

Metriplectic Geometries: Definitions and Applications

Philip J. Morrison

*Department of Physics, Institute for Fusion Studies, and ODEN Institute
The University of Texas at Austin*

morrison@physics.utexas.edu <https://web2.ph.utexas.edu/~morrison/>

WGMP Białystok, Poland

July 1, 2025

In Memory of Arno Bohm

Metriplectic 4-Bracket: undergrad, **Michael Updike** (geometry), now Princeton grad student, grad student intern, **Azeddine Zaidni** (UT Algorithm), UM6, Marrakesh-Safi, Morocco; **Naoki Sato** (collision operators), NIFS, Nagoya, Japan, **William Barham** (numerics), UT Austin.

Other Collaborators:

G. Flierl, M. Furukawa, C. Bressan, O. Maj, M. Kraus, E. Sonnendrücker; T. Ratiu, A. Bloch, B. Coquinot & M. Materassi.

Theory of thermodynamically consistent theories. (Theory = dynamical system).

Overview

I. Motivation

II. Metriplectic 4-Bracket

III. Unified Thermodynamic (UT) Algorithm

IV. Examples

V. Final Comments

I. Motivation

Thermodynamic Consistency (TC)– Examples

Navier-Stokes is **inconsistent**:

$$\partial_t \mathbf{v} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho_0} \nabla p + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad p[\mathbf{v}]$$

$$H = \int_{\Omega} \rho_0 |\mathbf{v}|^2 / 2 \quad \text{and} \quad \dot{H} \leq 0, \quad \nexists \text{ any thermodynamics!}$$

Navier-Stokes-Fourier (NSF) is **consistent**: (Eckart 1940):

$$\partial_t \mathbf{v} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \mathcal{T}$$

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v})$$

$$\partial_t s = -\mathbf{v} \cdot \nabla s - \frac{1}{\rho T} \nabla \cdot \mathbf{q} + \frac{1}{\rho T} \mathcal{T} : \nabla \mathbf{v} \quad \text{heat flux \& viscous heating}$$

$$H = \int_{\Omega} \rho |\mathbf{v}|^2 / 2 + \rho u(\rho, s), \quad \dot{H} = 0 \quad \text{and} \quad S = \int_{\Omega} \rho s \rightarrow \dot{S} \geq 0$$

Example of **Thermodynamic Completion**, i.e. NS $\rightarrow$ NSF.

Thermodynamic Consistency

The realization in a **dynamical system** of the first and second laws of thermodynamics:

First Law is energy conservation:

$$\dot{H} = 0$$

Second Law is entropy production:

$$\dot{S} \geq 0$$

Good models lift thermodynamics to **dynamical systems**. They have two functions H, S .

Theories & Models as Dynamical Systems

Main Scientific Goal:

Predict the future or explain the past $\Rightarrow$

$$\dot{z} = V(z), \quad \text{dynamical variable } z \in \mathcal{Z} \text{ the Phase Space}$$

Ultimately a dynamical system. Vector fields on manifolds and Cauchy problem (IVP).

Examples: Maps, ODEs, PDEs, etc. finite-dimensional, infinite-dimensional (field theories)

Whence vector field V ?

- Fundamental parent theory (microscopic, N interacting gravitating or charged particles, BBGKY hierarchy, Vlasov-Maxwell system, ...). Identify small parameters, limits, rigorous asymptotics, Hilbert's 6th $\rightarrow$ Reduced Computable Model for V .
- Phenomena based modeling using known properties, constraints, symmetries, etc. used to intuit $\rightarrow$ Reduced Computable Model V . $\leftarrow$ Here metriplectic structure can be useful.

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

Vector Field Splitting

$$V(z) = V_{nondissipative} + V_{dissipative}$$

How?

$$V_{nondissipative} \equiv \text{Hamiltonian} \quad \text{and} \quad V_{dissipative} \equiv ?$$

What is Dissipation?

