On the Closure of the BBGKY Hierarchy

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Main Points – Motivation

- **Conventional Plasma Derivation:**

$$\boxed{N\text{-Body} \to \text{Liouville} \to \text{BBGKY} \to \text{Vlasov or Vlasov-Landau-Lenard-Balescu (VLLB)}}$$

- **'Approximations'**:

$$\boxed{\text{largeness of the plasma parameter } g \text{ and Bogoliubov assumption}}$$

Recall g is # of particles in a Debye sphere and Bogo's assumption is about correlation decay.

- **Questions**:
  - ∘ g contains λ_D, which contains a temperature T. Whose T is that? Is the theory really only valid near thermal equilibrium? Unlike Boltzmann for gas, $\exists$ no basic length scale.
  - ∘ Is Bogoliubov's assumption correct? Can it be tested numerically? **Work in progress**.

- **Curse of dimensionality**: 13 dimensional PDE. Here propose 1D disk model to overcome.

- **Hamiltonian structure**: Provides invariants for quantification and maybe a numerical algorithm.

Outline

- Motivation
- Review of Vlasov and collision opperater derivation
- The plasma disk model and disk plasma physics
- Some disk Vlasov numerical computations. BBGKY work in progress.
- Final Comments

Relativistic N-Body Action Principle

Dynamical Variables: $\boldsymbol{q}_i(t), \phi(\boldsymbol{x},t), A(\boldsymbol{x},t)$, where $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}/c\,, \boldsymbol{B} = \nabla \times \boldsymbol{A}$

Action Principle:

$$S[q,\phi,A] = -\sum_{i=1}^{N}\int dt\;\, mc^2\sqrt{1-\frac{|\dot{\boldsymbol{q}}|_i^2}{c^2}} \qquad \longleftarrow \text{ptle kinetic energy}$$

$$\text{coupling} \longrightarrow \qquad -\,e\int dt\sum_{i=1}^{N}\int dx\left[\phi(\boldsymbol{x},t)+\frac{\dot{\boldsymbol{q}}_i}{c}\cdot\boldsymbol{A}(\boldsymbol{x},t)\right]\delta\left(\boldsymbol{x}-\boldsymbol{q}_i(t)\right)$$

$$\text{field contribution} \longrightarrow \qquad +\frac{1}{8\pi}\int dt\int dx\left[|\boldsymbol{E}|^2(\boldsymbol{x},t)-|\boldsymbol{B}|^2(\boldsymbol{x},t)\right].$$

Variations:

$$\frac{\delta S}{\delta \boldsymbol{q}^i(t)} = 0 \quad \Rightarrow \quad \text{Eqs. of Motion with Fields}\,; \qquad \frac{\delta S}{\delta\phi(\boldsymbol{x},t)} = 0\,, \frac{\delta S}{\delta \boldsymbol{A}(\boldsymbol{x},t)} = 0 \quad \Rightarrow \quad \text{Maxwell Eqs. with Sources}$$

Therefore, plasmas at a fundamental level are Hamiltonian.

All done? Useful? Develop approximations!

Liouville's Equation

Replace dynamical system of N interacting bodies by Liouville dynamics for N-particle distribution function $F^{(N)}(\mathbf{q_1}...\mathbf{q}_N, \mathbf{p_1}...\mathbf{p_N}, t)$ defined on $6N$-dimensional phase space with coordinates $(\mathbf{q_1}...\mathbf{q}_N, \mathbf{p_1}...\mathbf{p_N})$

Liouville's equation:

$$\frac{\partial F^{(N)}}{\partial t} + \sum_{i=1}^{N} \frac{\mathbf{p}_i}{m_i} \cdot \frac{\partial F^{(N)}}{\partial \mathbf{q}_i} + \sum_{i=1}^{N} \boldsymbol{\mathcal{F}}_i \cdot \frac{\partial F^{(N)}}{\partial \mathbf{p}_i} = 0\,, \qquad \text{or} \qquad \frac{\partial F^{(N)}}{\partial t} + \left[\mathcal{H}, F^{(N)}\right] = 0$$

Here the i^{th} particle has mass m_i and $\boldsymbol{\mathcal{F}}_i$ the total force acting on the i^{th} particle

$$\boldsymbol{\mathcal{F}}_i = \sum_{j=1, j\neq i}^{N} \boldsymbol{\mathcal{F}}_{i,j} = -\sum_{j=1, j\neq i}^{N} \frac{\partial V_{ij}}{\partial \mathbf{q}_i}, \qquad V_{ij} \quad \leftarrow \text{ interaction potential}\,.$$

