# Collision operators for non-Maxwellian equilibria

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## Theory of thermodynamically consistent theories

theory = dynamical system = $\mathfrak{X}(\mathcal{Z})$

Finite dimensions $\exists$ rigor. Infinite dimensions $\exists$ wishful thinking.

# Overview

**I.** Motivation and Review

**II.** Metriplectic 4-Bracket

**III.** Unified Thermodynamic (UT) Algorithm

**IV.** Collision Operator Examples

**V.** Final Comments

# I. Motivation

# Thermodynamic Consistency – Example I

Navier-Stokes is **inconsistent:**

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{v}\,, \qquad \nabla\cdot\boldsymbol{v} = 0 \quad \Rightarrow \quad p[\boldsymbol{v}]$$

$$H = \int_\Omega \rho_0\,|\boldsymbol{v}|^2/2 \quad \text{and} \quad \dot{H} \leq 0\,, \quad \nexists \text{ any thermodynamics!}$$

Navier-Stokes-Fourier (NSF) is **consistent:** (Eckart 1940):

$$\partial_t \boldsymbol{v} = -\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathcal{T} \qquad \text{viscous stress tensor is } \mathcal{T}$$

$$\partial_t \rho = -\nabla\cdot(\rho\boldsymbol{v})$$

$$\partial_t s = -\boldsymbol{v}\cdot\nabla s - \frac{1}{\rho T}\nabla\cdot\boldsymbol{q} + \frac{1}{\rho T}\mathcal{T}:\nabla\boldsymbol{v} \qquad \text{heat flux \& viscous heating}$$

$$H = \int_\Omega \rho\,|\boldsymbol{v}|^2/2 + \rho u(\rho, s)\,, \quad \dot{H} = 0 \quad \text{and} \quad S = \int_\Omega \rho s \;\rightarrow\; \dot{S} \geq 0$$

Example of **Thermodynamic Completion**, i.e. NS $\rightarrow$ NSF.

# Thermodynamic Consistency (TC)– Example II

Landau Collision Operator:

$$\frac{\partial f}{\partial t} = \text{Vlasov} + \frac{\partial}{\partial v_i} \int w_{ij} \left[ f(v) \frac{\partial f(v')}{\partial v'_j} - f(v') \frac{\partial f(v)}{\partial v_j} \right] dv'$$

where

$$w_{ij} = (\delta_{ij} - g_i g_j / g^2) \delta(\mathbf{x} - \mathbf{x}')/g \qquad \text{with} \qquad g_i = v_i - v'_i$$

$$w_{ij}(z, z') = w_{ji}(z, z'), \quad w_{ij}(z, z') = w_{ij}(z', z), \quad g_i w_{ij} = 0$$

**Thermodynamic Consistency:**

$$H[f] = m \int dx dv \, |v|^2 f/2 + \int dx |E|^2/2 \qquad \rightarrow \qquad \dot{H} = 0$$

$$S[f] = -k_B \int dz \, f \ln \, f \qquad \rightarrow \qquad \dot{S} \geq 0$$

# Thermodynamic Consistency

The realization in a **dynamical system** of the first and second laws of thermodynamics:

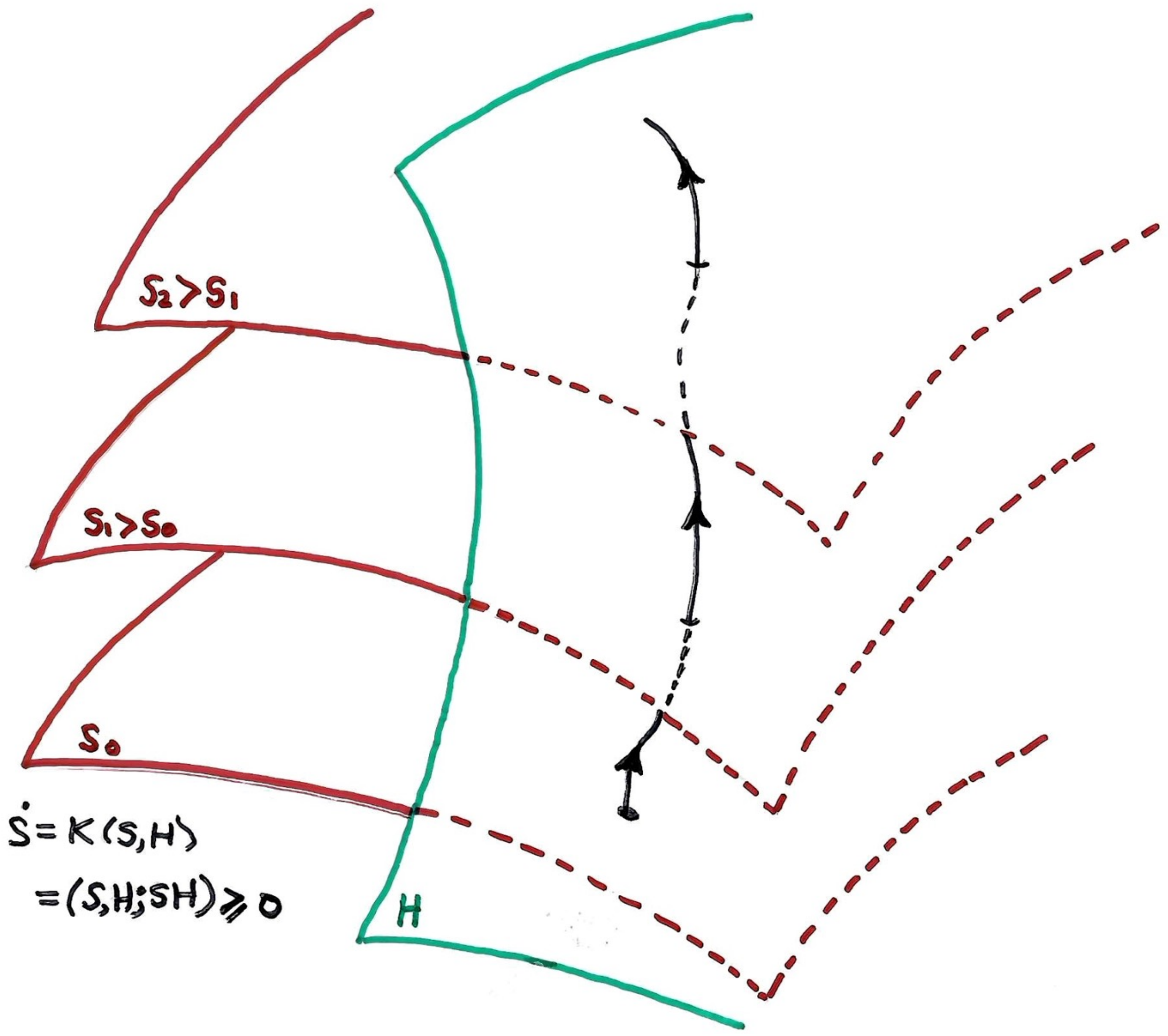
<u>First Law</u> is energy conservation:

$$\dot{H} = 0$$

<u>Second Law</u> is entropy production:

$$\dot{S} \geq 0$$

Good models lift thermodynamics to **dynamical systems.** They have two functions $H, S$.



$S_2 > S_1$
$S_1 > S_0$
$S_0$
H
$\dot{S} = K(S,H)$
$= (S,H;SH) \geqslant 0$

# Counting

Ways of arranging $N$ particles in $k$ boxes

$$W = \frac{N!}{n_1!n_2!....n_k!}$$

Boltzmann entropy:

$$S = k_B \ln W$$

Sterlings Formula:

$$S \sim -\sum_{i=1}^{N} n_i \ln n_i \qquad \rightarrow \qquad \int dxdv\, f \ln f$$

## Counting states

Pauli exclusion $E$

No Pauli exclusion $\neg E$

Distinguishable $D$

Not Distinguishable $\neg D$

| | $E$ | $\neg E$ | $D$ | $\neg D$ |
|---|---|---|---|---|
| | | X | X | |
| | | | | |
| | | | | |
| | | | | |

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| | ? | | ? | |

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|---|---|---|---|---|
| MB | | X | X | |
| FD | X | | | X |
| BE | | X | | X |
| LB | X | | X | |

MB = Maxwell-Boltzmann

FD = Fermi-Dirac

BE = Bose-Einstein

LB = Lynden-Bell ← astrophysicist

LB ∝ FD

Vlasov for stars

## Collision operator

Kadomstev & Poguste (1970)

Vlasov + fluctuations ⇒ FD

# F-D $\sim$ L-B Collision Operators

F-D well-known, Kadomstev and Pogutse (1970) obtained same for L-B collision operator.