- Not all conservative systems are Hamiltonian
- Not all Hamiltonian systems are conservative
- Not all reversible systems are Hamiltonian
- All finite dynamical systems (autonomous ODEs) are reversible (1 parameter Lie group)
- Some infinite systems (PDEs) are reversible and some irreversible (group vs. semigroup)
- Not all Hamiltonian systems have time reversal symmetry
- Not all systems with time reversal symmetry are Hamiltonian
- $\exists$ systems with time reversible symmetry and global asymptotic stability

Thermodynamically Consistent Dissipation:

Energy conserving systems with an increasing entropy that implies global asymptotic stability.

Such systems have a 'vector field' that naturally splits in Hamiltonian and dissipative parts. Hamiltonian is an unambiguous way to define nondissipative. The Metriplectic 4-bracket is an unambiguous way to define dissipative. Together they $\Rightarrow$ thermodynamic consistency.

Toward a Thermodynamically Consistent Split

$$V(z) = V_H + V_D$$

Hamiltonian Form:

$$V_H = J \frac{\partial H}{\partial z}$$

where $J(z)$ is Poisson tensor/operator and H is the Hamiltonian. Basic product decomposition.

Dissipative Form:

$$V_D = \dots ? \quad \rightarrow \quad V_D = G \frac{\partial F}{\partial z}$$

General degenerate ‘metric tensor’ of some kind for gradient flow?

Frameworks for Dissipation – Some History

Lagrangian/Action Based: Rayleigh (1873),: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} \right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu} \right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu} \right) = 0$

Linear dissipation e.g. of sound waves. *Theory of Sound*.

Gay-Balmaz & Yoshimura (2017) & Eldred (2020).

Gradient Based: Cahn-Hilliard (1958): $\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta \mathcal{F}}{\delta n} = \nabla^2 (n^3 - n - \nabla^2 n)$

Phase separation, nonlinear diffusive dissipation, binary fluid ..

Otto, Ricci Flows, Poincarè conjecture on S^3 , Hamilton, Perelman (2002)... : $\frac{\partial \psi}{\partial t} = \mathcal{G} \frac{\delta \mathcal{F}}{\delta \psi}$

Bracket Based: Kaufman & pjm (1982, 1984), Grmela (1984), pjm (1986), Öttinger & Grmela (1997), ... : $\frac{\partial \psi}{\partial t} = (\psi, \mathcal{F}) \leftarrow$ emerges here

Plasma models, kinetic theory, fluids.

New Metriplectic 4-Bracket Based: pjm, Updike, Zaidni (2024,2025):

An encompassing theory.

Metriplectic Dynamics

(Metric $\cup$ Symplectic Flows (pjm 1986) $\leftrightarrow V_D + V_H$)

- Formalism for natural split of vector fields
- Enforces thermodynamic consistency: $\dot{H} = 0$ the 1st Law and $\dot{S} \geq 0$ the 2nd Law.
- Other invariants? E.g., collision operators preserve, mass, momentum, There exists some theory for building in, but won't discuss today.
- **Encompassing 4-bracket:** Entropy is a Casimir is & “curvature” is dissipation rate

Ideas of Casimirs are candidates for entropy, multibracket, curvature, etc. in pjw (1984).
Metriplectic in pjw (1986).

Metriplectic 4-Bracket Dynamics

Dynamical System (finite or infinite):

$$\dot{z} = \{z, H\} + (z, H; S, H)$$

Dynamics for any observable (functional of dynamical variables), z , is generated by multilinear brackets, Poisson bracket + 4-bracket (2024), with Hamiltonian H and entropy = Casimir S .

Hamiltonian Review

Poisson Bracket: $\{f, g\}$

Noncanonical Poisson Brackets – Flows on Poisson Manifolds

Definition. A Poisson manifold $\mathcal{Z}$ has bracket

$$\{, \}: C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \rightarrow C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{, \}$ is a Lie algebra realization, i.e., is

- $\mathbb{R}$ -bilinear,
- antisymmetric,
- Jacobi identity
- Leibniz, i.e., acts as a derivation $\Rightarrow$ vector field.

