Introduction to Hamiltonian Chaos

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Magnetic Field as Hamiltonian System

Consider the simple configuration of a current ring. It is well known that the magnetic field lines near the ring are little curls.

\[ \mathbf{B} = \mathbf{B}_T + \mathbf{B}_p \]

Field lines are helical and lie on nested toroidal surfaces.
Suppose \( a/R \ll 1 \implies \text{straight torus} \)

on periodic cylinder

\[ z = 0 \Leftrightarrow z = 2\pi \]

\[ B = B_0 \hat{z} + B_\rho = B_0 \hat{z} + \nabla \times A_\rho \]

\[ A_\rho = -\psi \hat{z} \]

\[ B = B_0 \hat{z} + \hat{z} \times \nabla \psi \]

Field lines

\[ \frac{B_z}{dz} = \frac{B_\theta}{r d\theta} = \frac{B_r}{dr} \]

\[ \nabla \psi = \frac{1}{r} \frac{d\psi}{dr} \]

\[ \frac{d\psi}{dz} = -\frac{1}{r} \frac{d\psi}{d\theta} \]
Define \( P_\theta = \frac{r^3}{2} \)

\[
\frac{d\theta}{dz} = \frac{\delta \psi}{\delta P_\theta} \quad \frac{dP_\theta}{dz} = -\frac{\delta \psi}{\delta \theta}
\]

Hamilton's Equations

\[
\begin{align*}
\mathfrak{F} & \quad \text{plays the role of time} \\
\psi & \quad \text{" Hamiltonian"
}\end{align*}
\]

Before using this simple system to introduce a couple of ideas - let me remind you that for the configuration described \( B \neq 0 \implies \psi(r) = H_0(P_\theta) \) only

The functional form of this depends upon the current dist. inside

\[ J = J(r) \]

We will be interested in one where

\[ H_0 \propto r^4 \]

although the form in not that important \( \implies \]

\[ H_0 = \frac{P_\theta^2}{2m} \]

some constant.
Two Ideas

1. Surface of Section

If we consider the $\theta - \phi$ plane at $z = 0$ and plot the path where the field line intersects we get a set of points. (pseudoc plot)

$\nabla \cdot B = 0 \Rightarrow \text{Area preserving Ronja section}$

Picture 1

There are two types of surfaces depending on the pitch of the helix. One important quantity in the rotational transverse is the rotational change in $\theta$. Roughly speaking, it is the change in $\theta$ after one turn:

$\Delta \theta \sim L$

More precisely:

$$\lim_{n \to \infty} \frac{(\Delta \theta)}{(\Delta z)^n} = \frac{L}{2\pi} \Rightarrow t = \left( \frac{\kappa}{2\pi} \right)$$
The quantity that characterizes the surface. From the picture we observe that there are two types of surfaces:

(i) toroidal ⇒ the field line closes upon itself ⇒ fixed points;
   i.e., an entire surface of area.

(ii) spheroid ⇒ ergodic coverage of surface.
2. Integrability

The picture we have painted so far describes what is referred to as an integrable Hamiltonian system. This is the exceptional case, usually all the field lines don't close upon themselves in a nice way. Consider the following def.

Integrable

An $n$ degree of freedom Ham. System is integrable if $\exists$ $m$ indep. cons., in involution

$$[F_i, H] = 0 \quad [F_i, F_j] = 0 \quad i, j = 1, \ldots, n$$

If the surface in $\mathbb{R}^d$ phase space is connected & compact $\Rightarrow$ phase space foliated by invariant $n$-tori.