Entropy:

$$S[f] = \int f \ln f + (1-f)\ln(1-f)$$

Collision Operator:

$$\frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial}{\partial v_i} \int w_{ij} \left[ f(v)\big(1 - f(v)\big)\frac{\partial f(v')}{\partial v'_j} - f(v')\big(1 - f(v')\big)\frac{\partial f(v)}{\partial v_j} \right] dv'$$

which is thermodynamically consistent.

Fluctuation Spectrum:

$$\langle \delta f \delta f \rangle_{k,\omega} = \delta(v - v')\delta(\omega - k \cdot v) f(1 - f)$$

# Theories & Models as Dynamical Systems

**Main Scientific Goal:**

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z)\,, \qquad \text{dynamical variable } z \in \mathcal{Z} \text{ the Phase Space}$$

Ultimately a dynamical system. Vector fields on manifolds and Cauchy problem (IVP).

Examples: Maps, ODEs, PDEs, etc. finite-dimensional, infinite-dimensional (field theories)

**Whence vector field $V$?**

- Fundamental parent theory (microscopic, $N$ interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, limits, rigorous asymptotics, Hilbert's 6th $\rightarrow$ Reduced Computable Model for $V$.

- Phenomena based modeling using known properties, constraints, symmetries, etc. used to intuit $\rightarrow$ Reduced Computable Model $V$. $\leftarrow$ Here metriplectic structure can be useful.

# Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

**How?**

# Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

**How?**

$V_{nondissipative} \equiv$ Hamiltonian and $V_{dissipative} \equiv$ ?

# What is Dissipation?

- Not all conservative systems are Hamiltonian
- Not all Hamiltonian systems are conservative
- Not all reversible systems are Hamiltonian
- All finite dynamical systems (autonomous ODEs) are reversible (1 parameter Lie group)
- Some infinite systems (PDEs) are reversible and some irreversible (group vs. semigroup)
- Not all Hamiltonian systems have time reversal symmetry
- Not all systems with time reversal symmetry are Hamiltonian
- $\exists$ systems with time reversible symmetry and asymptotic stability

**Thermodynamically Consistent Dissipation:**
Energy conserving systems with an increasing entropy that implies global asymptotic stability.

Such systems have a 'vector field' that naturally splits in Hamiltonian and dissipative parts. Hamiltonian is an unambiguous way to define nondissipative. The metriplectic 4-bracket is an unambiguous way to define dissipative. Together they $\Rightarrow$ thermodynamic consistency.

# Toward a Thermodynamically Consistent Split

$$V(z) = V_H + V_D$$

**Hamiltonian Form:**

$$V_H = \{z, H\} = J\,\frac{\partial H}{\partial z} \qquad \text{where} \quad J^T = -J$$

where $J(z)$ is Poisson tensor/operator and $H$ is the Hamiltonian. Has product decomposition.

**Dissipative Form:**

$$V_D = \ldots\ ? \qquad \to \qquad V_D = (z, G) = G\,\frac{\partial F}{\partial z} \qquad \text{where} \quad G^T = G$$

General degenerate 'metric tensor' $G$ of some kind for gradient system?

## Metriplectic Dynamics

Metric ∪ Symplectic Flows (pjm 1986) $\leftrightarrow V_D + V_H$

- Formalism for natural split of vector fields

- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.

- Other invariants? E.g., collision operators preserve, mass, momentum, .... There exists some theory for building in, but won't discuss today.

- **Encompassing 4-bracket**: Entropy is a Casimir is & "curvature" is dissipation rate

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in pjm (1984). Metriplectic in pjm (1986).

# Metriplectic 4-Bracket Dynamics

Dynamical System (finite or infinite):

$$\dot{z} = \{z, H\} + (z, H; S, H)$$

Dynamics for any observable (functional of dynamical variables), $z$, is generated by multi-linear brackets, Poisson bracket + 4-bracket (2024), with Hamiltonian $H$ and entropy = Casimir $S$.

