A collision operator for describing dissipation in Poisson manifolds, noncanonical Hamiltonian phase spaces

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- Metriplectic dynamics, a theory of thermodynamically consistent theories.
- An algorithm for constructing such theories.
- Use algorithm to construct new collision operator*.

* With **Naoki Sato**, National Institute for Fusion Science, Toki-city, Japan

Old and New

Old:

- A. N. Kaufman and P. J. Morrison, "Algebraic Structure of the Plasma Quasilinear Equations," Physics Letters A 88, 405–406 (1982).
- P. J. Morrison, "Bracket Formulation for Irreversible Classical Fields," Physics Letters A **100**, 423–427 (1984).
- P. J. Morrison, "Some Observations Regarding Brackets and Dissipation," arXiv:2403.14698v1 [mathph] 15 Mar 2024 (1984 CPAM report).
- P. J. Morrison, "A Paradigm for Joined Hamiltonian and Dissipative Systems," Physica D **18**, 410–419 (1986).

New:

- **N. Sato** and P. J. Morrison, "A Collision Operator for Describing Dissipation in Noncanonical Phase Space," Fundamental Plasma Physics **10**, 100054 (18pp) (2024).
- A. Zaidni, P. J. Morrison, and S. Benjelloun,, "Thermodynamically Consistent Cahn-Hilliard-Navier-Stokes Equations using the Metriplectic Dynamics Formalism," Physica D **468**, 134303 (11pp) (2024).
- P. J. Morrison and M. Updike, "Inclusive Curvature-Like Framework for Describing Dissipation: Metriplectic 4-Bracket Dynamics," Physical Review E **109**, 045202 (22pp) (2024).

Thermodynamic Consistency – Examples

Navier-Stokes (**inconsistent**):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathcal{T} \qquad \leftarrow \ \mathcal{T}\ \text{viscous stress tensor} \sim \nabla v$$

$$\partial_t \rho = -\nabla\cdot(\rho\boldsymbol{v})$$

$$H = \int_\Omega \rho\,|\boldsymbol{v}|^2/2 + \rho U(\rho) \quad \text{and} \quad \dot{H} \neq 0$$

Thermodynamic Navier-Stokes (**consistent**) (Eckart 1940):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathcal{T}$$

$$\partial_t \rho = -\nabla\cdot(\rho\boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v}\cdot\nabla s - \frac{1}{\rho T}\nabla\cdot\boldsymbol{q} + \frac{1}{\rho T}\mathcal{T}:\nabla\boldsymbol{v} \qquad \text{heat flux \& viscous heating}$$

$$H = \int_\Omega \rho\,|\boldsymbol{v}|^2/2 + \rho U(\rho, s)\,, \quad \dot{H} = 0 \quad \text{and} \quad S = \int_\Omega \rho s \ \rightarrow\ \dot{S} \geq 0$$

Thermodynamic Consistency

Dynamical System:

$$\frac{\partial \Psi}{\partial t} = \mathcal{V}[\Psi]$$

Ψ set of dynamical variables; system of ODEs, PDEs, ...

First Law:

