Using the Metriplectic 4-Bracket for Constructing Thermodynamically Consistent Models and Structure Preserving Numerical Algorithms

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Metriplectic 4-Bracket: undergrad, **Michael Updike** (geometry), now Princeton grad student, grad student intern, **Azeddine Zaidni** (UTA), UM6, Marrakesh-Safi, Morocco; **Naoki Sato** (collision operators), NIFS, Nagoya, Japan, **William Barham** (numerics), UT Austin.

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Recent Papers

- A. Zaidni and P. J. Morrison, "Metriplectic 4-Bracket Algorithm for Constructing Thermodynamically Consistent Dynamical Systems," arXiv:2501.00159v1 [physics.flu-dyn] 30 Dec 2024
- W. Barham, P. J. Morrison, and A. Zaidni, "A Thermodynamically Consistent Discretization of 1D Thermal-Fluid Models Using their Metriplectic 4-Bracket Structure," arXiv:2410.11045v2 [physics.comp-ph] 19 Oct 2024
- A. Zaidni, P. J. Morrison, and S. Benjelloun, "Thermodynamically Consistent Cahn-Hilliard-Navier-Stokes Equations Using the Metriplectic Dynamics Formalism," Physica D **468**, 134303 (11pp) (2024).
- N. Sato and P. J. Morrison, "A Collision Operator for Describing Dissipation in Noncanonical Phase Space," Fundamental Plasma Physics **10**, 100054 (18pp) (2024).
- P. J. Morrison and M. Updike, "Inclusive Curvature-Like Framework for Describing Dissipation: Metriplectic 4-Bracket Dynamics," Physical Review E **109**, 045202 (22pp) (2024).
- B. Coquinot and P. J. Morrison, "A General Metriplectic Framework with Application to Dissipative Extended Magnetohydrodynamics," Journal of Plasma Physics **86**, 835860302 (32pp) (2020).

Thermodynamic Consistency – Examples

Navier-Stokes (**inconsistent):**

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho_0} \nabla p + \nu \nabla^2 \boldsymbol{v} \,, \qquad \nabla \cdot \boldsymbol{v} = 0 \quad \Rightarrow \quad p[\boldsymbol{v}]$$

$H = \int_\Omega \rho_0 \, |\boldsymbol{v}|^2/2$ and $\dot{H} \leq 0$, $\nexists$ any thermodynamics!

Navier-Stokes-Fourier (NSF) (**consistent)** (Eckart 1940):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \mathcal{T}$$

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v} \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot \boldsymbol{q} + \frac{1}{\rho T} \mathcal{T} : \nabla \boldsymbol{v}$$ heat flux & viscous heating

$H = \int_\Omega \rho \, |\boldsymbol{v}|^2/2 + \rho u(\rho, s) \,, \quad \dot{H} = 0$ and $S = \int_\Omega \rho s \;\rightarrow\; \dot{S} \geq 0$

Thermodynamic Consistency and the UTA

The **dynamical** realization of the first and second laws of thermodynamics: energy conservation and entropy production,

$$\dot{H} = 0 \qquad \text{and} \qquad \dot{S} \geq 0$$

Recall, thermodynamics has no real dynamics – only vague statements like adiabatic compression or or isothermal expansion, etc.

Here we present **Unified Thermodynamic Algorithm** (UTA) for construction. Important **by-product** re 'Force-Flux' relations of nonequilibrium thermodynamics:

$$\boldsymbol{J}^\alpha = L^{\alpha\beta} X_\beta \quad \rightarrow \quad \boldsymbol{J}^\alpha = -L^{\alpha\beta} \nabla(\delta H/\delta\xi^\beta)$$

'Forces': $\boldsymbol{X} \sim \nabla T, \nabla p, \nabla \boldsymbol{v}$ etc., UTA removes ambiguous selection. $L^{\alpha\beta}$ phenomenological coefficients, ξ^β dynamical variables.

What is Dissipation?

