Hamiltonian Methods in Weakly Nonlinear Vlasov-Poisson Dynamics

by

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To Julie, Justin and Gram

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The Vlasov-Poisson equation describes the evolution of probability distributions of electrically charged particles in a plasma, subject only to electrostatic forces. Even in the linear limit, solutions of this equation display rich behavior: the singular nature of the inverse of the linearized Vlasov-Poisson operator endows it with a continuous spectrum. The normal modes of this operator, called Van Kampen modes, form a complete set, and so provide a general solution to the linearized initial value problem. With the well-developed linear theory as a starting point, we explore the nonlinear behavior of small amplitude disturbances about linearly stable, homogeneous equilibria of the Vlasov-Poisson equation. Specifically, we truncate the equation to include only the lowest-order interactions between Van Kampen modes.

In contrast to previous attempts, we take advantage of the noncanonical, infinite-dimensional Hamiltonian structure of this equation. To this end, we import concepts from the perturbation theory of finite-dimensional Hamiltonian systems to our infinite-dimensional context. In particular, we modify the technique known as partial averaging to apply it to a system with a continuous linear spectrum, by introducing the idea of a *layer of resonances.* As this technique has been developed for canonical Hamiltonian equations, we show how to canonize the Hamiltonian to the order of interest.

From this retooled perturbation theory, we derive an approximate *resonance Hamiltonian* that retains the dominant characteristics of the weakly nonlinear behavior. We find that, in spite of the presence of negative-energy modes in the linear spectrum, the solutions to the weakly nonlinear equation remain stable. We also show that the weakly nonlinear equation can be truncated to an integrable system when used to describe the transient behavior of a system starting with only two excited modes. We integrate this system, and discuss its solutions.

Contents

Chapter 1

Introduction

Finding physically relevant simplifications of equations derived from first principles is one of the primary, if not highly celebrated, tasks performed by theoretical physicists. This task is difficult, and is especially so for complex physical systems. Simplification becomes somewhat easier if the physical system can be shown to have a mathematical structure in common with other, already studied systems.

A mathematical structure that underlies a vast number of physical systems is the Hamiltonian structure. The basic equations of mechanics are but one class of equations that can be written in the familiar canonical Hamiltonian form. Many other systems, including ideal fluids [1]-[2], also have a Hamiltonian structure, except of a more general, noncanonical type. In either case, many physical inferences, such as the preservation of phase space volume, can be drawn simply from fact that the systems are Hamiltonian. Also, what is more pertinent for this dissertation, a Hamiltonian structure gives a convenient framework for a theory of perturbations around an equilibrium.

It is well-known that the equation which governs collisionless plasmas, the Maxwell-Vlasov system, has a Hamiltonian structure [2]-[7]. This structure persists when the plasma is one-dimensional (is uniform in two spatial dimensions). The behavior of such a plasma is now governed by the Vlasov-Poisson system, given in equations (2.1) and (2.2). This system, though already describing an artificially ideal situation for a plasma, is still a nonlinear integro-partial-differential equation. So, further simplification is needed to make the problem tractable.

The simplification we consider in this dissertation is that of weakly nonlinear dynamics near an equilibrium. This is most easily understood in terms of the Hamiltonian. In the vicinity of the equilibrium, the Hamiltonian (or Free Energy for noncanonical systems- see chapter 3) can be expanded in a Taylor series in the displacement from equilibrium. The lowest nonzero term is quadratic. Truncating at that order yields the linearized system; truncating at the cubic order yields the weakly nonlinear system. Roughly speaking, the weakly nonlinear approximation captures the nonlinear behavior displayed by small, but finite perturbations. To understand the weakly nonlinear dynamics though, it is necessary to thoroughly understand the linearized dynamics.

The linearized Vlasov-Poisson problem, has been studied for more than a half-century. Owing to the singular nature of the inverse of the linear operator that arises from linearization, the linear theory is rich, and gives rise to some surprising behavior, the most famous example being Landau damping of Langmuir (electrostatic) waves [8],[9]. At first glance, the fact that a wave in a conservative system damps away (without the simultaneous presence of an unstable wave) seems paradoxical. The paradox is resolved by realizing that the Landau damped wave is not an eigenmode of the linear operator, but rather a “quasimode,” which furthermore, only attains a wave-like character at large times. In contrast, the undamped, or neutral Langmuir wave which exists when the equilibrium distribution has an inflection point *is* an eigenmode.

In fact, when the equilibrium is linearly stable (without neutral modes), the linear operator has no true eigenmodes, and yields no true dispersion relation. However, as shown by Van Kampen [10]-[11], and developed by Case [12], it is fruitful to consider generalized eigenmodes, or eigenmodes that can only be understood in a distributional sense. These Van Kampen modes make up the *continuous spectrum* of the linear Vlasov-Poisson equation. In a spatially periodic system, there are actually a countable infinity of such modes for each velocity value at which the background distribution is non-zero. This continuous spectrum is complete, and can be used as a basis to solve the linearized initial value problem for all times. Therefore, unlike the asymptotically valid quasimodes, the Van Kampen modes can also be used to understand the early, or transient behavior of the perturbed plasma.