Geometrically $C^\infty(\mathcal{Z}) \equiv \Lambda^0(\mathcal{Z})$ and d exterior derivative.

$$\{f, g\} = \langle df, Jdg \rangle = J(df, dg) = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}.$$

J the Poisson tensor/operator. Flows are integral curves of noncanonical Hamiltonian vector fields, JdH , i.e.,

$$\dot{z}^i = J^{ij}(z) \frac{\partial H(z)}{\partial z^j}, \quad \mathcal{Z}'s \text{ coordinate patch } z = (z^1, \dots, z^N)$$

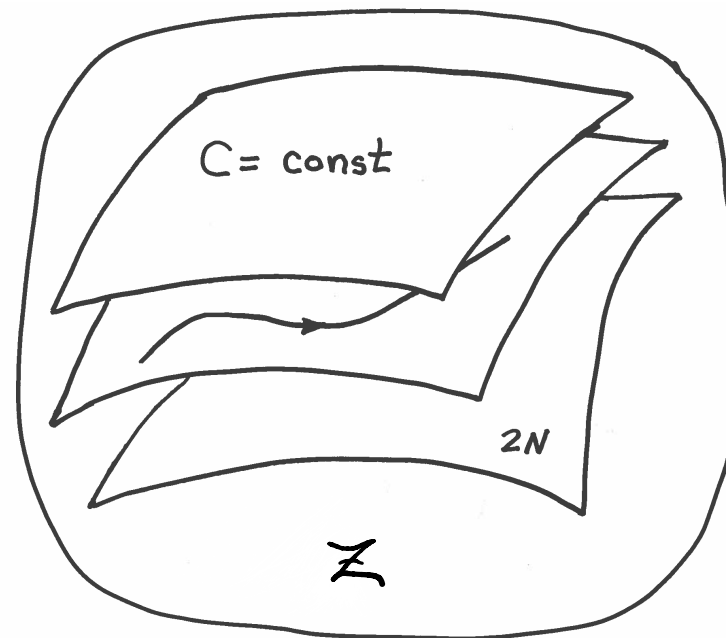
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{Z})$, called Casimir invariants. (Lie's distinguished functions!) **Casimir are candidate entropies!**

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in $J \Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall f : \mathcal{Z} \rightarrow \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (noncanonical 1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{z}^i = \{z^i, H\} = J^{ij}(z) \frac{\partial H}{\partial z^j}$$

Noncanonical Poisson Bracket:

$$\{f, g\} = \frac{\partial f}{\partial z^i} J^{ij}(z) \frac{\partial g}{\partial z^j}$$

Bilinear Poisson Bracket Properties:

antisymmetry $\rightarrow \{f, g\} = -\{g, f\} \rightarrow J^{ij} = -J^{ji}$

Jacobi identity $\rightarrow \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \rightarrow$ Jacobiator $S^{ijk} = J^{il} \partial_l J^{jk} + \text{cyc} \equiv 0$

Leibniz $\rightarrow \{fh, g\} = f\{h, g\} + \{h, g\}f, \quad fg$ pointwise

G. Darboux: $\det J \neq 0 \implies J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $\det J = 0 \implies$ Canonical Coordinates plus Casimirs $\leftarrow$ G. Sudarshan
(Lie's distinguished functions!)

Hamilton's Canonical Equations

Phase Space with Canonical Coordinates: (q, p)

Hamiltonian function: $H(q, p)$ ← the energy

Equations of Motion:

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}, \quad \dot{q}^\alpha = \frac{\partial H}{\partial p_i}, \quad \alpha = 1, 2, \dots, N$$

Phase Space Coordinate Rewrite: $z = (q, p), \quad i, j = 1, 2, \dots, 2N$

$$\dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j} = \{z^i, H\}_c, \quad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

$J_c :=$ Poisson tensor, Hamiltonian bi-vector, cosymplectic form

II. Metriplectic 4-Bracket: $(f, k; g, n)$

Why a 4-Bracket?