Or [,] is the **particle** Poisson bracket and the **particle** Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{N} \frac{|\boldsymbol{p}_i|^2}{2m_i} + \sum_{i=1, j\neq i}^{N} V_{ij}(\boldsymbol{q}_i, \boldsymbol{q}_j)$$

BBGKY Hierarchy

Marginal Distributions:

$$F^{(s)}(\mathbf{q}_1 ... \mathbf{q}_s, \mathbf{p}_1 ... \mathbf{p}_s, t) = \int d\mathbf{q}_{s+1} .. d\mathbf{q}_N, d\mathbf{p}_{s+1} .. d\mathbf{p}_N \;\; F^{(N)}(\mathbf{q}_1 ... \mathbf{q}_N, \mathbf{p}_1 ... \mathbf{p}_\mathbf{N}, t),$$

Hierarchy of Marginals:

$$\frac{\partial F^{(s)}}{\partial t} + \sum_{i=1}^{s} \frac{\mathbf{p}_i}{m_i} \cdot \frac{\partial F^{(s)}}{\partial \mathbf{q}_i} + \sum_{i=1}^{s} \sum_{j=1, j \neq i}^{s} \boldsymbol{\mathcal{F}}_{i,j} \cdot \frac{\partial F^{(s)}}{\partial \mathbf{p}_i} = -(N-s) \sum_{i=1}^{s} \int \boldsymbol{\mathcal{F}}_{i,s+1} \cdot \frac{\partial F^{(s+1)}}{\partial \mathbf{p}_i} d\mathbf{q}_{s+1} d\mathbf{p}_{s+1}.$$

First and Second:

$$\frac{\partial F^{(1)}}{\partial t} + \frac{\mathbf{p}_1}{m_1} \cdot \frac{\partial F^{(1)}}{\partial \mathbf{q}_1} = -(N-1) \int \boldsymbol{\mathcal{F}}_{1,2} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{p}_1} d\mathbf{q}_2 d\mathbf{p}_2,$$

$$\frac{\partial F^{(2)}}{\partial t} + \frac{\mathbf{p}_1}{m_1} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m_2} \cdot \frac{\partial F^{(2)})}{\partial \mathbf{q}_2} + \boldsymbol{\mathcal{F}}_{1,2} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{p}_1} + \boldsymbol{\mathcal{F}}_{2,1} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{p}_2} = -(N-2) \int \left[\boldsymbol{\mathcal{F}}_{1,3} \cdot \frac{\partial F^{(3)}}{\partial \mathbf{p}_1} + \boldsymbol{\mathcal{F}}_{2,3} \cdot \frac{\partial F^{(3)}}{\partial \mathbf{p}_2} \right] d\mathbf{q}_3 d\mathbf{p}_3.$$

$\boxed{\textbf{How to close the system?}}$

Closure and Bogoliubov's Assumption

Introduce Correlations $\Delta F^{(2)}$:

$$F^{(2)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,t) = F^{(1)}(\mathbf{q}_1,\mathbf{p}_1,t)F^{(1)}(\mathbf{q}_2,\mathbf{p}_2,t) + \Delta F^{(2)}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2,t)$$

$$\begin{aligned} F^{(3)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,t) = {} & F^{(1)}(\mathbf{q}_1,\mathbf{p}_1,t)F^{(1)}(\mathbf{q}_2,\mathbf{p}_2,t)F^{(1)}(\mathbf{q}_3,\mathbf{p}_3,t) \\ & + F^{(1)}(\mathbf{q}_1,\mathbf{p}_1,t)\,\Delta F^{(2)}(\mathbf{q}_2,\mathbf{p}_2,\mathbf{q}_3,\mathbf{p}_3,t) + F^{(1)}(\mathbf{q}_2,\mathbf{p}_2,t)\,\Delta F^{(2)}(\mathbf{q}_3,\mathbf{p}_3,\mathbf{q}_1,\mathbf{p}_1,t) \\ & + F^{(1)}(\mathbf{q}_3,\mathbf{p}_3,t)\,\Delta F^{(2)}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2,t) \quad + \quad \mathcal{O}(g^2) \end{aligned}$$

$\Rightarrow$ closed system of evolution equations for $\Delta F^{(2)}$!