The Van Kampen modes also turn out to play an important role in the Hamiltonian formulation of the linearized Vlasov-Poisson equation: the amplitudes of the modes *diagonalize* the Hamiltonian [13]-[15], making them analogous to a continuous set of uncoupled oscillators. As a result, the Van Kampen modes are a useful foundation for approaching the weakly nonlinear problem, an idea stated explicitly in [15].

Various strategies have been used to study particular nonlinear effects in the Vlasov-Poisson (and Maxwell-Vlasov) equation [16]-[17]. One of the earliest led to the construction of the exact nonlinear solutions known as BGK modes [18].

Also worth mentioning are quasilinear theory [19]-[21], which assumed the electron distribution function to be a sum of an equilibrium and a set of traveling waves with phases distributed randomly, and the consideration of the effect of trapped electrons on Landau damped waves [22]. (Recently, O'Neil's result that a Landau damped wave saturates in a BGK mode has been disputed, and consistency problems have been noted in his calculation, see [23].)

The general approach that comes closest to matching ours in flavor is the consideration of resonant three-wave coupling [16]-[17], [24]-[27]. Of special note are references [24]-[26], which considered the contribution of negative energy modes to nonlinear instability. We must make clear, however, that we do not consider the *resonant* coupling of three Landau damped "waves;" as is pointed out in [16], these waves cannot couple resonantly in one dimension. Instead, and this is a point in which the present work differs from the references just mentioned, we are concerned with the resonant coupling of three Van Kampen modes. And, by the very nature of Van Kampen modes as generalized functions, we cannot restrict ourselves to a single resonant triplet as is usual in three-wave problems. Instead, we retain an infinity of resonant triplets.

Two studies that also explore the resonant coupling of Van Kampen modes are those by Best et. al. [28], and Trocheris [29]-[30]. Like this dissertation, [28] restricts consideration to linearly stable one dimensional plasmas. Unlike the present work, that paper assumes an ansatz for the second-order (nonlinear) solutions. References [29] and [30], on the other hand, assume no nonlinear ansatz, but they restrict their attention to neutrally stable and unstable equilibria, emphasizing the dynamics of the discrete modes. They mostly neglect the dynamics of the continuous spectrum, only considering it in enough detail to assess the conditions of validity of the theory they develop for discrete modes.

Another, and perhaps more important way, in which this dissertation differs from references [28]-[30] is its use of the Hamiltonian formalism. By using it, we can study the (infinite-dimensional) weakly nonlinear problem by importing concepts from finite dimensional canonical perturbation theory. In particular, we use a modification of the technique of partial averaging to derive an approximate "Resonance Hamiltonian" which simplifies the problem considerably [31]-[33]. The modification involves including the nearly resonant terms as well as the exactly resonant terms in the interaction term of the Hamiltonian. In finite dimensions, the use of partial averaging amounts to finding the nonlinear normal form of a Hamiltonian near an equilibrium point [34]-[35]. The even more relevant (for this dissertation) problem of finding nonlinear normal forms for an equation with a continuous spectrum was treated in [36].

We present two results concerning the partially averaged weakly nonlinear Vlasov-Poisson system. First, (for the linearly stable equilibria we consider here) we show the existence of a positive definite constant of motion, related to the momentum of the linearized system. This implies the stability of the weakly nonlinear system, even when negative energy modes are present. Second, we find that by truncating to two Fourier modes (still an infinite number of continuous spectrum modes) and passing to the limit of small-k_0, we arrive at an integrable system that describes the weakly nonlinear interaction between transients. This system is solved, and the nature of its solutions briefly discussed.

The structure of the dissertation is as follows. We begin in chapter 2 with an overview of the Hamiltonian formulation of finite degree of freedom systems, and its generalization to systems with infinite degrees of freedom. Then, in chapter 3, we consider how to apply the Darboux theorem to find a set of coordinates suitable for studying the weakly nonlinear limit of a problem that is originally stated in noncanonical coordinates. Chapter 4 covers the diagonalization of the Hamiltonian of the linear Vlasov-Poisson system, and the interpretation of the interaction term. In chapter 5, we describe the method of partial averaging, how it must be modified to cope with the Vlasov-Poisson continuous spectrum, and the result of applying it to the weakly nonlinear Hamiltonian. In chapter 6, we present the results we have derived from the partially averaged Hamiltonian, and we end the dissertation with a concluding summary in chapter 7.

Chapter 2

The Hamiltonian Formulation of the Vlasov-Poisson System

The object of study is the one-dimensional Vlasov–Poisson system for a single species of electric charge e and mass m (see, for instance, [37]). This system is composed of two equations. One is the Vlasov equation which describes the evolution of the phase space distribution function, f, of the species subject to an electric field E:

$$\frac{\partial f}{\partial t}(x,v,t) = -v\frac{\partial f}{\partial x}(x,v,t) - \frac{e}{m}E(x,t)\frac{\partial f}{\partial v}(x,v,t)\,. \tag{2.1}$$

The other equation is Poisson's equation which determines E in terms of f and a fixed background charge density N:

$$\frac{\partial E}{\partial x}(x,t) = 4\pi e\left(-N + \int_{-\infty}^{\infty} \mathrm{d}v\, f(x,v,t)\right)\,. \tag{2.2}$$

Because of equation (2.2), equation (2.1) is actually a nonlinear, integro-partial-differential equation for f. And even though these two equations together describe an already artificially ideal situation for a plasma, they are still too complicated to solve for general initial conditions. Still, much information can be gained from the fact that equation (2.1) has a noncanonical Hamiltonian structure. Since one of the aims of this investigation is to find a method applicable to a wide range of Hamiltonian systems, a summary of the general properties of a Hamiltonian system is an appropriate start. The summary will first be done for finite-dimensional systems, and then a way of generalizing to infinite-dimensional systems will be given.