# Hamiltonian Review

Poisson Bracket: $\{f, g\}$

# Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: $(q, p)$

Hamiltonian function: $H(q, p)$ $\leftarrow$ the energy

Equations of Motion:

$$\boxed{\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}\,, \qquad \dot{q}^\alpha = \frac{\partial H}{\partial p_i}\,,} \qquad \alpha = 1, 2, \ldots N$$

Phase Space Coordinate Rewrite: $z = (q, p)\,, \quad i, j = 1, 2, \ldots 2N$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = \{z^i, H\}_c\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

# Noncanonical Hamiltonian Structure

S. Lie (1890) $\longrightarrow$ pjm (noncanonical 1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^i = \{z^i, H\} = J^{ij}(z)\frac{\partial H}{\partial z^j}$$

Noncanonical Poisson Bracket:

$$\{f, g\} = \frac{\partial f}{\partial z^i} J^{ij}(z) \frac{\partial g}{\partial z^j}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow$ $\{f, g\} = -\{g, f\}$ $\rightarrow$ $J^{ij} = -J^{ji}$

Jacobi identity $\rightarrow$ $\{f, \{g, h\}\} + \{b, \{h, f\}\} + \{h, \{f, g\}\} = 0$ $\rightarrow$ Jacobiator $S^{ijk} = J^{i\ell}\partial_\ell J^{jk} + cyc \equiv 0$

Leibniz $\rightarrow$ $\{fh, g\} = f\{h, g\} + \{h, g\}f\,,$ $fg$ pointwise

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan
(Lie's distinguished functions!)

# Noncanonical Poisson Brackets – Flows on Poisson Manifolds

**Definition.** A Poisson manifold $\mathcal{Z}$ has bracket

$$\{\,,\,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$-bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and $\boldsymbol{d}$ exterior derivative.

$$\{f,g\} = \langle \boldsymbol{d}f, J\boldsymbol{d}g\rangle = J(\boldsymbol{d}f,\boldsymbol{d}g) = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}\,.$$

$J$ the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, $J\boldsymbol{d}H$, i.e.,

$$\dot{z}^i = J^{ij}(z)\frac{\partial H(z)}{\partial z^j}\,, \qquad \mathcal{Z}'s \text{ coordinate patch } z = (z^1,\dots,z^N)$$

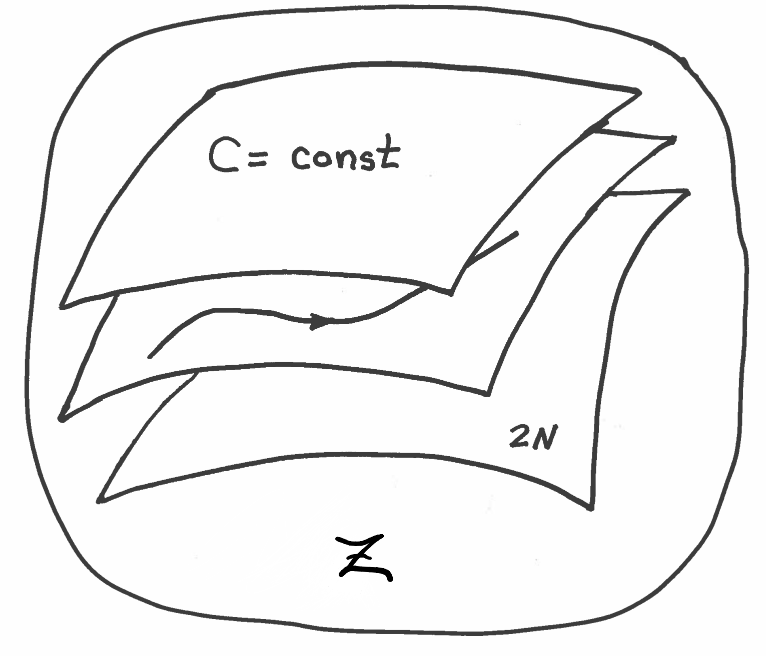
Because of degeneracy, $\exists$ functions $C$ st $\{f,C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$, called Casimir invariants. **Casimir are candidate entropies!**

# Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in $J \;\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



# Hamiltonian Structure of Vlasov-Poisson (pjm 1980)

Hamiltonian:

$$H = \frac{1}{2}\int |v|^2 f(z)\, dz + \frac{1}{2}\int |E|^2\, dx \qquad \text{where } z = (x, v) \in \mathbb{R}^6$$
$$= \frac{1}{2}\int |v|^2 f\, dz + \frac{1}{2}\int\int G(x|x')f(z)f(z')\; dz\, dz'\,,$$

Bracket:

$$\{F, G\} = \int f\left(\nabla_x F_f \cdot \nabla_v G_f - \nabla_x G_f \cdot \nabla_v F_f\right) dz = \int f\left[F_f, G_f\right] dz$$

where $F_f = \delta F/\delta f$ means functional derivative of $F$ with respect to $f$ etc.