1. Action-Angle coordinates

$$(q, p) \rightarrow (J, \theta)$$

$$H(q, p) = H(J)$$

$${\dot{J}} = 2H = \sum \omega_i(J) \dot{\theta} \quad {\dot{\theta}} = 2H = 0$$

$${\dot{J}} = \text{const}; \quad {\dot{\theta}} = \omega_0 t + \Theta_0$$
Let us now return to the way field problems and add a new twist to the configuration. Suppose e.g. that the coils that create $B_\text{r}$ are not perfect. The windings may have a helical pitch, with misalignment and gaps too. We will represent this "misp" by adding a piece to the vector potential, which recall is the Hamiltonian:

$$\Psi_\text{R} = \sum_{m,n} \Psi_{m,n}(r) \cos(m\theta - n\phi)$$

$$\Rightarrow \quad \vec{\Psi}_\text{H} = P_0$$

Non perfectini

$$H = H_0 + H_1$$

$$H = \frac{P_0^2}{2m} + \sum_{m,n} \Psi_{m,n}(P_0) \cos(m\theta - n\phi)$$

This is a form for a typical, i.e. non-integrable, system

Pict sequence 2 $\Rightarrow$ Slides
The goal of perturbation theory is to transfer to a new system of canonical variables where the Hamiltonian becomes ignorable, i.e., doesn't depend upon the count.

For example,

\[ H(p, q) = H_0(p) + \varepsilon H_1(p, q) \]

In action-angle variables,

\[ (J, \theta) \leftrightarrow (p, q) \]

we want

\[ H(p, q) = \bar{H}(J) \]

\[ \Rightarrow \quad \frac{\partial \bar{H}}{\partial J} = \dot{\theta} = \omega_J \]

\[ \Rightarrow \quad \frac{\partial \bar{H}}{\partial \theta} = \dot{J} = 0 \]

\[ \Rightarrow \quad \oint \dot{\theta} = \omega_J J = t + C_0 \]

\[ J = \text{const} \]
One way of generating canonical trans is via a mixed variable generating function

\[ S(q, J) \]

A new mom.

The trans eq. are

\[ p = \frac{\partial S}{\partial q} \quad \theta = \frac{\partial S}{\partial J} \]

Plus

\[ H \left( \frac{\partial S}{\partial q}, q \right) = \tilde{H}(J) \]

Eq. for \( S \); Hamilton-Jacobi Eq.

\[ H \downarrow \text{small} \Rightarrow \text{mean identity} \]

\[ S = q \cdot J + \varepsilon S_1(q, J) + \ldots \Rightarrow \]

\[ H_0(J + \varepsilon \frac{\partial S_1}{\partial q} + \ldots) + \varepsilon H_1(J + \ldots, q) = \tilde{H}(J) \]
to order $\varepsilon$

$$H_0(J) + \varepsilon \frac{dH_0(J)}{dJ} \frac{2S_1}{dJ} + \varepsilon H_1(J, q)$$

$$= \bar{H}(J) = H_0(J) + H_1(J, q)$$

$$H_1 = \sum_{m} H_m^{(1)}(q) \, e^{i \, m \cdot q}$$

$$S_1 = \sum_{m} S_m^{(1)}(J) \, e^{i \, m \cdot q}$$

$$\Rightarrow \quad S(q, J) = q \cdot J + \varepsilon \sum_{m \neq 0} \frac{H_m^{(1)}(J) \, e^{i \, m \cdot q}}{m \cdot \omega_0}$$

m-khice

Weierstrass

Poincare

Kolmogorov

Arnold

Moser

Converges

W. Saffman

Green

Small divisor
\[ \omega_0 = (\omega_{01}, \omega_{02}) \]

\[ \omega = \frac{\omega_{01}}{\omega_{02}} \]

\[ \left| \omega - \frac{1}{5} \right| < \frac{1}{5} \alpha \]

\[ \alpha \approx 2.5 \]

Continued fraction expansion:

\[ \left| \omega - \frac{m}{n} \right| < \frac{1}{n^2} \text{ for all } n \]

Had one to approach closure.
Surface of Section - Integrable

\[ P_0 \text{ (radius)} \]

\[ \Theta \]

Rational Surface: \( t \) rational \&

every point a fixed point

Irrational Surface: \( t \) irrational \&

field lines ergodically fill surface
* Charged particle orbits in fixed $\mathbf{E}$ & $\mathbf{B}$ fields
  confinement
  heating

* Magnetic field lines \( \nabla \cdot \mathbf{B} = 0 \)

* Nonlinear stability
  arnold diffusion

* Self-consistent problem

References

M. Berry

A. Lichtenberg & M. Lieberman, "Regular & Stochastic Moti" Somium - Valag