2.1 Finite Degree of Freedom Hamiltonian Systems

Let z^i denote a component of an n-tuple z of dynamical variables. A system of evolution equations for the z^i is said to have a Hamiltonian structure if it can be written in terms of a Hamiltonian function H, and a Poisson bracket $\{\cdot,\cdot\}$ in the form

$$\frac{\partial z^i}{\partial t} = \{z, H\}\,. \tag{2.3}$$

A Poisson bracket is a bilinear, antisymmetric binary operation on functions that further satisfies the following two properties:

$$\{F, GK\} = \{F, G\}K + G\{F, K\} \tag{2.4}$$

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0 \tag{2.5}$$

Condition (2.4) implies that a Poisson bracket can be written in terms of an antisymmetric rank two contravariant tensor $J^{ij}(z)$ called the cosymplectic form:

$$\{F, G\} = \sum_{i,j=1}^{n} \frac{\partial F}{\partial z^i} J^{ij}(z) \frac{\partial G}{\partial z^j} \tag{2.6}$$

And condition (2.5) implies that the components J^{ij} obey

$$J^{il}\frac{\partial J^{jk}}{\partial z^l} + J^{jl}\frac{\partial J^{ki}}{\partial z^l} + J^{kl}\frac{\partial J^{ij}}{\partial z^l} = 0\,. \tag{2.7}$$

While there exist many different types of Poisson brackets, a particularly important type, called a *canonical* bracket, can arise in systems with an even number of dimensions. In such cases, the $2n$ components of z can be divided up into two classes of variables denoted q^i and p_i $(i = 1 \ldots n)$. Then the canonical bracket has the form

$$\{F, G\} = \sum_i^n \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right) . \tag{2.8}$$

For present purposes, canonical brackets are most important because an extensive perturbation theory has been developed for systems that can be written in terms of them. But they are also generally important in light of the Darboux theorem. This theorem states that even noncanonical brackets can be locally written in the form (2.8).

Of course, not all Poisson brackets are canonical. And unlike canonical brackets, noncanonical brackets may be degenerate. Suppose the corank of the matrix of the values $J^{ij}(z)$ is r. Then there exists a set of r linearly independent covectors $\chi_l^{(\alpha)}$:

$$J^{il}(z)\chi_l^{(\alpha)} = 0 \qquad \alpha = 1 \ldots r . \tag{2.9}$$

The $\chi^{(\alpha)}$ are called null covectors. For later convenience, we also note that given a particular set of null covectors, we can specify their duals $\chi^l_{(\alpha)}$ through the relation

$$\chi^l_{(\beta)}\chi_l^{(\alpha)} = \delta^\alpha_\beta . \tag{2.10}$$

Now, as proved in [39], the fact that all Poisson brackets satisfy equation (2.5) implies that each of the r null covectors (2.9) can be chosen to be an exact differential. In other words, there exist r functions $C^{(\alpha)}$ such that

$$\chi^{(\alpha)} = \mathrm{d}C^{(\alpha)} . \tag{2.11}$$

The functions C^α are called *Casimir invariants* or simply *Casimirs* [7],[38]-[39]. From (2.11) and (2.9), it is obvious that

$$\{F, C^\alpha\} = 0 \qquad \forall F\,. \tag{2.12}$$

And so, because of equation (2.3), Casimirs are constants of motion irrespective of the Hamiltonian that specifies the dynamics.

2.2 Infinite Degree of Freedom Hamiltonian Systems

The Hamiltonian formalism as given above applies to discrete systems. With some qualifications (for instance, the Darboux theorem is false in general), and some technical care, it can be generalized to apply to continuous systems as well simply by recognizing that the dynamical variables that describe continuous systems are fields [1]. Then, instead of the Hamiltonian being a function of dynamical variables, it is a functional of the dynamical fields. The Poisson bracket is now a binary operation on functionals. In equation (2.6), J^{ij} becomes an operator, the derivatives $\partial/\partial z^i$ become functional derivatives $\delta/\delta\psi(x)$, and the sum becomes an integral over the continuous index x. Of course, there may be more than one field, in which case another sum over all possible pairs of fields needs to be added. In other words, a Poisson bracket for a continuous system described by l fields $\psi^l(x)$ has the form

$$\{\mathcal{F}, \mathcal{G}\}[\psi^1, \ldots, \psi^l] = \sum_{i,j=1}^{l} \int \mathrm{d}x\, \frac{\delta F}{\delta\psi^i} \mathcal{J}^{ij}[\psi^1, \ldots, \psi^l] \frac{\delta G}{\delta\psi^j}\,. \tag{2.13}$$

Equation (2.1) is such a continuous Hamiltonian system. The distribution function $f(x, v, t)$ is the sole dynamical variable, and, like most textbook examples of Hamiltonian systems, the Hamiltonian functional is the total energy of the plasma, given by

$$\mathcal{H}[f] = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x\, mv^2 f + \frac{1}{8\pi}\int_{-L}^{L} \mathrm{d}x\, E^2\,. \tag{2.14}$$

Of course, for (2.14) to generate the Vlasov–Poisson system specifically, $E(x,t)$ must satisfy equation (2.2).