Equation of Motion:

$$\frac{\partial f}{\partial t} = \{f, H\}\,,$$

Casimirs invariants:

$$C[f] = \int \mathcal{C}(f)\, dz\,, \qquad st \quad \{F, C\} = 0 \quad \forall F\,.$$

where $\mathcal{C}$, an arbitrary function; thus $C$ that includes entropy.

## II. Metriplectic 4-Bracket: $(f,k;g,n)$ or $(F,K;G,N)$

Finite dimensions (functions):

$$f,k,g,n \in C^{\infty}(\mathcal{Z})$$

Infinite dimensions (functionals):

$$F,K,G,N : \mathcal{B} \to \mathbb{R}$$

where $\mathcal{B}$ is some function space.

# Why a 4-Bracket?

- One slot for dynamical variables (observables), $z$.

- Two slots for two fundamental functions: Hamiltonian, $H$, and Entropy (Casimir), $S$.

- There remains one slot for $\mathcal{F}$, free energy like generator $\mathcal{F} = H - TS$. Better argument: Needed to have multilinearity.

**Comments:**

- Provides natural reductions to other bilinear & binary brackets.

- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

# The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\,\cdot\,,\,\cdot\,;\,\cdot\,,\,\cdot\,)\colon \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \times \Lambda^0(\mathcal{Z}) \to \Lambda^0(\mathcal{Z})$$

For functions $f, k, g, n \in \Lambda^0(\mathcal{Z})$ in a coordinate patch the 4-bracket has the form:

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \qquad \leftarrow \text{quadravector?}$$

• Metriplectic manifolds have both Poisson tensor, $J^{ij}$, and compatible quadravector $R^{ijkl}$, where $S$ (selected from set of Casimirs) and $H$ comes from Hamiltonian part.

A blend of my previous early ideas 1980s: Two important functions $H$ and $S$, symmetries, curvature idea, multi-brackets.

# Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$-linearity in all arguments, e.g, for $\lambda \in \mathbb{R}$

$$(f + \lambda h, k; g, n) = (f, k; g, n) + \lambda(h, k; g, n)$$

(ii) algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n)\,, \qquad (f, k; g, n) = -(f, k; n, g)\,, \qquad (f, k; g, n) = (g, n; f, k)$$

(iii) derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h$$

where as usual, $fh$ denotes pointwise multiplication.

**Symmetries of algebraic curvature** without torsion identity. **Minimal Metriplectic.**

Observation: Often see ${R^l}_{ijk}$ or $R_{lijk}$ but not $R^{lijk}$! Never 4-bracket, i.e. action on 1-forms?

# Properties – Existence – General Construction Methods

- Thermodynamic Consistency Built-in:

$$\dot{H} = \{H, H\} + (H, H; S, H) = 0 \qquad \text{and} \qquad \dot{S} = (S, H; S, H) \geq 0$$

Reduces to metriplectic 2-bracket (1984): $(F, G)_H = (F, H; G, H)$.

- For any Riemannian manifold ∃ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

- If Riemannian, entropy production rate is positive contravariant sectional curvature. For closed $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H) \geq 0\,,$$

where the second equality follows from $\sigma = \boldsymbol{d}S$ and $\eta = \boldsymbol{d}H$.

- Two methods of construction? **Kulkarni-Nomizu** (K-N) product and **Lie algebra** based. $K(\sigma, \eta) \geq 0$ automatic for K-N and easily made minimally degenerate!