Bogoliubov's assumption:

$$\Delta F^{(2)}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2,t) \;\rightarrow\; \tilde{\Delta} F^{(2)}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2,t) \qquad \leftarrow \qquad \text{Rapidly!}$$

where

$$\tilde{\Delta} F^{(2)}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2,t) = \tilde{\Delta} F^{(2)}(\mathbf{q}_1,\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_2|F^{(1)}(\mathbf{q},\mathbf{p},t))$$

Thus at each t, $\Delta F^{(2)}$ is uniquely determined by the instantaneous form of $F^{(1)}(\mathbf{q},\mathbf{p},t)$.

This time scale separation leads to dissipation.

Hamiltonian Systems and Closures

Wave equation as canonical Hamiltonian system:

$$\{F,G\} = \int dx \left(\frac{\delta F}{\delta \psi}\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta \psi}\frac{\delta F}{\delta \pi}\right), \qquad H = \int dx \left(\pi^2 + \psi_x^2\right)/2\,, \qquad \text{No Casimirs}$$

$$\partial_t \pi = \{\pi, H\} = \psi_{xx} \quad \text{and} \quad \partial_t \psi = \{\psi, H\} = \pi$$

Vlasov-Poisson as noncanonical Hamiltonian system:

$$\{F,G\} = \int dxdv\, f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right], \qquad H = \int dxdv \left(fv^2 + E^2\right)/2\,, \qquad C = \int dxdv\, \mathcal{C}(f)$$

$$\partial_t f = \{f, H\} = -v \cdot \nabla f - E \cdot \partial_v f$$

Liouville as noncanonical Hamiltonian system:

$$\{F,G\} = \int \prod_i d\boldsymbol{q}_i d\boldsymbol{p}_i\; F^{(N)} \left[\frac{\delta F}{\delta F^{(N)}}, \frac{\delta G}{\delta F^{(N)}}\right], \qquad H = \int \prod_i d\boldsymbol{q}_i d\boldsymbol{p}_i \mathcal{H} F^{(N)}\,, \qquad C = \int \prod_i d\boldsymbol{q}_i d\boldsymbol{p}_i\, \mathcal{C}(F^{(N)})$$

$$\partial_t F^{(N)} = \{F^{(N)}, H\} = [F^{(N)}, \mathcal{H}]$$

Pre-Bogoliubov as noncanonical Hamiltonian system:

$\exists$ a bracket in terms of $F^{(1)}$ and $\Delta F^{(2)}$, with H, that has two Casimir families.
Such invariants are useful for checking computations. This system sufficient to derive LLB.

If Bogoliubov is correct, then VLLB behavior happens within a Hamiltonian system.

The Curse of Dimensionality

• In 3-D space the equation for time evolution of this two-point distribution function $F^{(2)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2, t)$ would be 13-dimensional (one time dimension plus the 12-dimensional two-particle phase space).

• In 1-D space the equation for the two-point (actually two-sheet) distribution would be 5-dimensional (time plus 4-dimensional two-sheet phase space). Possibility.

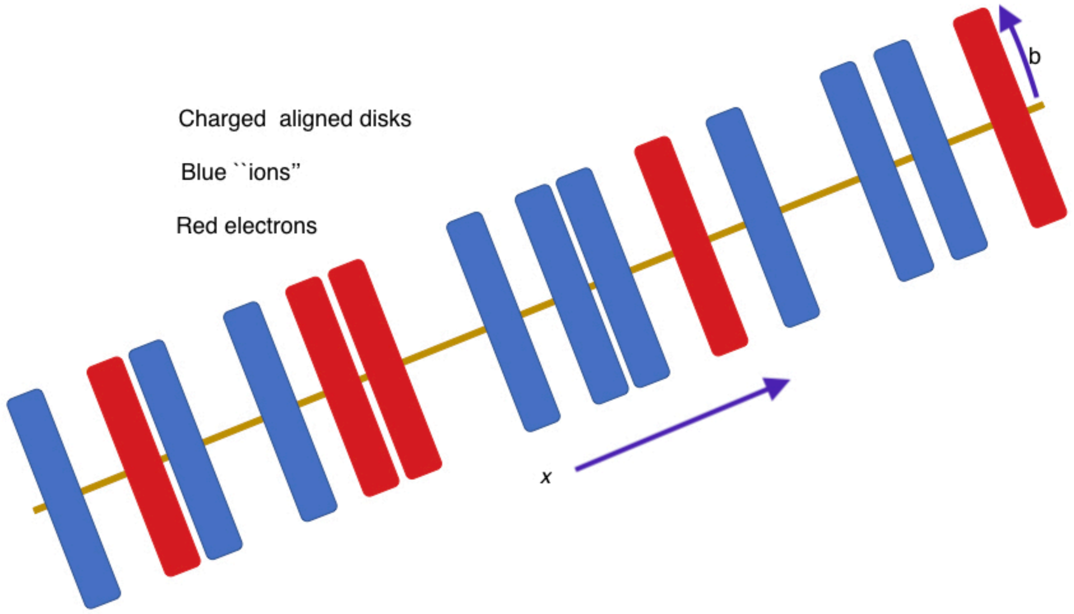
• However in 1-D the electric field generated by a charge sheet does not decay with distance so that the two-sheet interaction energy diverges at infinity. This makes it unclear how to define a finite correlation length and thus to define binary collisions between sheets.