Unlike most textbook examples, however, the Vlasov-Poisson system is non-canonical. Its Poisson bracket is given by

$$\{\mathcal{F},\mathcal{G}\}[f] = \int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x\, f\left[\frac{\delta\mathcal{F}}{\delta f},\frac{\delta\mathcal{G}}{\delta f}\right]\,. \tag{2.15}$$

The square brackets denote the (mass-scaled) canonical bracket on the $x-v$ phase space:

$$[F,G] = \frac{1}{m}\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial v} - \frac{\partial F}{\partial v}\frac{\partial G}{\partial x}\right)\,. \tag{2.16}$$

An integration by parts in equation (2.15) yields the following cosymplectic operator:

$$\mathcal{J}[f](\cdot) = -[f,\cdot]\,. \tag{2.17}$$

Equation (2.1) then can be written in the compact form

$$\frac{\partial f}{\partial t} = \mathcal{J}[f]\frac{\delta\mathcal{H}}{\delta f} = -\left[f,\frac{\delta\mathcal{H}}{\delta f}\right] \tag{2.18}$$

Bracket (2.15) is also extremely degenerate: for an arbitrary function C, a functional of the form

$$\mathcal{C}[f] = \int \mathrm{d}x\,\mathrm{d}v\, C(f) \tag{2.19}$$

is a Casimir. Physically, these constants of motion (sometimes called generalized entropies) constrain the evolution of f to rearrangements, or "reshufflings" of the graph of f over the $x-v$ plane. Also, as will soon be apparent, the constraints imposed by these constants of motion must be kept in mind when studying these equations perturbatively.

2.3 Discussion

Possibly the most powerful aspect of the (noncanonical) Hamiltonian formulation is its generality. This is evident above in the result (2.19). Not only are these Casimirs constants of motion for the Vlasov-Poisson system, but for any other system (for example, the 2D Euler system) which uses the same Poisson bracket. Because of this generality, any of the results in this dissertation that deal only with the Poisson bracket can be directly applied to these analogous systems. Furthermore, the results specific to the Hamiltonian (2.14) should, at the very least, provide inspiration as to how to handle other, similar Hamiltonians.

A surprising feature of equation (2.1) is that although it has an uncountable family of constants of motion represented by equation (2.19), it is not integrable. In fact, one of the motivations of this dissertation was to try to find integrable approximations to (2.1) on a single symplectic leaf defined by the "level sets" of the Casimirs. One candidate for an integrable approximation is given in equation (6.11), but its status with regard to integrability is still unknown. However, the long wavelength limit of equation (6.11), given in equations (6.27)–(6.30), *is* integrable.

Lastly, the bracket defined by equations (2.15) and (2.16) is an example of a Lie-Poisson bracket [40] (also known as a Kostant-Kirillov bracket [41]-[42]), perhaps the most commonly occurring type of noncanonical Poisson bracket. And by virtue of having this type of bracket, the Vlasov-Poisson system does find (in spite of what was stated above) a good analogy among textbook finite dimensional Hamiltonian systems: the free rigid body [43].

Chapter 3

Noncanonical Perturbation Theory

The Hamiltonian structure given in equations (2.14) and (2.15) is valid for the full nonlinear evolution of f. Part of the problem now is to determine how this structure looks in a weakly nonlinear limit. The Darboux theorem, though not generally valid for infinite dimensional systems, suggests that canonical variables exist.

One advantage that comes with knowing a canonical Hamiltonian formulation for a finite degree-of-freedom system is the existence of a detailed perturbation theory. In particular, when a canonical Hamiltonian can be written as a sum of a completely integrable Hamiltonian and a small modification, a general method can be used to study this system. One of the goals of the present thesis is to attempt to extend this method to a class of infinite degree-of-freedom systems that are characterized by having a continuous linear spectrum.

Canonical perturbation theory essentially follows a two-step approach. The

Hamiltonian is assumed to have the following form:

$$H(q,p) = H_0(q,p) + \epsilon H_1(q,p) + \epsilon^2 H_2(q,p) + \cdots , \tag{3.1}$$

where $H_0(q,p)$ is a completely integrable Hamiltonian. The first step is to find *action-angle variables* for H_0. Action-angle variables are canonical variables, the conjugate pairs of which are often denoted J, θ, such that only the actions J, appear in the Hamiltonian. In other words, in action-angle variables, $H_0 = H_0(J)$. From Hamilton's equations, it is clear that each J is a constant of motion. Once such variables are found, the whole Hamiltonian (3.1) is expressed in terms of them:

$$H(J,\theta) = H_0(J) + \epsilon H_1(J,\theta) + \epsilon^2 H_2(J,\theta) + \cdots . \tag{3.2}$$

The second step in canonical perturbation theory is to try to find a new set of action-angle variables for the Hamiltonian up to $\mathcal{O}(\epsilon)$. The usual technique is to consider the Hamiltonian averaged over each θ, and find the function which generates the canonical transformation to the new variables. This is not always possible because of the existence of resonances. When resonances exist, a partial average is taken. This step is the subject of chapter 5, and much more will be said about it there.