# Methods of Construction

# Construction via Kulkarni-Nomizu Product

Given $\sigma$ and $\mu$, two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) $\boldsymbol{d}f, \boldsymbol{d}k$ and $\boldsymbol{d}g, \boldsymbol{d}n$, the K-N product is

$$\begin{aligned} \sigma \owedge \mu\,(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n) &= \sigma(\boldsymbol{d}f, \boldsymbol{d}g)\,\mu(\boldsymbol{d}k, \boldsymbol{d}n) - \sigma(\boldsymbol{d}f, \boldsymbol{d}n)\,\mu(\boldsymbol{d}k, \boldsymbol{d}g) \\ &+ \mu(\boldsymbol{d}f, \boldsymbol{d}g)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}n) - \mu(\boldsymbol{d}f, \boldsymbol{d}n)\,\sigma(\boldsymbol{d}k, \boldsymbol{d}g)\,. \end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \owedge \mu(\boldsymbol{d}f, \boldsymbol{d}k, \boldsymbol{d}g, \boldsymbol{d}n)\,.$$

In coordinates:

$$R^{ijkl} = \sigma^{ik}\mu^{jl} - \sigma^{il}\mu^{jk} + \mu^{ik}\sigma^{jl} - \mu^{il}\sigma^{jk}\,.$$

If $\sigma$ or $\mu$ defines inner product, then minimally degenerate, one fixed point on $H=$ constant.

Infinite dimensions: $\mu \to M\,, \quad \sigma \to \Sigma$ 'operators'.

# Lie Algebra Based Metriplectic 4-Brackets

- For structure constants $c^{kl}{}_{s}$:

$$(f, k; g, n) = c^{ij}{}_{r} c^{kl}{}_{s}\, g^{rs} \frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,.$$

Lacks cyclic symmetry, but $\exists$ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' $g^{rs}$. Dynamics does not see torsion, but manifold does.

- For $g^{rs}_{CK} = c^{rl}{}_{k}\, c^{sk}{}_{l}$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra. For $\mathfrak{so}(3)$ reproduces relaxing free rigid body (pjm 1986).

- Covariant connection $\nabla\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$. A contravariant connection $D\colon \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \to \Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = f D_\alpha \gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij} \partial f / \partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$, $f \in \Lambda^0(\mathcal{Z})$. Build 4-bracket like curvature from connection $\Rightarrow$?

## III. Unified Thermodynamic (UT) Algorithm

UT Algorithm is an algorithm or recipe for constructing metriplectic (thermodynamically consistent) systems! Akin to building Lagrangians. Applied to many systems. **So far UT Algorithm either reproduces, corrects, or extends for every case considered!**

• Cahn-Hilliard-Navier-Stokes: agrees with Anderson et al.; corrects Guo and Lin

• Brenner-Navier-Stokes: UT Algorithm produces Brenner's equations, plus corrects statements, e.g., that the results are most general.

• Generalization of Brenner-Navier-Stokes: UT Algorithm produces equations of Reddy et al. (2019). All are generalizations of Navier-Stokes-Fourier with modified dissipation.

• **Collision Operators**: Landau, Fermi-Dirac, generalization to any monotonic equilibrium, collisions in noncanonical phase space, e.g. drift kinetics, **Landau-Fisher**.

# Four Steps of the UT Algorithm

**1.** Identify dynamical variables

$$\text{Fluid} \to \boldsymbol{\xi}(x,t) = (\boldsymbol{m} = \rho\boldsymbol{v}, \rho, \sigma = \rho s) \qquad \text{or} \qquad \text{Kinetic} \to f(z,t)$$

**2.** Propose energy and entropy functionals, $H[\boldsymbol{\xi}]$ and $S[\boldsymbol{\xi}]$

$$\text{Fluid} \to H = \int dx \left( \frac{|\mathbf{m}|^2}{2\rho} + \rho U(\rho, \sigma/\rho) \right) \quad \text{and} \quad S = \int dx\, \sigma$$

$$\text{Kinetic} \to H = m \int dz\, f|v|^2/2 + \int dx\, |E|^2/2 \quad \text{and} \quad S = \int dz\, f \ln f$$

**3.** Find Poisson bracket $\{F, G\}$ for which entropy $S$ is a Casimir invariant, $\{F, S\} = 0 \,\forall\, F$

**4.** Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product via physical reasoning that **separates local thermodynamics from phenomenological quantities**, giving the EoMs as Poisson bracket + 4-bracket:

$$\partial_t \boldsymbol{\xi} = \{\boldsymbol{\xi}, H\} + (\boldsymbol{\xi}, H; S, H)$$

Result automatically **thermodynamically consistent** for **any** choices of $H$ and $S$!