Note that charged sheets can be consistently treated within the collisionless Vlasov equation, but in this 1-D case, contrary to what is envisioned here, the limit of zero collisionality is taken (or the collision operator is derived in the weak collisionality limit) first, reducing the problem to a single particle 6-dimensional particle phase space, and then the 1-D limit of the resulting equation is considered.

A Plasma of Charged Disks

Introduce a interaction potential that behaves as a 1D sheet at short distances and decays as the inverse of the distance at large distances. Any interaction potential of this type introduces a **length scale** *that is not present in the Coulomb interaction.*

An example is the interaction of aligned, uniformly charged disks (an abacus) where the characteristic length is the disk radius b.



In this case the electrostatic interaction energy between two charged disks is finite both at zero and at infinite separation ξ and its absolute value is a decreasing function of ξ.

Redo All of Plasma Physics for Plasma of Disks

Coulomb interaction $\to$ disk interaction, Coulomb Green's $\to$ disk Green's function

Thus the interaction energy $\mathcal{V}(x_1, x_2)$ between two uniformly charged, infinitely thin disks of radius b located at x_1 and x_2 and charge Q_1 and Q_2 respectively can be written as

$$\mathcal{V}(x_1, x_2) = 4\frac{Q_1 Q_2}{b} V(\xi) = Q_1 \varphi_2 = Q_2 \varphi_1 \,,$$