But before the first step can be applied to the weakly nonlinear Vlasov-Poisson system, it must be cast in the form (3.1).

3.1 Noncanonical Perturbation Theory - Finite Degrees of Freedom

Given a noncanonical Hamiltonian structure for a dynamical system, how can we find a canonical structure for the system that is valid near an equilibrium point?

The Darboux theorem says that such canonical variables must exist (at least for finite dimensional systems). And since we are considering the problem of a small nonlinear correction to a linearized system, it is natural to first look for canonical variables applicable to the linear system, and then hope a small modification will make them appropriate for the weakly nonlinear system.

We begin by considering a general finite degree-of-freedom Hamiltonian system with a noncanonical structure:

$$\dot{\xi} = J(\xi) \cdot \nabla H(\xi)\,. \tag{3.3}$$

We can investigate the weakly nonlinear dynamics near an equilibrium point $\xi = z_0$ by writing $\xi = z_0 + \epsilon z$ and truncating the equations at $\mathcal{O}(\epsilon^2)$. In finite dimensions, the resulting equations are

$$\begin{aligned}
\epsilon \dot{z}^i &= \left[J^{ij}(z_0) + \epsilon \frac{\partial J^{ij}}{\partial \xi^k}(z_0) z^k + \frac{\epsilon^2}{2} \frac{\partial^2 J^{ij}}{\partial \xi^l \partial \xi^m}(z_0) z^l z^m \right] \\
&\times \frac{1}{\epsilon} \frac{\partial}{\partial z^j} \left[H(z_0) + \epsilon \frac{\partial H}{\partial \xi^n}(z_0) z^n + \frac{\epsilon^2}{2} \frac{\partial^2 H}{\partial \xi^p \partial \xi^q}(z_0) z^p z^q \right. \\
&+ \left. \frac{\epsilon^3}{6} \frac{\partial^3 H}{\partial \xi^r \partial \xi^s \partial \xi^t}(z_0) z^r z^s z^t \right] + \mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{3.4}$$

The Hamiltonian is here expanded to third order because of the factor of ϵ^{-1} that arises from changing variables in the derivative. Upon taking the derivative and dividing through by ϵ, equation (3.4) simplifies to

$$\begin{aligned}
\dot{z}^i &= \left[J^{ij}(z_0) + \epsilon \frac{\partial J^{ij}}{\partial \xi^k}(z_0) z^k + \frac{\epsilon^2}{2} \frac{\partial^2 J^{ij}}{\partial \xi^l \partial \xi^m}(z_0) z^l z^m \right] \\
&\times \left[\frac{1}{\epsilon} \frac{\partial H}{\partial \xi^j}(z_0) + \frac{\partial^2 H}{\partial \xi^j \partial \xi^q}(z_0) z^q + \frac{\epsilon}{2} \frac{\partial^3 H}{\partial \xi^j \partial \xi^s \partial \xi^t}(z_0) z^s z^t \right] + \mathcal{O}(\epsilon^2)\,.
\end{aligned} \tag{3.5}$$

To obtain the linearized system from (3.5), we keep the first two terms in each factor in square brackets on the RHS:

$$\dot{z}^i = \left[J^{ij}(z_0) + \epsilon \frac{\partial J^{ij}}{\partial \xi^k}(z_0) z^k \right] \left[\frac{1}{\epsilon} \frac{\partial H}{\partial \xi^j}(z_0) + \frac{\partial^2 H}{\partial \xi^j \partial \xi^q}(z_0) z^q \right] + \mathcal{O}(\epsilon) \,. \tag{3.6}$$

But in general, the truncation of the cosymplectic form in (3.6) does not satisfy the Jacobi identity (2.7). In other words, (3.6) is not manifestly Hamiltonian.

However, since the (obviously antisymmetric) first term in the expansion of the cosymplectic form is constant, it does satisfy (2.7). Furthermore, since it is constant, we can easily transform to variables that canonize it. The only fact that prevents us from using the lowest-order term of the expansion of J^{ij} as the cosymplectic form for equation (3.6) is that the first derivatives of the Hamiltonian do not vanish at the equilibrium. We can overcome this obstacle if we use the Free Energy, a quantity composed of the energy and the Casimir invariants, as Hamiltonian.

3.1.1 The Linear System: Free Energy

The definition of Free Energy is motivated by the fact that in a canonical Hamiltonian system, an equilibrium is a critical point of the Hamiltonian. In the general noncanonical system (3.3), z_0 is not necessarily a critical point of H.