- One slot for dynamical variables (observables), z .
- Two slots for two fundamental functions: Hamiltonian, H , and Entropy (Casimir), S .
- There remains one slot for $\mathcal{F}$, free energy like generator $\mathcal{F} = H - TS$. Better argument: Needed to have multilinearity.

Comments:

- Provides natural reductions to other bilinear & binary brackets.
- The three slot brackets of pjm 1984 were not trilinear. Four needed to be multilinear.

The Metriplectic 4-Bracket

4-bracket on 0-forms (functions):

$$(\cdot, \cdot; \cdot, \cdot): C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \rightarrow C^\infty(\mathcal{Z})$$

For functions $f, k, g, n \in C^\infty(\mathcal{Z})$ in a coordinate patch the 4-bracket has the form:

$$(f, k; g, n) = R^{ijkl}(z) \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l}. \quad \leftarrow \text{quadravector?}$$

- Metriplectic manifolds have both Poisson tensor, J^{ij} , and compatible quadravector R^{ijkl} , where S (selected from set of Casimirs) and H come from Hamiltonian part.

A blend of my previous early ideas 1980s: Two important functions H and S , symmetries, curvature idea, multi-brackets.

Metriplectic 4-Bracket Properties

(i) $\mathbb{R}$ -linearity in all arguments, e.g, for $\lambda \in \mathbb{R}$

$$(f + \lambda h, k; g, n) = (f, k; g, n) + \lambda(h, k; g, n)$$

(ii) algebraic identities/symmetries

$$(f, k; g, n) = -(k, f; g, n), \quad (f, k; g, n) = -(f, k; n, g), \quad (f, k; g, n) = (g, n; f, k)$$

(iii) derivation in all arguments, e.g.,

$$(fh, k; g, n) = f(h, k; g, n) + (f, k; g, n)h$$

where as usual, fh denotes pointwise multiplication.

Symmetries of algebraic curvature without torsion identity. **Minimal Metriplectic.**

Observation: Often see R^l_{ijk} or R_{lijk} but not R^{lijk} ! Never 4-bracket, i.e. action on 1-forms?

Properties – Existence – General Construction Methods

- Thermodynamic Consistency Built-in:

$$\dot{H} = \{H, H\} + (H, H; S, H) = 0 \quad \text{and} \quad \dot{S} = (S, H; S, H) \geq 0$$

Reduces to metriplectic 2-bracket (1984): $(F, G)_H = (F, H; G, H)$.

- For any Riemannian manifold $\exists$ metriplectic 4-bracket. This means there is a wide class of them, but the bracket tensor does not need to come from Riemann tensor only needs to satisfy the bracket properties.

- If Riemannian, entropy production rate is positive contravariant sectional curvature. For closed $\sigma, \eta \in \Lambda^1(\mathcal{Z})$, entropy production by

$$\dot{S} = K(\sigma, \eta) := (S, H; S, H) \geq 0,$$

where the second equality follows from $\sigma = dS$ and $\eta = dH$.

- Two methods of construction? **Kulkarni-Nomizu** (K-N) product and **Lie algebra** based. $K(\sigma, \eta) \geq 0$ automatic for K-N and easily made minimally degenerate!

Methods of Construction

Construction via Kulkarni-Nomizu Product

Given σ and μ , two symmetric rank-2 tensor fields operating on 1-forms (assumed exact) df, dk and dg, dn , the K-N product is

$$\begin{aligned}\sigma \otimes \mu(df, dk, dg, dn) &= \sigma(df, dg) \mu(dk, dn) - \sigma(df, dn) \mu(dk, dg) \\ &+ \mu(df, dg) \sigma(dk, dn) - \mu(df, dn) \sigma(dk, dg) .\end{aligned}$$

Metriplectic 4-bracket:

$$(f, k; g, n) = \sigma \otimes \mu(df, dk, dg, dn) .$$

In coordinates:

$$R^{ijkl} = \sigma^{ik} \mu^{jl} - \sigma^{il} \mu^{jk} + \mu^{ik} \sigma^{jl} - \mu^{il} \sigma^{jk} .$$

If σ or μ defines inner product, then minimally degenerate, one fixed point on $H = \text{constant}$.