# IV. Collision Operator Examples

# General 4-Bracket Collision Operator

Phase space $z = (x, v) \in \mathbb{R}^6$, density $f(z, t)$

Define operator on $w \colon \mathbb{R}^6 \to \mathbb{R}$ (at fixed time)

$$P[w]_i = \frac{\partial w(z)}{\partial v_i} - \frac{\partial w(z')}{\partial v'_i}$$

$$\begin{aligned}(F, K; G, N) \;&=\; \int dz \int dz' \, \mathcal{G}(z, z') \\ &\quad \times \, (\delta \owedge \delta)_{ijkl} \, P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l \,,\end{aligned}$$

where simplest K-N

$$(\delta \owedge \delta)_{ijkl} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \,.$$

Choose $H$ and $S$

$$\frac{\partial f}{\partial t} = (f, H; SH) = (f, S)_H = G \, \frac{\delta S}{\delta f} \quad \leftarrow \quad \text{degenerate gradient system}$$

# Landau Collision Operator

(Dropping Hamiltonian part for now.)

**Choose:** $H = \int dz f\, |v|^2/2$

$$(F,H;G,H) = (F,G)_H = \int dz \int dz' \left[\frac{\partial}{\partial v_i}\frac{\delta F}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta F}{\delta f(z')}\right] T_{ij}(z,z') \left[\frac{\partial}{\partial v_j}\frac{\delta G}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta G}{\delta f(z')}\right]$$

Reproduces metriplectic 2-bracket (gradient system) $(F,G)_H$ in pjm 1984. Let

$$T_{ij}(z,z') = w_{ij}(z,z') f(z) f(z')/2 \quad \& \quad w_{ij} = (\delta_{ij} - g_i g_j/g^2)\delta(\mathbf{x}-\mathbf{x}')/g$$

$$w_{ij}(z,z') = w_{ji}(z,z') \quad w_{ij}(z,z') = w_{ij}(z',z) \quad g_i w_{ij} = 0 \text{ with } g_i = v_i - v_i'$$

**Choose Entropy:**

$$S[f] = \int dz\, f \ln f$$

Landau Collision Operator:

$$\frac{\partial f}{\partial t} = (f,H;S,H) = (f,S)_H = G\frac{\delta S}{\delta f} \qquad \leftarrow \text{gradient system, pjm 1984}$$

# Kadomstev and Pogutse Collision Operator

Entropy:

$$S[f] = \int f \ln f + (1 - f) \ln(1 - f)$$

4-Bracket reproduces according to

$$(f, H; S, H)$$

Usual $H$.

# General Collision Operators

H-Theorem to **any** monotonic distribution.

Choose

$$T_{ij}(z,z') = w_{ij}(z,z')M(f(z))M(f(z'))/2$$

Entropy & Compatibility:

$$S[f] = \int dz\, s(f) \qquad \text{where} \quad M(f)\frac{d^2 s}{df^2} = 1$$

4-bracket gives

$$\begin{aligned}\frac{\partial f}{\partial t} &= (f,H;S,H) = (f,S)_H \\ &= \frac{\partial}{\partial v_i}\int w_{ij}\left[M(f(v))\frac{\partial f(v')}{\partial v'_j} - M(f(v'))\frac{\partial f(v)}{\partial v_j}\right] dv'\end{aligned}$$

Fluctuation Spectrum:

$$\langle \delta f \delta f \rangle_{k,\omega} = \delta(v-v')\delta(\omega - k\cdot v)M(f) \qquad \leftarrow \qquad \textbf{measurable?}$$

## Desiderata: Maxwell-Vlasov + Collisions

Desire to solve M-V with large inhomogeneous magnetic fields, $B$. Curse of dimensions and disparate time scales $\Rightarrow$ no way.

Motivates reductions: drift kinetic theory, gyrokinetic theory, ...

For example, make a kinetic theory where characteristics are drift orbits that remove the fast gyromotion in $B$. This is done by using near constancy of the magnetic moment $\mu$, which on the orbit level is an adiabatic invariant. Removes fast time and lowers the dimension.

What collisions should be used? Still Landau?

In practice codes use a variety, some better than others.