The noncanonical nature of the Poisson bracket leads to this problem: the existence of Casimirs (see equation (2.12)) constrains the dynamics. If we insist on the principle that an equilibrium extremize the Hamiltonian of the system, we must extremize the energy subject to the constraint that the Casimirs must remain constant (see Discussion at the end of this chapter); hence, the method of Lagrange multipliers is applicable. Given constraining functions $C^{(\alpha)}$, z_0 is an extremum of the constrained Hamiltonian if

$$\nabla(H + \lambda_{(\alpha)} C^{(\alpha)})(z_0) = 0\,. \tag{3.7}$$

The function $F = H + \lambda_{(\alpha)} C^{(\alpha)}$ that satisfies equation (3.7) is called the *free energy* at that equilibrium.

Now, since Casimirs $C^{(\alpha)}$ Poisson-commute with any function, they can be added onto a Hamiltonian H to give a new Hamiltonian that generates the same dynamics. In other words,

$$\dot{z} = \{z, H\} = \{z, H + C\}\,. \tag{3.8}$$

So, we could just as easily have used the free energy at $\xi = z_0$ as a Hamiltonian for the system (3.3). If we had, the lowest order term in the truncation of the Hamiltonian in equation (3.5) would vanish: changing H to F yields

$$\dot{z}^i = \left[J^{ij}(z_0) + \epsilon \frac{\partial J^{ij}}{\partial \xi^k}(z_0) z^k \right] \left[\frac{\partial^2 F}{\partial \xi^j \partial \xi^q}(z_0) z^q + \frac{\epsilon}{2} \frac{\partial^3 F}{\partial \xi^j \partial \xi^s \partial \xi^t}(z_0) z^s z^t \right] + \mathcal{O}(\epsilon^2)\,. \tag{3.9}$$

From equation (3.9), we obtain the system linearized about $\xi = z_0$ by keeping only the first term in each of the square brackets. And conveniently, the cosymplectic form and the Hamiltonian for the linearized system are given by the lowest order truncations of $J^{ij}(z_0 + z)$ and $F(\xi)$ respectively. Since this cosymplectic form is constant, a linear transformation suffices to map to canonical coordinates.

3.1.2 The Weakly Nonlinear System: Flattening the Bracket

Moving on to the next order, we see we must again keep two terms in the expansion of the cosymplectic form on the RHS of equation (3.9) to consistently truncate the system at $\mathcal{O}(\epsilon^2)$.

One solution that overcomes this difficulty is to find a transformation to new variables η that eliminates the term of $\mathcal{O}(\epsilon)$ from the truncation of the bracket. Let the components of the cosymplectic form in the new variables by $\bar{J}^{ij}$, we require

$$\bar{J}^{ij}(\eta) = J^{ij}(z_0) + \mathcal{O}(\epsilon^2)\,. \tag{3.10}$$

The resulting truncation would be constant through $\mathcal{O}(\epsilon)$, so would necessarily satisfy (2.5), and also would be easily canonizable. In some Poisson-geometrical sense, a constant bracket is "flat," so we say that the desired transformation *flattens* the bracket to higher order.

To find such new coordinates, we note that the transformation law of J^{ij} requires

$$\bar{J}^{ij}(\eta) = \frac{\partial \eta^i}{\partial z^k} J^{kl}(z_0 + \epsilon z) \frac{\partial \eta^j}{\partial z^l}\,. \tag{3.11}$$

Then, we simply set the RHS of equation (3.11) equal to the RHS of (3.10), and solve for η. Solutions for general finite dimensional Poisson brackets were found in [44]. (The details of determining these transformations will be given in an appendix.) The transformations and their inverses take the general form

$$\eta^i = z^i + \frac{\epsilon}{2} D^i_{kl} z^k z^l + \mathcal{O}(\epsilon^2)\,, \tag{3.12}$$

and

$$z^i = \eta^i - \frac{\epsilon}{2} D^i_{kl} \eta^k \eta^l \tag{3.13}$$

where the form of the tensor D^i_{kl} depends on whether $J^{ij}(z_0)$ is singular or not.

For nonsingular $J^{ij}(z_0)$, we have

$$D^i_{kl} = \frac{\partial J^{im}}{\partial \xi^k}(z_0) S_{ml} + \frac{\partial J^{im}}{\partial \xi^l}(z_0) S_{mk}\,, \tag{3.14}$$

where S_{mk} is proportional to the inverse of $J^{ij}(z_0)$:

$$J^{ij}(z_0)S_{jk} = -\frac{1}{3}\delta_l^k \,. \tag{3.15}$$

The case when $J^{ij}(z_0)$ is singular is more complicated. Although the inverse of $J^{ij}(z_0)$ does not exist in this case, a slightly more general object, the (Moore-Penrose) *pseudoinverse* does [45]. The pseudoinverse T_{ij} of $J^{ij}(z_0)$ is defined by the following equation:

$$J^{ij}(z_0)T_{jk} = \delta_k^i - \chi_{(\alpha)}^i \chi_k^{(\alpha)} \,, \tag{3.16}$$

where $\chi_{(\alpha)}$ and $\chi^{(\alpha)}$ are the null covectors and their duals defined in (2.9), (2.10). In terms of the null covectors, their duals, and the pseudoinverse, we can state the form of the tensor D^i_{jk} that defines the variable transformation (3.12) applicable to singular brackets:

$$\begin{aligned} D^i_{jk} \;=\; & \frac{-1}{3}\frac{\partial J^{il}}{\partial \xi^m}(z_0)T_{lj}\left(\delta_k^m + \frac{1}{2}\chi^m_{(\alpha)}\chi_k^{(\alpha)}\right) + \frac{1}{6}\chi^i_{(\beta)}\frac{\partial \chi_j^{(\beta)}}{\partial \xi^m}(z_0)\left(\delta_k^m + \frac{1}{2}\chi^m_{(\alpha)}\chi_k^{(\alpha)}\right) \\ & \frac{-1}{3}\frac{\partial J^{il}}{\partial \xi^m}(z_0)T_{lk}\left(\delta_j^m + \frac{1}{2}\chi^m_{(\alpha)}\chi_j^{(\alpha)}\right) + \frac{1}{6}\chi^i_{(\beta)}\frac{\partial \chi_k^{(\beta)}}{\partial \xi^m}(z_0)\left(\delta_j^m + \frac{1}{2}\chi^m_{(\alpha)}\chi_j^{(\alpha)}\right) \end{aligned} \tag{3.17}$$

3.1.3 Example: The Heavy Top

To demonstrate the usefulness of the formalism outlined above, we apply it to the example of perturbations of a heavy top at equilibrium. We begin by reviewing the so-called Euler-Poisson equations of motion for the heavy top, and describing their Hamiltonian structure [43].

A heavy top is defined as a rigid body fixed at one point in a gravitational field. In a coordinate frame fixed in the body, the state of such a system can be described by two 3-vectors: Γ will denote the gravitational field vector, and L will denote the angular momentum. Also, it is convenient to define two other 3-vectors. One, is the angular velocity Ω, which is related to the angular momentum by means of the inertia matrix $\mathcal{I}$ as follows.

$$L = \mathcal{I} \cdot \Omega\,. \tag{3.18}$$

The other is the vector R (stationary in the body frame) that points from the fixed point to the body's center of mass. In terms of these four vectors, the Euler-Poisson equations are:

$$\begin{aligned} \dot{\Gamma} &= \Gamma \times \Omega \\ \dot{L} &= L \times \Omega + \Gamma \times R\,. \end{aligned} \tag{3.19}$$

Equations (3.19) have a noncanonical Hamiltonian structure. Like "natural" mechanical systems, the Hamiltonian function is the total energy:

$$H = \frac{1}{2} L \cdot \Omega + \Gamma \cdot R\,. \tag{3.20}$$

The Poisson bracket, on the other hand, is not canonical: its cosymplectic form can be compactly represented as

$$J(\Gamma, L) = \begin{bmatrix} 0 & \Gamma\times \\ \Gamma\times & L\times \end{bmatrix}. \tag{3.21}$$

The Hamiltonian formulation of equations (3.19) is then

$$\begin{bmatrix} \dot{\Gamma} \\ \dot{L} \end{bmatrix} = J(\Gamma, L) \cdot \begin{bmatrix} \nabla_\Gamma H \\ \nabla_L H \end{bmatrix}. \tag{3.22}$$

Finally, we note that the bracket defined by (3.21) is degenerate. It is easy to verify that the following two functions are Casimirs of this bracket:

$$C^{(1)}(\Gamma, L) = \frac{1}{2}\Gamma \cdot \Gamma \qquad C^{(2)}(\Gamma, L) = \Gamma \cdot L\,. \tag{3.23}$$

Physically, $C^{(1)}$ is half the squared magnitude of the force of gravity, and $C^{(2)}$ is the component of angular momentum along the gravitational field. The fact that they are both Casimirs implies they are conserved; in effect, these conservation laws are built into the kinematics by means of the Poisson bracket.

We now choose a specific class of tops and a suitable coordinate system for our example. For simplicity we restrict ourselves to the tops for which the fixed point lies on one of its principal axes. In these cases, we can choose a coordinate system in which both the inertia matrix is diagonal and R has the form $R = (0, 0, 1)$. Though this is a restriction on the class of tops, it is a mild one. Along with the well-known Lagrange top, which is rotationally symmetric about the axis containing the fixed point, and has integrable dynamics, this restricted class contains many nonsymmetric tops that are known to be not integrable [43].

For the sake of uniformity, we denote the components of Γ in the above-mentioned coordinate system as (ξ^1, ξ^2, ξ^3), and the components of L as (ξ^4, ξ^5, ξ^6). Denoting the moments of inertia by I_1, I_2, I_3, we find that the angular velocity can be written $\Omega = (\xi^4/I_1, \xi^5/I_2, \xi^6/I_3)$. With this choice of coordinates, then, the Hamiltonian (3.20) has the form

$$H(\xi) = \frac{1}{2}\left(\frac{(\xi^4)^2}{I_1} + \frac{(\xi^5)^2}{I_2} + \frac{(\xi^6)^2}{I_3}\right) + \xi^3\,. \tag{3.24}$$

The cosymplectic form (3.21) is written

$$J(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 & -\xi^3 & \xi^2 \\ 0 & 0 & 0 & \xi^3 & 0 & -\xi^1 \\ 0 & 0 & 0 & -\xi^2 & \xi^1 & 0 \\ 0 & -\xi^3 & \xi^2 & 0 & -\xi^6 & \xi^5 \\ \xi^3 & 0 & -\xi^1 & \xi^6 & 0 & -\xi^4 \\ -\xi^2 & \xi^1 & 0 & -\xi^5 & -\xi^4 & 0 \end{bmatrix}. \tag{3.25}$$