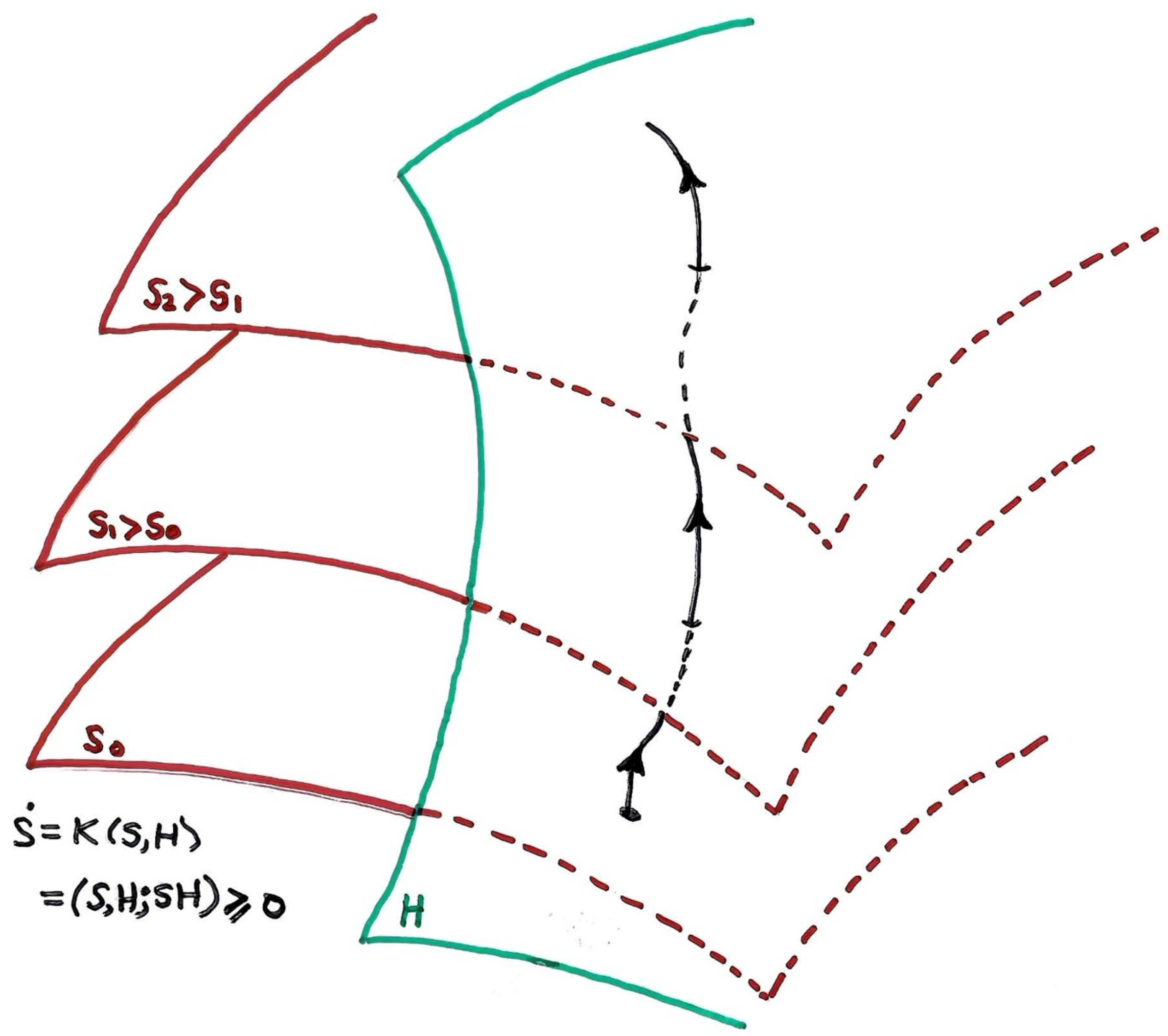
$$\exists\, H[\Psi] \;\; st \quad \dot{H} = 0$$

energy conservation

Second Law:

$$\exists\, S[\Psi] \;\; st \quad \dot{S} \geq 0$$

entropy production


S₂>S₁
S₁>S₀
S₀
H
Ṡ = K(S,H)
=(S,H;SH) ≥ 0

Metriplectic Dynamics $\Rightarrow$ Thermodynamic Consistency

(pjm 1984, 1986)

Metric (gradient) flow $\cup$ Symplectic Flow

Dynamical System:

$$\boxed{\frac{\partial \Psi}{\partial t} = \{\Psi, H\} + (\Psi, \mathcal{F}; H, S)}$$

Noncanonical Poisson Bracket:

$$\{f, g\}$$

Metriplectic 4-Bracket:

$$(f, k; g, n)$$

Alternatives:

$$\frac{\partial \Psi}{\partial t} = \{\Psi, \mathcal{F}\} + (\Psi, \mathcal{F})_H = \mathcal{J}\,\frac{\delta H}{\delta \Psi} + \mathcal{G}\,\frac{\delta S}{\delta \Psi}$$