- Not all conservative systems are Hamiltonian
- Not all Hamiltonian systems are conservative
- Not all reversible systems are Hamiltonian
- All finite dynamical systems (autonomous ODEs) are reversible
- Some infinite systems (PDEs) are reversible and some irreversible
- Not all Hamiltonian systems have time reversal symmetry
- Not all systems with time reversal symmetry are Hamiltonian
- $\exists$ systems with time reversible symmetry and global asymptotic stability

Dissipation:
Systems with an increasing entropy and global asymptotic stability have dissipation.

Such systems have a 'vector field' that naturally splits in Hamiltonian and dissipative parts. Hamiltonian is an unambiguous way to define nondissipative. The Metriplectic 4-bracket is an unambiguous way to define dissipative. Together they provide thermodynamic consistency.

Metriplectic Dynamics

(Metric ∪ Symplectic Flows)

Dynamical System:

$$\dot{O} = \{O, H\} + (O, H; S, H) \qquad \leftarrow \text{Poisson} \ + \ 4-\text{bracket}$$

- O any observable (functional of dynamical variables)
- Can be finite or infinite. We will not distinguish much.

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

- bilinear,
- antisymmetric,
- Jacobi, and
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f, \boldsymbol{d}g)\,.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot{z}^i = J^{ij}(z)\frac{\partial H(z)}{\partial z^j}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \dots, z^N)$$

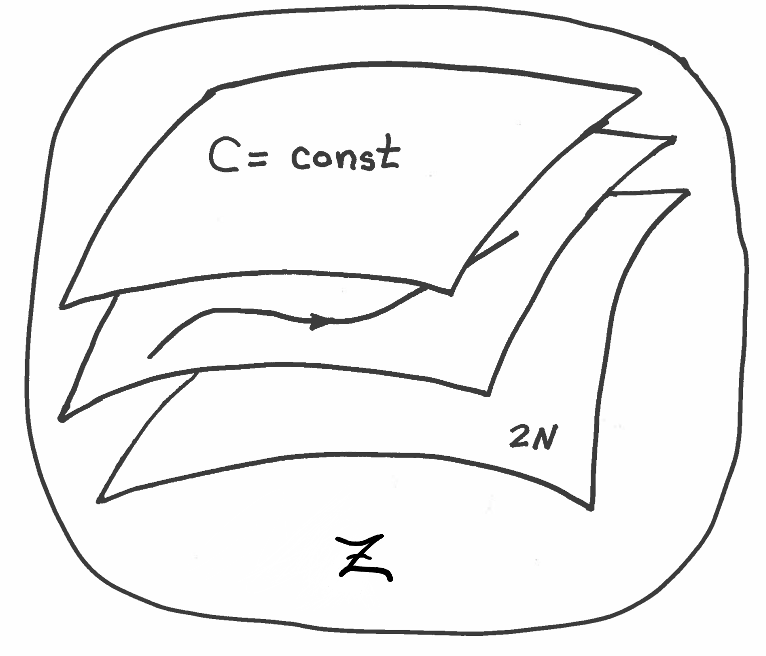
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$. Casimir invariants (Lie's distinguished functions! They can be entropies.).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

- Two slots for two fundamental functions: Hamiltonian, H, and Entropy (Casimir), S.

- There remains two slots for bilinear bracket: one for observable one for generator ($\mathcal{F}$?) s.t. $\dot{H} = 0$ and $\dot{S} \geq 0$.

- Provides natural reductions to other bilinear & binary brackets.

- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\times\Lambda^0(\mathcal{Z})\to\Lambda^0(\mathcal{Z})$$

For functions $f,k,g,n\in\Lambda^0(\mathcal{Z})$

$$(f,k;g,n) := R(\boldsymbol{d}f,\boldsymbol{d}k,\boldsymbol{d}g,\boldsymbol{d}n)\,,$$

In a coordinate patch the metriplectic 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

• A blend of my previous early ideas: Two important functions H and S, symmetries, curvature idea, multilinear brackets.