Infinite dimensions: $\mu \rightarrow M$, $\sigma \rightarrow \Sigma$.

Lie Algebra Based Metriplectic 4-Brackets

- For structure constants c_s^{kl} :

$$(f, k; g, n) = c_r^{ij} c_s^{kl} g^{rs} \frac{\partial f}{\partial z^i} \frac{\partial k}{\partial z^j} \frac{\partial g}{\partial z^k} \frac{\partial n}{\partial z^l}.$$

Lacks cyclic symmetry, but $\exists$ procedure to remove torsion (Bianchi identity) for any symmetric 'metric' g^{rs} . Dynamics does not see torsion, but manifold does.

- For $g_{CK}^{rs} = c_k^{rl} c_l^{sk}$ the Cartan-Killing metric, torsion vanishes automatically. Completely determined by Lie algebra. For $\mathfrak{so}(3)$ reproduces relaxing free rigid body (pjm 1986).

- Covariant connection $\nabla: \mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X}$. A contravariant connection $D: \Lambda^1(\mathcal{Z}) \times \Lambda^1(\mathcal{Z}) \rightarrow \Lambda^1(\mathcal{Z})$ satisfying Koszul identities, but Leibniz becomes $D_\alpha(f\gamma) = fD_\alpha\gamma + J(\alpha)[f]\gamma$ where $J(\alpha)[f] = \alpha_i J^{ij} \partial f / \partial z^j$ is a 0-form that replaces the term $\mathbf{X}(f)$ (Fernandes, 2000). Here $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{Z})$, $f \in \Lambda^0(\mathcal{Z})$. Build 4-bracket like curvature from connection.

III. Unified Thermodynamic (UT) Algorithm

UT Algorithm is an algorithm for constructing metriplectic systems! Applied to many systems. **So far UT Algorithm either reproduces, corrects, or extends for every case considered!**

Examples: Cahn-Hilliard-Navier-Stokes, Brenner-Navier-Stokes, Generalized Brenner-Navier-Stokes, generalization of Landau collision operator

Four Steps of the UT Algorithm

1. Identify dynamical variables defined on $\Omega \subset \mathbb{R}^3$; e.g. for NSF

$$\xi = (m = \rho v, \rho, \sigma = \rho s)$$

2. Propose energy and entropy functionals, $H[\xi]$ and $S[\xi]$; for NSF

$$H = \int_{\Omega} \frac{|m|^2}{2\rho} + \rho u(\rho, \sigma/\rho) \quad \text{and} \quad S = \int_{\Omega} \sigma$$

3. Find Poisson bracket $\{F, G\}$ for which entropy S is a Casimir invariant, $\{F, S\} = 0 \forall F$

4. Construct metriplectic 4-bracket $(F, K; G, N)$ via Kulkarni-Nomizu product by a **new method that separates local thermodynamics from phenomenological quantities**, giving the EoMs as Poisson bracket + 4-bracket:

$$\partial_t \xi = \{\xi, H\} + (\xi, H; S, H)$$

Result automatically Thermodynamically consistent!

3. For NSF Ideal Fluid Poisson Bracket Dynamics

Hamiltonian:

$$H = \int_{\Omega} \frac{\rho |\mathbf{v}|^2}{2} + \rho u(\rho, s), \quad T = \frac{\partial u}{\partial s}, \quad p = \rho^2 \frac{\partial u}{\partial \rho}.$$