Hamiltonian Systems

Poisson Bracket: $\{f, g\}$

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ ← the energy

Equations of Motion:

$$\boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}\,, \qquad \dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}\,,} \qquad \alpha = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad i, j = 1, 2, \ldots 2N$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = \{z^i, H\}_c\,, \qquad J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\, ,\, \} : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with, $\{\, ,\, \}$ defines a Poisson algebra $=$ Lie algebra realization $+$ Leibniz, i.e., is

- bilinear,
- antisymmetric,
- Jacobi identity, and
- Leibniz, i.e., acts as a derivation with pointwise multiplication $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f, g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g \rangle = J(\boldsymbol{d}f, \boldsymbol{d}g) = \frac{\partial f(z)}{\partial z^a} J^{ab}(z) \frac{\partial g(z)}{\partial z^b}\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot{z}^a = J^{ab}(z)\frac{\partial H(z)}{\partial z^b}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \ldots, z^M)$$

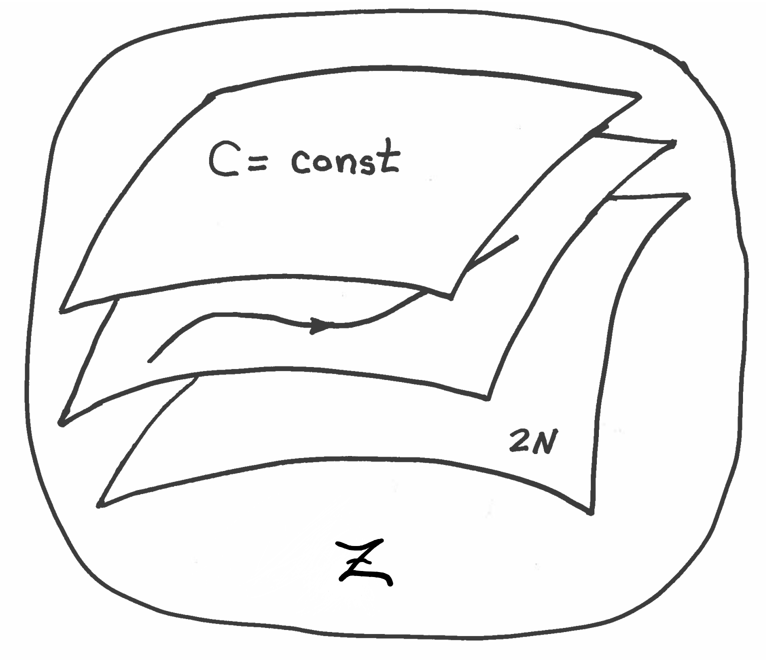
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$. Casimir invariants.

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Metriplectic 4-Bracket

$$(f, k; g, n)$$

Why a 4-Bracket?

- Slot for Hamiltonian (energy), H
- Slot for Entropy (Casimir), S.
- Slot for Free energy, $\mathcal{F} = H - \mathcal{T}S$,
- Slot for dynamical variable, Ψ or any observable

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \to \Lambda^0(\mathcal{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathcal{Z})$

$$(f, k; g, n) := R(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f, k; g, n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

- A blend of my previous ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets.
- Manifolds with both Poisson tensor, J^{ij}, and compatible quadravector R^{ijkl}, where S and H come from Hamiltonian part.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g,

$$(f+h,k;g,n) = (f,k;g,n) + (h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}(f,k;g,n) &= -(k,f;g,n)\\ (f,k;g,n) &= -(f,k;n,g)\\ (f,k;g,n) &= (g,n;f,k)\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n) = f(h,k;g,n) + (f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature without cyclic identity. Often see $R^{l}{}_{ijk}$ or R_{lijk} but not R^{lijk}! **Minimal Metriplectic.**

General Constructions

• For any Riemannian manifold $\exists$ metriplectic finite-dimensional 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

• Symmetries imply thermodynamic consistency; e.g., entropy production rate is

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H) \geq 0\,,$$

where if Riemannian K is sectional curvature for $\sigma, \eta \in \Lambda^1(\mathcal{Z})$ with $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$

• Methods of construction? We describe two: Kulkarni-Nomizu and Lie algebra based. Goal is to develop intuition like building Lagrangians.