• Metriplectic manifolds have both Poisson tensor, J^{ij}, and compatible quadravector R^{ijkl}, where S (Casimir) and H come from Hamiltonian part.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g,

$$(f+h,k;g,n) = (f,k;g,n) + (h,k;g,n)$$

(ii) algebraic identities/symmetries

$$\begin{aligned}(f,k;g,n) &= -(k,f;g,n)\\ (f,k;g,n) &= -(f,k;n,g)\\ (f,k;g,n) &= (g,n;f,k)\end{aligned}$$

(iii) derivation in all arguments, e.g.,

$$(fh,k;g,n) = f(h,k;g,n) + (f,k;g,n)h$$

which is manifest when written in coordinates. Here, as usual, fh denotes pointwise multiplication. Symmetries of algebraic curvature without cyclic identity. Often see $R^l{}_{ijk}$ or R_{lijk} but not R^{lijk}! **Minimal Metriplectic.**

Properties – Existence – General Construction Methods

- Thermodynamic Consistency:

$$\dot{H} = \{H, H\} + (H, H; S, H) = 0 \qquad \text{and} \qquad \dot{S} = (S, H; S, H) \geq 0$$

- For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

- If Riemannian, entropy production rate is positive contravariant sectional curvature. For $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H)\,,$$

where the second equality follows if $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

- Methods of construction? We describe two: **Kulkarni-Nomizu and Lie algebra based**. Goal is to develop intuition like building Lagrangians. $K(\sigma, \eta) \geq 0$ automatic for K-N!

Construction via Kulkarni-Nomizu Product

Given σ and μ, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned}
\sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) \\
&- \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\
&+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) \\
&- \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,.
\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

Infinite dimensions: $\mu \to M\,, \quad \sigma \to \Sigma$.

Lie Algebras and Lie-Poisson Brackets

Lie Algebras: Denoted $\mathfrak{g}$, is a vector space (over $\mathbb{R}, \mathbb{C}$, for us $\mathbb{R}$) with binary, bilinear product $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. In basis $\{e_i\}$, $[e_i, e_j] = c_{ij}{}^{k}\, e_k$. Structure constants $c_{ij}{}^{k}$. For example $\mathfrak{so}(3)$, which has $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) + \boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A}) + \boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B}) \equiv 0$.

Lie-Poisson Brackets: special noncanonical Poisson brackets associated with any Lie algebra, $\mathfrak{g}$.

Natural phase space $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned} \{f,g\} &= \langle z, [\nabla f, \nabla g]\rangle \\ &= \frac{\partial f}{\partial z^i} c^{ij}{}_{k}\, z_k \frac{\partial g}{\partial z^j}\,, \qquad\qquad i,j,k = 1,2,\ldots,\dim\mathfrak{g} \end{aligned}$$

Pairing $<\ ,\ >\colon \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$, z^i coordinates for $\mathfrak{g}^*$, and $c^{ij}{}_{k}$ structure constants of $\mathfrak{g}$. Note

$$J^{ij} = c^{ij}{}_{k}\, z_k\,.$$

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants ${c^{kl}}_s$:

$$(f,k;g,n) = {c^{ij}}_r {c^{kl}}_s \, g^{rs} \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l} \,.$$

Lacks cyclic symmetry, but $\exists$ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs}. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = {c^{rl}}_k \, {c^{sk}}_l$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra.

- Covariant connection $\nabla\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$. A contravariant connection $D\colon \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = f D_\alpha \gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij} \partial f/\partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$, $f \in \Lambda^0(\mathcal{Z})$. Add a metric, build 4-bracket like curvature from connection.

Unified Thermodynamic Algorithm (UTA)

UTA is an algorithm for constructing metriplectic systems! Applied to many systems. **So far UTA either reproduces, corrects, or extends for every case considered!**

- Cahn-Hilliard-Navier-Stokes: agrees with Anderson et al.; corrects Guo and Lin

- Brenner-Navier-Stokes: UTA produces Brenner's equations, plus corrects statements, e.g., that the results are most general.

- Generalization of Brenner-Navier-Stokes: UTA produces equations of Reddy et al. (2019). All are generalizations of Navier-Stokes-Fourier with modified dissipation.

- Collisions on noncanonical phase space: Generalization of Landau for drift kinetic, ...