Lie-Poisson Bracket (pjm-Greene, 1980):

$$\begin{aligned} \{F, G\} = & - \int_{\Omega} \mathbf{m} \cdot [F_{\mathbf{m}} \cdot \nabla G_{\mathbf{m}} - G_{\mathbf{m}} \cdot \nabla F_{\mathbf{m}}] + \rho [F_{\mathbf{m}} \cdot \nabla G_{\rho} - G_{\mathbf{m}} \cdot \nabla F_{\rho}] \\ & + \sigma [F_{\mathbf{m}} \cdot \nabla G_{\sigma} - G_{\mathbf{m}} \cdot \nabla F_{\sigma}]. \end{aligned}$$

Equations of Motion:

$$\partial_t \mathbf{v} = \{\mathbf{v}, H\} = -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla p / \rho, \quad \partial_t \rho = \{\rho, H\} = -\nabla \cdot (\rho \mathbf{v}), \quad \partial_t \sigma = \{\sigma, H\} = -\nabla \cdot (\sigma \mathbf{v}).$$

Casimir:

$$S = \int_{\Omega} \rho s = \int_{\Omega} \sigma.$$

Note: $F_{\mathbf{m}} = \delta F / \delta \mathbf{m}$, etc., functional derivatives.

4. Metriplectic 4-Bracket

Old method (early 2024): guess M and Σ .

New Method

Theorem: Order dynamical variables st

$$\begin{aligned}\partial_t \xi^\alpha &= \{\xi^\alpha, H\} + \nabla \cdot \mathbf{J}^\alpha, & \alpha &= 1, \dots, N-1, \\ \partial_t \xi^N &= \{\xi^N, H\} + \nabla \cdot \mathbf{J}^N + \mathbf{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \mathbf{Z}_\beta.\end{aligned}$$

where $\xi^N = \sigma$, the entropy density. Above splits Hamiltonian and conservative.

Then

$$\dot{S} = \int_{\Omega} \mathbf{Z}_\alpha \cdot \tilde{L}^{\alpha\beta} \cdot \mathbf{Z}_\beta =: \int_{\Omega} \dot{\sigma}^{prod} \geq 0.$$

and $\dot{H} \Rightarrow$

$$\mathbf{Z}_\alpha = \nabla H_{\xi^\alpha}, \quad \mathbf{J}^\alpha = -H_{\xi^N} \tilde{L}^{\alpha\beta} \nabla H_{\xi^\beta} = -L^{\alpha\beta} \nabla H_{\xi^\beta}.$$

which leads naturally to

$$M(dF, dG) = F_{\xi^N} G_{\xi^N}, \quad \Sigma(dF, dG) = \nabla(F_{\xi^\alpha}) \frac{L^{\alpha\beta}}{H_{\xi^N}} \nabla(G_{\xi^\beta}).$$

Important Byproduct of UT Algorithm

- Special **ordering of dynamical variables** and concomitant ‘Force-Flux’ relations of nonequilibrium thermodynamics:

$$\mathbf{J}^\alpha = L^{\alpha\beta} X_\beta \quad \rightarrow \quad \mathbf{J}^\alpha = -L^{\alpha\beta} \nabla(\delta H / \delta \xi^\beta)$$

‘Forces’: $\mathbf{X} \sim \nabla T, \nabla p, \nabla \mathbf{v}$ etc., UT Algorithm removes ambiguous selection of forces and provides definition of phenomenological coefficients, $L^{\alpha\beta}$, for dynamical variables ξ^β .

- Separates dependences on thermodynamical variables that come from internal energy U (local thermodynamic equilibrium) from those that appear in the phenomenological coefficients $L^{\alpha\beta}$. For example in the Fourier heat law entropy production expression

$$\dot{\sigma}^{prod} = \nabla T \cdot \frac{\bar{\kappa}}{T^2} \cdot \nabla T$$

one T comes from Fourier’s law $\mathbf{q} = -\bar{\kappa} \nabla T / T$ while the other comes from the phenomenological coefficient.