# Drift Kinetics on Noncanonical Phase Space

Drift orbits are governed by noncanonical Hamiltonians system, with Poisson tensor $J \rightarrow$ noncanonical Poisson bracket $[f,g]_{NC}$.

Noncanonical kinetic theory:

$$\frac{\partial f}{\partial t} + [f, \mathcal{E}]_{NC} = 0$$

Usual Vlasov $\mathcal{E} = v^2/2 + \phi$ and $[f,g]$ canonical. For noncanonical $[f,g]_{NC}$ PDE bracket has new Casimirs, inner Casimirs

$$\{F,G\} = \int dz\, f [F_f, G_f]_{NC}$$

$[c(z), f] = 0 \;\forall f$

$$C = \int dz\, c(z) f$$

Drift kinetic theories have broader choice 'entropies' for metriplectic formalism.

# Collision Operator on Noncanonical Phase Space

Kulkarni-Nomizu with Poisson tensor:

$$\sigma^{ij} = \mu^{ij} = J^{ij} \Rightarrow$$
$$\mathcal{R}^{ijkl} = J^{ij}J^{kl} + J^{il}J^{kj} - J^{ki}J^{jl} - J^{ji}J^{kl}$$

Metriplectic 4-bracket:

$$(F,K;G,N) = \int dz \int dz' \, f f' \, \Gamma R^{ijkl} P\left[F_f\right]_i P\left[K_f\right]_j P\left[G_f\right]_k P\left[N_f\right]_l ,$$

where Γ determined by interaction potential gives Π and

$$P\left[F_f\right]_i = \frac{\partial}{\partial z^i}\frac{\delta F}{\delta f} - \frac{\partial}{\partial z'^i}\frac{\delta F}{\delta f'} ,$$

where $z$ here is a mixture of $x$ and $v$.

• Conservation laws, entropy production, equilibria generalization of Maxwellian involving Casimirs. Reduces to Landau. ...

# Collision Operator on Noncanonical Phase Space – GC Kinetics

Why? Clusters appear and interact on shorter time scales. Drift kinetic theories and gyrokinetic theories are noncanonical.

Collision operator:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} = \mathcal{C}\,(f,f) = \frac{\partial}{\partial z}\cdot\left[f J\cdot\int f'\,\Pi\left(J'\cdot\frac{\partial \log f'}{\partial z'} - J\cdot\frac{\partial \log f}{\partial z}\right) dz'\right]$$

where $\Pi$ is a symmetric covariant (interaction) tensor, determined by type of binary interactions, and $J$ is the Poisson tensor/operator of noncanonical Poisson bracket. Here $f' = f\,(z',t)$ and $J' = J\,(z')$.

$H$-Theorem $\Rightarrow$ relaxation to

$$f_\infty = \frac{1}{Z}\exp(-\beta(\mu B_0 + \frac{1}{2}\mu^2 + q\Phi) + g(\mu)) = f^{\mathrm{MB}} e^{g(\mu)}$$

Actually fits experiments, e.g. RT-1 levitated dipole equilibria in Tokyo. On intermediate time scales $\mu \approx$ constant.

# V. Final Comments

• Metriplectic 4-bracket describes thermodynamically consistent theories. Fluids, magnetofluids, multiphase fluids, kinetic theories, ... It is rich in geometry and produces interesting dynamical systems. Tons of interesting geometry already ... more to explore.

• The UT Algorithm based on the metriplectic 4-bracket, is a proven framework, provides a direct method for constructing thermodynamically consistent systems. Useful for constructing such models, even though complicated. See refs. for lots of them. Here kinetic guiding center theory. Operator ready to be put into gyrokinetic code. Simpler version linearized.

• Metriplectic 4-brackets are easy to discretize while maintaining symmetries. First numerical implementation via 4-bracket discretization (Barham et al. 2025) for 1-D Navier-Stokes-Fourier. Finite element projection of PDE to thermodynamically consistent finite-dimensional 4-bracket, i.e., ODEs. For example, for the density $\rho(x,t)$

$$\rho_h(x,t) = \sum_{i=1}^{N} \rho_i(t)\phi_i(x) \quad \rightarrow \quad \dot{\rho}_i(t) = \{\rho_i, H\} + (\rho_i, H; S, H) \ldots$$

Results use Firedrake library, implicit midpoint, Irksome module ...