4 Steps of the UTA

1. Identify dynamical variables defined on $\Omega \subset \mathbb{R}^3$; e.g. for NSF

$$\boldsymbol{\xi} = (\boldsymbol{m} = \rho \boldsymbol{v}, \rho, \sigma = \rho s)$$

2. Propose energy and entropy functionals, $H[\boldsymbol{\xi}]$ and $S[\boldsymbol{\xi}]$; for NSF

$$H = \int_\Omega \frac{|\mathbf{m}|^2}{2\rho} + \rho u(\rho, \sigma/\rho) \quad \text{and} \quad S = \int_\Omega \sigma$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S is a Casimir invariant, $\{F, S\} = 0 \, \forall \, F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product by a **new method that separates local thermodynamics from phenomenological quantities**, giving the EoMs as Poisson bracket + 4-bracket:

$$\partial_t \boldsymbol{\xi} = \{\boldsymbol{\xi}, H\} + (\boldsymbol{\xi}, H; S, H)$$

Result automatically Thermodynamically consistent!

3. For NSF Ideal Fluid Poisson Bracket Dynamics (pjm-Greene (1980)

Hamiltonian:

$$H = \int_\Omega \frac{\rho|\mathbf{v}|^2}{2} + \rho u\,(\rho, s)\,, \qquad T = \frac{\partial u}{\partial s}\,, \qquad p = \rho^2 \frac{\partial u}{\partial \rho}\,.$$

M-G Poisson Bracket:

$$\{F, G\} = -\int_\Omega \mathbf{m}\cdot[F_{\boldsymbol{m}}\cdot\nabla G_{\boldsymbol{m}} - G_{\boldsymbol{m}}\cdot\nabla F_{\boldsymbol{m}}] + \rho\,[F_{\boldsymbol{m}}\cdot\nabla G_\rho - G_{\boldsymbol{m}}\cdot\nabla F_\rho] + \sigma\,[F_{\boldsymbol{m}}\cdot\nabla G_\sigma - G_{\boldsymbol{m}}\cdot\nabla F_\sigma]\,.$$

Equations of Motion:

$$\partial_t \mathbf{v} = \{\boldsymbol{v}, H\} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p/\rho\,, \quad \partial_t\rho = \{\rho, H\} = -\nabla\cdot(\rho\,\boldsymbol{v})\,, \quad \partial_t\sigma = \{\sigma, H\} = -\nabla\cdot(\sigma\,\boldsymbol{v})\,.$$

Casimir:

$$S = \int_\Omega \rho s = \int_\Omega \sigma\,.$$

Note: $F_{\boldsymbol{m}} = \delta F/\delta\boldsymbol{m}$, etc., functional derivatives.

4. Metriplectic 4-Bracket

Old method: guess M and Σ.

New Method

Theorem: Order dynamical variables st

$$\begin{aligned}\partial_t \xi^\alpha &= \{\xi^\alpha, H\} + \nabla \cdot \boldsymbol{J}^\alpha\,, \qquad\qquad \alpha = 1, \ldots, N-1\,,\\ \partial_t \xi^N &= \{\xi^N, H\} + \nabla \cdot \boldsymbol{J}^N + \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta\,.\end{aligned}$$

where $\xi^N = \sigma$, the entropy density. Above splits Hamiltonian and conservative.

Then

$$\dot{S} = \int_\Omega \boldsymbol{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \boldsymbol{Z}_\beta =: \int_\Omega \dot{s}^{prod} \geq 0\,.$$

and $\dot{H} \Rightarrow$

$$\boldsymbol{Z}_\alpha = \nabla H_{\xi^\alpha}\,, \qquad \boldsymbol{J}^\alpha = -H_{\xi^N} \tilde{L}^{\alpha\beta} \nabla H_{\xi^\beta} = -L^{\alpha\beta} \nabla H_{\xi^\beta}\,.$$

which leads naturally to

$$M(dF, dG) = F_{\xi^N} G_{\xi^N}\,, \qquad \Sigma(dF, dG) = \nabla(F_{\xi^\alpha}) \frac{L^{\alpha\beta}}{H_{\xi^N}} \nabla(G_{\xi^\beta})\,.$$