where

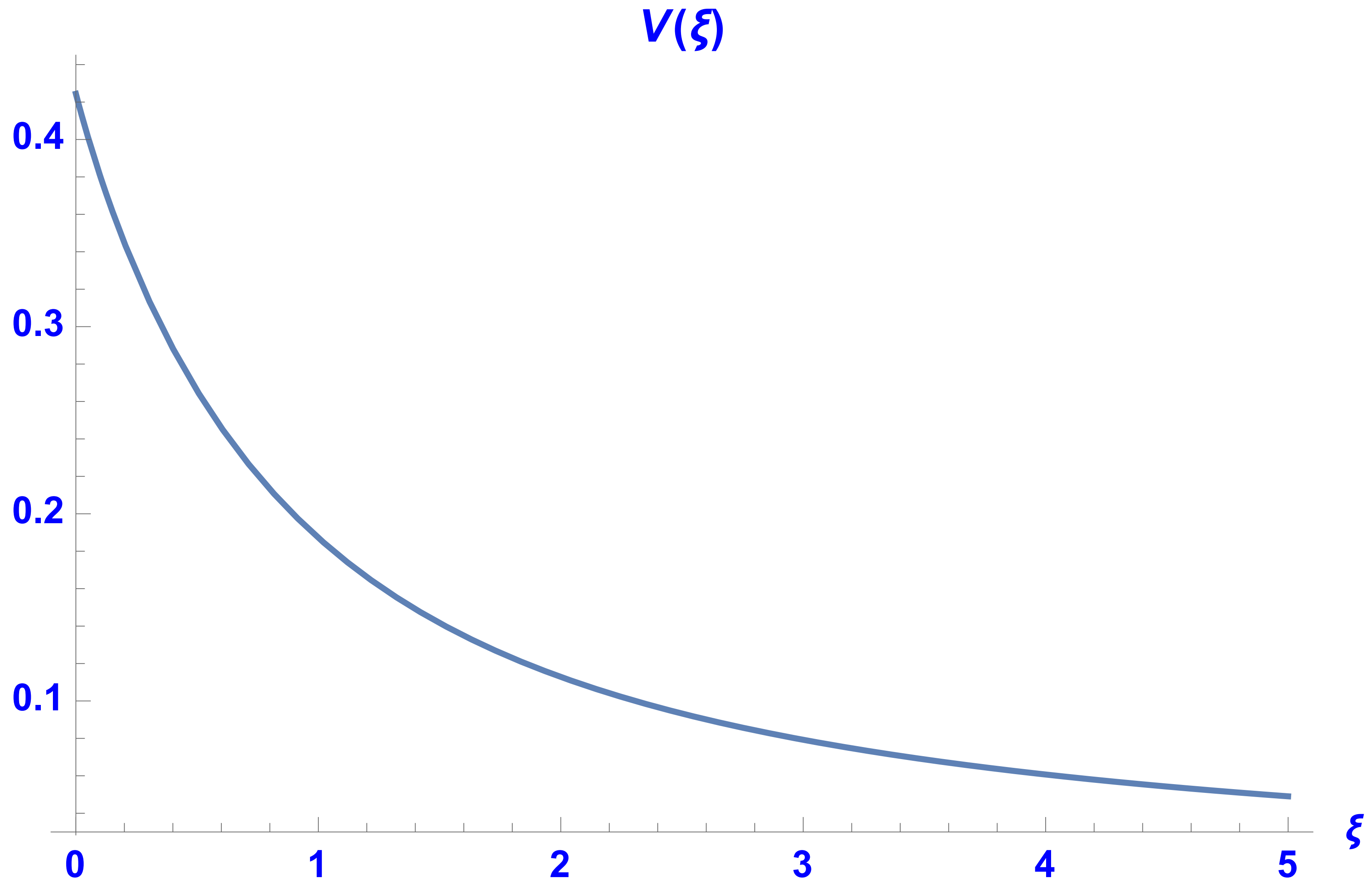
$$V(\xi) = -\frac{\xi}{2}\left\{1 - \frac{1}{3\pi}\left[(4-\xi^2)E(-4/\xi^2) + (4+\xi^2)K(-4/\xi^2)\right]\right\},$$

Here φ_1, φ_2 are the potentials generated by the disks 1, 2, $\xi(x_1, x_2) = |x_1 - x_2|/b$, $Q_1 = \pi b^2 \sigma_1$ and $Q_2 = \pi b^2 \sigma_2$ are the (fixed) charges of the two disks, $\sigma_{1,2}$ are their surface densities, $E(-4/\xi^2)$ and $K(-4/\xi^2)$ are elliptic integrals.

Consequences:

Poisson's equation $\to$ disk Poisson equation, plasma oscillations $\to$ disk plasma oscillations (cold, warm, kinetic), ion acoustic waves (linear and nonlinear) $\to$ disk ion acoustic waves (linear and nonlinear), Vlasov-Poisson dynamics $\to$ disk Vlasov-Poisson dynamics, disk Landau damping ....

Disk Interaction Potential



Note the $1/(4\xi)$-behavior for large ξ and the finite value $4/(3\pi)$ at $\xi = 0$.

Cold Disk Plasma Waves

Can derive the linear dispersion relation for cold disk plasma waves as usual.

Linearization of static equilibrium with density n_0 and immobile ion-disks yields

$$\omega^2 = \omega_{Dpe}^2\,[(kb)^2\,\widehat{V}(kb)], \quad \text{where} \quad \omega_{Dpe}^2 = \frac{4Q^2 n_0}{M_e b^2}, \quad \widehat{V} \text{ FT of V.}$$

The disk-plasma waves are dispersive for finite values of kb.

Short wavelength limit $kb \gg 1$, waves become Langmuir waves with frequency ω_{Dpe}, where $n_0/(\pi b^2)$ has dimension 'volume' density.

Long wavelength limit obtain (logarithmically corrected) "cold electron-disk sound" waves *

$$\omega^2 = k^2[C_1 + C_2 \ln{(kb)^2}]v_D^2\,,$$

where $v_D^2 = b^2\omega_{Dpe}^2 = \frac{4Q^2 n_o}{M_e}$ and C_1 and C_2 are known constants.