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric or antisymmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned} \sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) \\ &- \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\ &+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) \\ &- \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,. \end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

Examples

Landau Collision Operator

Phase space $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v})$, density $f(\boldsymbol{z}, t)$, functional derivative $F_f = \delta F/\delta f$.

Define operator on $w\colon \mathbb{R}^6 \to \mathbb{R}$ (at fixed time)

$$P[w]_i = \frac{\partial w(\boldsymbol{z})}{\partial v_i} - \frac{\partial w(\boldsymbol{z}')}{\partial v_i'}$$

$$\begin{aligned}(F, K; G, N) \;&=\; \int d^6 z \int d^6 z' \, \mathcal{G}(\boldsymbol{z}, \boldsymbol{z}') \\ &\times \, (\delta \bigcirc\!\!\!\!\wedge\; \delta)_{ijkl} \, P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l \,,\end{aligned}$$

where simplest K-N

$$(\delta \bigcirc\!\!\!\!\wedge\; \delta)_{ijkl} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,.$$

with $S = -\int d^6 f \ln f$

$$(f, H; S, H) = ??$$

Landau-Lenard-Balescu collision operator!

Metriplectic 2-bracket $(f, g)_H = (f, H; g, H)$ gives 'gradient' flow of pjm 1984!

Collision Operator on Noncanonical Phase Space

Why? Clusters appear and interact on shorter time scales. Drift kinetic theories and gyrokinetic theories are noncanonical.

Collision operator:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathsf{coll}} = \mathcal{C}\left(f, f\right) = \frac{\partial}{\partial \boldsymbol{z}} \cdot \left[f \mathcal{J} \cdot \int f' \Pi \left(\mathcal{J}' \cdot \frac{\partial \log f'}{\partial \boldsymbol{z}'} - \mathcal{J} \cdot \frac{\partial \log f}{\partial \boldsymbol{z}} \right) d\boldsymbol{z}' \right], \tag{1}$$

where Π is a symmetric covariant (interaction) tensor, determined by type of binary interactions, and $\mathcal{J}$ is the Poisson tensor/operator of noncanonical Poisson bracket. Here $f' = f\left(\boldsymbol{z}', t\right)$ and $\mathcal{J}' = \mathcal{J}\left(\boldsymbol{z}'\right)$.

Collision Operator on Noncanonical Phase Space (cont)

Kulkarni-Nomizu with Poisson tensor:

$$\sigma^{ij} = \mu^{ij} = \mathcal{J}^{ij} \Rightarrow$$
$$\mathcal{R}^{ijkl} = \mathcal{J}^{ij}\mathcal{J}^{kl} + \mathcal{J}^{il}\mathcal{J}^{kj} - \mathcal{J}^{ki}\mathcal{J}^{jl} - \mathcal{J}^{ji}\mathcal{J}^{kl}$$

Metriplectic 4-bracket:

$$(F,K;G,N) = \int d^6z \int d^6z'\, f f'\, \Gamma \mathcal{R}^{ijkl} P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l ,$$

where Γ determined by interaction potential gives Π and

$$P\left[F_f\right]_i = \frac{\partial}{\partial z^i}\frac{\delta F}{\delta f} - \frac{\partial}{\partial z'^i}\frac{\delta F}{\delta f'}$$

- Conservation laws, entropy production, equilibria generalization of Maxwellian involving Casimirs. Reduces to Landau. ...

Final Comments

- Metriplectic 4-bracket describes thermodynamically consistent theories. Fluids, magnetofluids, multiphase fluids, kinetic theories, ...

- Useful for constructing such models, even though complicated. See refs. for lots of them.

- Useful for computations? Structure preservation.