4. Metriplectic 4-Bracket General and NSF

General flux expressions:

$$\boldsymbol{J}_\rho = -L^{\rho\rho}\cdot \nabla H_\rho - L^{\rho \boldsymbol{m}} : \nabla H_{\boldsymbol{m}} - L^{\rho\sigma}\cdot \nabla H_\sigma \,,$$
$$\bar{J}_{\mathbf{m}} = -L^{\boldsymbol{m}\rho} \otimes \nabla H_\rho - L^{\boldsymbol{m}\boldsymbol{m}} : \nabla H_{\boldsymbol{m}} - L^{\boldsymbol{m}\sigma} \otimes \nabla H_\sigma \,,$$
$$\boldsymbol{J}_\sigma = -L^{\sigma\rho}\cdot \nabla H_\rho - L^{\sigma \boldsymbol{m}} : \nabla H_{\boldsymbol{m}} - L^{\sigma\sigma}\cdot \nabla H_\sigma \,,$$

where $\boldsymbol{J}_\rho$ is mass flux, $\bar{J}_{\mathbf{m}}$ is momentum flux 2-tensor, and $\boldsymbol{J}_\sigma$ is entropy flux.

For **NSF** all zero except:

$$L^{\boldsymbol{m}\boldsymbol{m}} = \bar{\bar{\Lambda}} \quad \text{and} \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T}\,.$$

$\bar{\bar{\Lambda}}$ isotropic 4-tensor, $\bar{\kappa}$ conduction 2-tensor

$$\dot{S} = (S,H;S,H) = \int_\Omega \Sigma(dH,dH) = \int_\Omega \nabla \mathbf{v} : \frac{\bar{\bar{\Lambda}}}{T} : \nabla \mathbf{v} + \nabla T \cdot \frac{\bar{\kappa}}{T^2} \cdot \nabla T \geq 0\,.$$

Note in $\bar{\kappa}/T^2$ one T from H one from $L^{\alpha\beta}$. Σ sectional curvature density?

4. Metriplectic 4-Bracket for NSF Generalizations

For **BNSF** all zero except:

$$L^{m\rho} = \tilde{D}\rho\, \boldsymbol{m}\,, \quad L^{m\sigma} = \tilde{D}\hat{\sigma}\, \boldsymbol{m}\,, \quad L^{mm} = \bar{\bar{\Lambda}} + \tilde{D}\, \boldsymbol{m} \otimes \bar{I} \otimes \boldsymbol{m}\,.$$

$$L^{\sigma\rho} = \tilde{D}\rho\hat{\sigma}\, \bar{I}\,, \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T} + \tilde{D}\hat{\sigma}^2\, \bar{I} \quad L^{\sigma m} = \tilde{D}\hat{\sigma}\, \bar{I} \otimes \boldsymbol{m}$$

$$\dot{S} = \int_\Omega \frac{1}{T}\left[\frac{\tilde{D}}{\kappa_T^2\rho^2}|\nabla\rho|^2 + \nabla T \cdot \frac{\bar{\kappa}}{T} \cdot \nabla T \; + \nabla \boldsymbol{v} : \bar{\bar{\Lambda}} : \nabla \boldsymbol{v}\right] \geq 0\,.$$

Similarly, the generalization of Reddy et al. (2019) falls out. It can be further generalized.

Final Comments

• UTA based on the metriplectic 4-bracke, a proven framework, provides a direct method for constructing thermodynamically consistent systems.

• Metriplectic 4-brackets are easy to discretize while maintaining symmetries. First numerical implementation via 4-bracket discretization (Barham et al. 2024) for 1-D Navier-Stokes-Fourier. Finite element projection of PDE to thermodynamically consistent finite-dimensional 4-bracket, i.e., ODEs. For example, for the density $\rho(x,t)$

$$\rho_h(x,t) = \sum_{i=1}^{N} \rho_i(t)\phi_i(x) \quad \rightarrow \quad \dot{\rho}_i(t) = \{\rho_i, H\} + (\rho_i, H; S, H) \ldots$$

Results use Firedrake library, implicit midpoint, Irksome module ...