Note that in a Coulomb plasma there is no velocity directly corresponding to v_D.

*Not to be confused with ion sound waves, which we have done.

Disk Vlasov equation

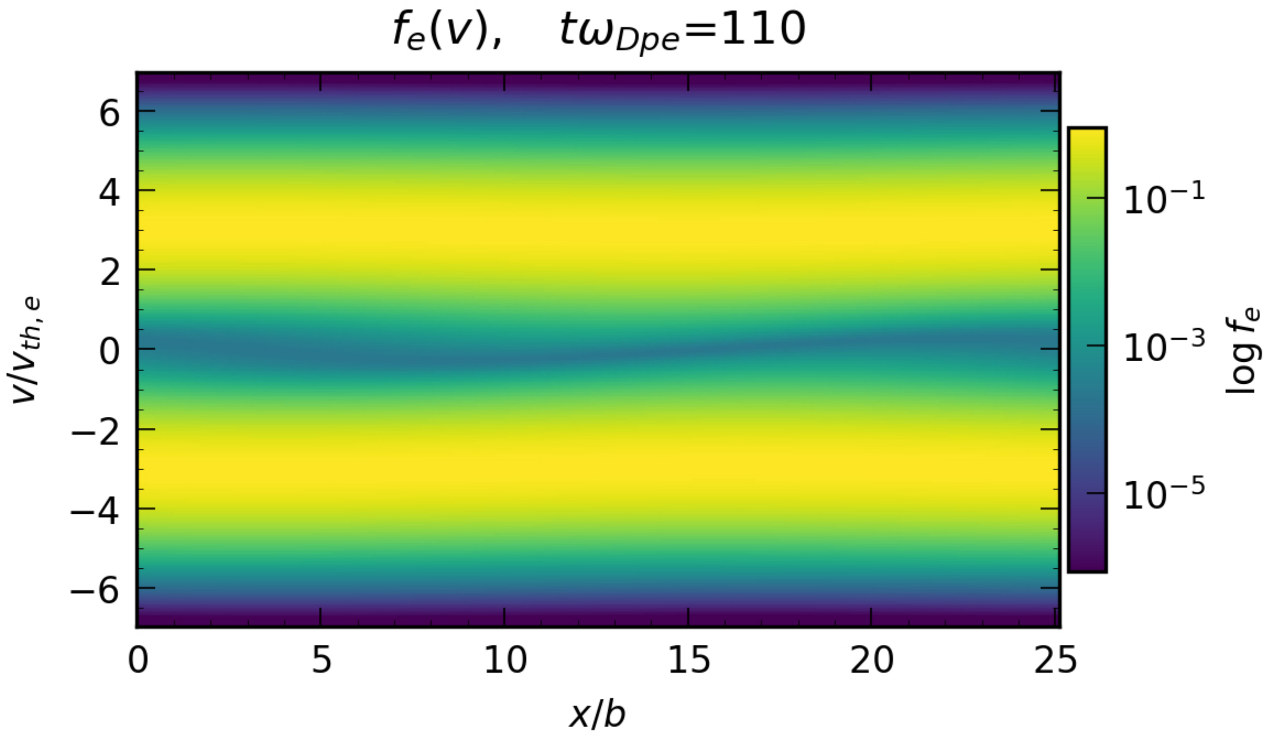
Replace Coulumb interaction by the disk interaction potential $V(\xi)$. For two species

