanonical Hamiltonian Approach and Casimirs

MHD Eulerian variables $\Psi = (\boldsymbol{v}, \rho, s, \boldsymbol{B})$

pjm &Greene Poisson Bracket:

$$\frac{\partial \Psi}{\partial t} = \{\Psi, H\} = \mathfrak{J} \frac{\delta H}{\delta \Psi}$$

Unlike canonical Poisson brackets, $\mathfrak{J}$ is degenerate, i.e. $\exists C$, such that $\{C, H\} = 0 \ \forall H$.

Energy-Casimir variational principle:

$$0 = \frac{\partial \Psi}{\partial t} = \mathfrak{J} \frac{\delta H}{\delta \Psi} = \mathfrak{J} \frac{\delta (H + C)}{\delta \Psi}$$

Helicities are in set of Casimirs. Explains Taylor and other variational principles.

Flux constraint built into null space of $\mathfrak{J}$!

Counting Casimirs and Dynamical Accessibility

For finite-dimensional systems $\mathfrak{J}$ is a matrix and there is dim(corank($\mathfrak{J}$)) number of Casimirs. Variational Principle:

$$\frac{\partial(H + \sum C)}{\partial z} = 0 \quad \Rightarrow \text{"All" Equilibria}$$

For infinite-dimensional systems (field theories)

$$\{\text{Energy} - \text{Casimir equilibria}\} \neq \{\text{Dynamical equilibria}\}$$

The null space of $\mathfrak{J}$ is more difficult to understand. Deep math problem $\rightarrow$ what to do?

Dynamically accessible variations:

$$\delta\Psi_{DA} = \mathfrak{J}G \quad \leftarrow \text{whatever the nullspace, it is preserved!}$$

G an arbitrary generator. Constrained variations discovered by pjm & Pfirsch (1989).

$$\delta H[\Psi, \delta\Psi_{DA}] = 0 \qquad \Rightarrow \qquad \text{All Equilibria}$$

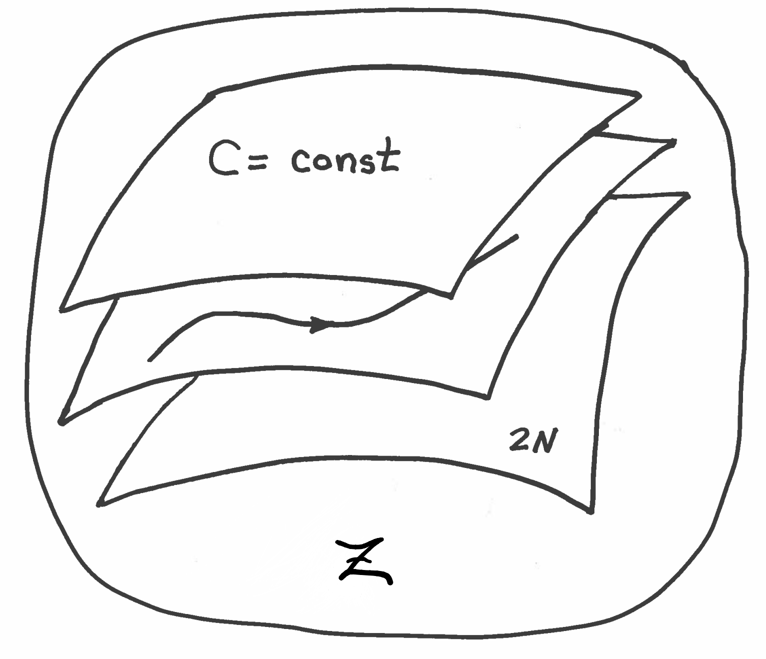
Despite Casimir deficit problem, all constraints maintained including flux preservation. See several papers on MHD by **Andreussi, pjm, & Pegoraro** 2010 – 2020

Poisson Manifold (phase space) finite $\mathcal{Z}$ Cartoon

Degeneracy in $\mathfrak{J}$ $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



$\delta\Psi_{DA}$ is variation within Casimir leaf.

Alternative Methods – Tools

Removing Casimirs:

Yoshida & pjm (2014),
"Unfreezing Casimir Invariants: Singular Perturbations Giving Rise to Forbidden Instabilities"

We showed breaking of flux constraint and island formation and instability.

Why Relaxation:

Metriplectic 4-Bracket (pjm, Updike, Zaidni, Sato 2024) $\leftarrow$ Similar but different from Furukawa:

$$\frac{\partial \Psi}{\partial t} = \{\Psi, H\} + (\psi, H; S, H)$$

Symmetries/properties of the 4-bracket $\Rightarrow$ extremize S at constant H or vice versa $H \leftrightarrow S$.

There is a **algorithm** (physical or practical) for constructing the 4-bracket. $\leftarrow$ removes mystery.

Camilla Bressan's thesis, Omar Maj, ... solves the energy-Casimir VP in time