- Physically identify the sectional curvature

$$\dot{S} = (S, H; S, H) = K(H, S) = \int_{\Omega} \Sigma(dH, dH) = \int_{\Omega} \nabla H_{\xi^\alpha} \cdot \tilde{L}^{\alpha\beta} \cdot \nabla H_{\xi^\beta} \geq 0.$$

4. Metriplectic 4-Bracket: General and NSF

General flux expressions:

$$\begin{aligned} \mathbf{J}_\rho &= -L^{\rho\rho} \cdot \nabla H_\rho - L^{\rho m} : \nabla H_m - L^{\rho\sigma} \cdot \nabla H_\sigma, \\ \bar{\mathbf{J}}_m &= -L^{m\rho} \otimes \nabla H_\rho - L^{mm} : \nabla H_m - L^{m\sigma} \otimes \nabla H_\sigma, \\ \mathbf{J}_\sigma &= -L^{\sigma\rho} \cdot \nabla H_\rho - L^{\sigma m} : \nabla H_m - L^{\sigma\sigma} \cdot \nabla H_\sigma, \end{aligned}$$

where $\mathbf{J}_\rho$ is mass flux, $\bar{\mathbf{J}}_m$ is momentum flux 2-tensor, and $\mathbf{J}_\sigma$ is entropy flux.

For **NSF** all zero except:

$$L^{mm} = \bar{\bar{\Lambda}} \quad \text{and} \quad L^{\sigma\sigma} = \frac{\bar{\kappa}}{T}.$$

$\bar{\bar{\Lambda}}$ isotropic 4-tensor, $\bar{\kappa}$ conduction 2-tensor

$$\dot{S} = (S, H; S, H) = \int_{\Omega} \Sigma(dH, dH) = \int_{\Omega} \nabla \mathbf{v} : \frac{\bar{\bar{\Lambda}}}{T} : \nabla \mathbf{v} + \nabla T \cdot \frac{\bar{\kappa}}{T^2} \cdot \nabla T \geq 0.$$

Note in $\bar{\kappa}/T^2$ one T from H one from $L^{\alpha\beta}$. Σ sectional curvature density?

4. Metriplectic 4-Bracket for NSF Generalizations

For **Brenner NSF** all zero except:

$$\begin{aligned} L^{m\rho} &= \tilde{D}_\rho m, & L^{m\sigma} &= \tilde{D}\hat{\sigma} m, & L^{mm} &= \bar{\bar{\Lambda}} + \tilde{D} m \otimes \bar{I} \otimes m. \\ L^{\sigma\rho} &= \tilde{D}_\rho \hat{\sigma} \bar{I}, & L^{\sigma\sigma} &= \frac{\bar{\kappa}}{T} + \tilde{D}\hat{\sigma}^2 \bar{I} & L^{\sigma m} &= \tilde{D}\hat{\sigma} \bar{I} \otimes m \end{aligned}$$

$$\dot{S} = \int_{\Omega} \frac{1}{T} \left[\frac{\tilde{D}}{\kappa_T^2 \rho^2} |\nabla \rho|^2 + \nabla T \cdot \frac{\bar{\kappa}}{T} \cdot \nabla T + \nabla v : \bar{\bar{\Lambda}} : \nabla v \right] \geq 0.$$

Generalization of Brenner by **Reddy et al.** (2019) falls out. We further generalized.

V. Final Comments

- The UT Algorithm based on the metriplectic 4-bracket, is a proven framework, provides a direct method for constructing thermodynamically consistent systems.
- Tons of interesting geometry already ... more to explore.
- Metriplectic 4-brackets are easy to discretize while maintaining symmetries. First numerical implementation via 4-bracket discretization (Barham et al. 2025) for 1-D Navier-Stokes-Fourier. Finite element projection of PDE to thermodynamically consistent finite-dimensional 4-bracket, i.e., ODEs. For example, for the density $\rho(x, t)$

$$\rho_h(x, t) = \sum_{i=1}^N \rho_i(t) \phi_i(x) \quad \rightarrow \quad \dot{\rho}_i(t) = \{\rho_i, H\} + (\rho_i, H; S, H) \dots$$

Results use Firedrake library, implicit midpoint, Irksome module ...