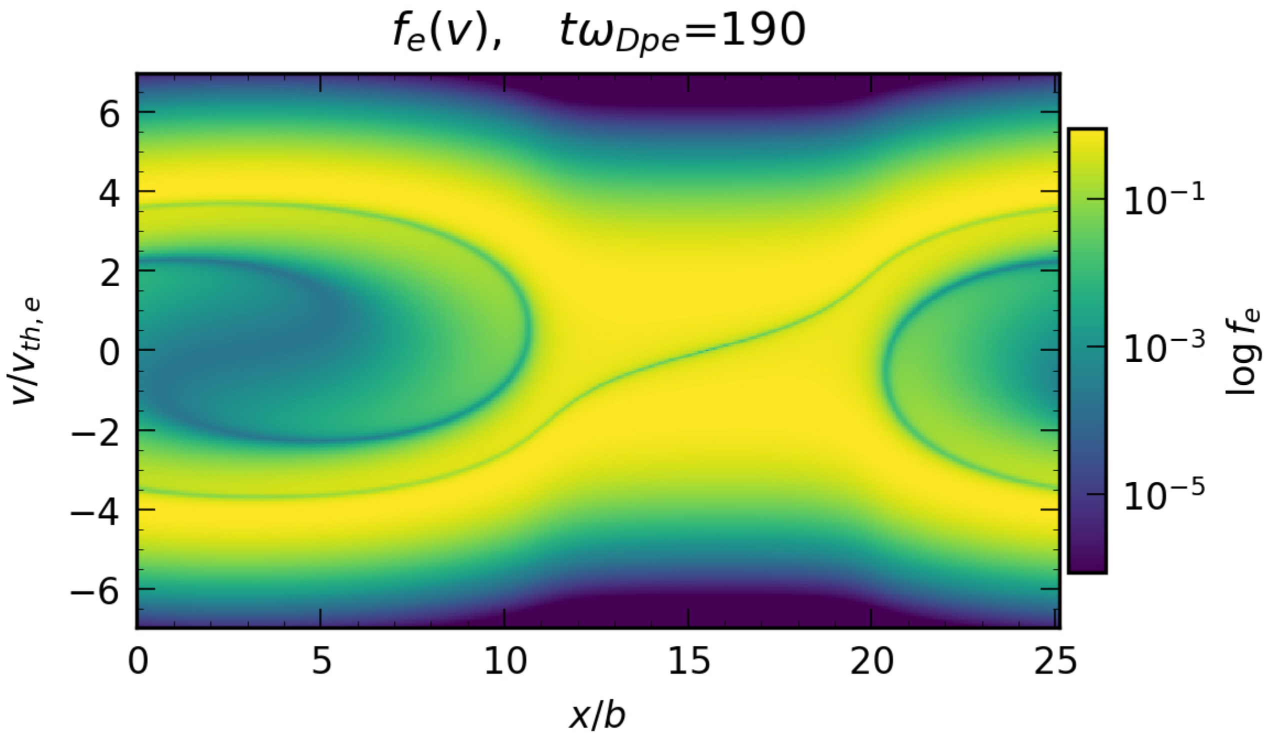
$$\frac{\partial f_j(x,v,t)}{\partial t} + v\frac{\partial f_j(x,v,t)}{\partial x} + \frac{\mathcal{F}_j(x,t)}{M_j}\frac{\partial f_j(x,v,t)}{\partial v} = 0,$$

where $f_j(x,v,t)$ is the distribution function of the j species $(j = e, i)$, v is the disk velocity coordinate in phase space and $\mathcal{F}_j(x,t)$ is the electric force due to the disk density

$$\rho(x,t) = Q\left[\int_{-\infty}^{+\infty} f_i(x,v,t)\,dv - \int_{-\infty}^{+\infty} f_e(x,v,t)\,dv\right].$$

Two Stream Instability and Formation of Electron Holes





Finite amplitude disk plasma waves

1D BBGKY Hierarchy (work in progress)

Use disk interaction potential in the time evolution equation for 1-point function $F^{(1)}(x_1, v_1, t)$ coupled to 2-point function of $F^{(2)}(x_1, v_1, x_2, v_2, t)$

$$\frac{\partial F^{(1)}(x_1, v_1, t)}{\partial t} + v_1 \frac{\partial F^{(1)}(x_1, v_1, t)}{\partial x_1}$$
$$- \frac{2}{M} \frac{4Q^2}{b} \frac{\partial}{\partial v_1} \int_{-\infty}^{+\infty} dx_s \frac{\partial V(|x_1 - x_s|/b)}{\partial x_1} \int_{-\infty}^{+\infty} dv_s\, F^{(2)}(x_1, v_1, x_s, v_s, t) = 0.$$

Similarly for the $\Delta F^{(2)}$ equation with closure.

The numerical project uses Eulerian integration of the disk kinetic equations for $F^{(1)}(x_1, v_1)$ and $\Delta F^{(2)}(x_1, x_2, v_1, v_2, t)$ in 4-D phase space using a method based on a multi-advection in phase space.

• A. Mangeney F. Califano, C. Cavazzoni, *A numerical scheme for the integration of the Vlasov - Maxwell system of equations, J. Comp. Physics 179 495 (2002) .*

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