And the Casimirs (3.23) take the form

$$C^{(1)}(\xi) = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2) \qquad C^{(2)}(\xi) = \xi^1\xi^4 + \xi^2\xi^5 + \xi^3\xi^6\,. \tag{3.26}$$

All that remains in setting up this example is identifying an equilibrium about which we can study perturbations. As can be seen from equations (3.19), when Γ, Ω, L, and R are colinear, the top is in equilibrium. In terms of the coordinates ξ, such an equilibrium is given by setting all but ξ^3 and ξ^6 to zero. We denote this equilibrium by z_0 :

$$z_0 := (0, 0, g, 0, 0, l_0)\,. \tag{3.27}$$

Physically, this equilibrium corresponds to a sleeping top: the top is standing straight up, and spinning at a constant angular velocity.

Now, the first step in finding the canonical weakly nonlinear Hamiltonian structure near $\xi = z_0$ is to find the Free Energy for that equilibrium (see [46] for a determination of the Free Energy of a Lagrange Top). Substituting (3.26) into equation (3.7) and evaluating at $\xi = z_0$ allows us to find $\lambda_{(1)}$ and $\lambda_{(2)}$, and hence the relevant Free Energy. It is

$$F(\xi) \;=\; \frac{1}{2}\left(\frac{(\xi^4)^2}{I_1} + \frac{(\xi^5)^2}{I_2} + \frac{(\xi^6)^2}{I_3}\right) + \xi^3$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\left(\frac{l_0^2 - gI_3}{g^2 I_3}\right)((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2) \\
&- \quad \frac{l_0}{gI_3}(\xi^1\xi^4 + \xi^2\xi^5 + \xi^3\xi^6)\,. \qquad (3.28)
\end{aligned}
$$

To study perturbations near the equilibrium, we assume $\xi^i = z_0^i + \epsilon z^i$, where ϵ is small. Substituting this expression into (3.28), we obtain

$$
\begin{aligned}
F(z) \quad &= \quad \frac{\epsilon^2}{2}\left[\left(\frac{l_0^2 - gI_3}{g^2 I_3}\right)((z^1)^2 + (z^2)^2 + (z^3)^2)\right. \\
&+ \quad \frac{(z^4)^2}{I_1} + \frac{(z^5)^2}{I_2} + \frac{(z^6)^2}{I_3} \\
&- \quad \left.\left(\frac{2l_0}{gI_3}\right)(z^1z^4 + z^2z^5 + z^3z^6)\right] + \frac{1}{2}\frac{gI_3 - l_0^2}{I_3}\,. \qquad (3.29)
\end{aligned}
$$

Finding the linearization of equations (3.19) about z_0 is now simply a matter of substituting (3.29) into the Poisson bracket truncated at lowest order. This is also obtained by substituting $\xi^i = z_0^i + z^i$ into (3.25):

$$
J(z) = \frac{1}{\epsilon^2}\begin{bmatrix} 0 & 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & -l_0 & 0 \\ g & 0 & 0 & l_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(1/\epsilon)\,. \qquad (3.30)
$$

(The factor of ϵ^{-2} comes from the fact that J transforms as a tensor.) Using (3.29) and (3.30) we find the linearized equations to be

$$
\begin{aligned}
\dot{z}^1 \quad &= \quad \frac{l_0}{I_3}z^2 - \frac{g}{I_2}z^5 \\
\dot{z}^2 \quad &= \quad \frac{g}{I_1}z^4 - \frac{l_0}{I_2}z^1 \\
\dot{z}^3 \quad &= \quad 0 \qquad (3.31)
\end{aligned}
$$

$$\begin{aligned}
\dot{z}^4 &= \frac{l_0(I_2 - I_3)}{I_2 I_3} z^5 + z^2 \\
\dot{z}^5 &= \frac{l_0(I_3 - I_1)}{I_1 I_3} z^4 - z^1 \\
\dot{z}^6 &= 0\,.
\end{aligned}$$

The second step in canonizing to weakly nonlinear order is finding the new variables η for which the form of the bracket given by equation (3.30) is correct to second order. To this end, we apply the results given in equations (3.12) and (3.17). To compute the tensor D^i_{jk} in (3.17), we require the null covectors $\chi^{(\alpha)}$, (and their duals), the derivatives of the null covectors, and the pseudoinverse T of the bracket (3.25) all evaluated at $\xi = z_0$. For convenience in manipulations, the two null covectors are taken to be

$$\chi^{(1)}_i = \frac{\partial C^{(1)}}{\partial \xi^i}; \qquad \chi^{(2)}_i = \frac{\partial}{\partial \xi^i}\left[C^{(2)} - \left(\frac{2l_0}{g}\right) C^{(1)}\right], \tag{3.32}$$

using $C^{(1)}$ and $C^{(2)}$ from (3.26). The only nonzero components of the null vectors and their duals at z_0 are then given by

$$\chi^{(1)}_3(z_0) = g; \qquad \chi^3_{(1)}(z_0) = g^{-1}; \qquad \chi^{(2)}_6(z_0) = g; \qquad \chi^6_{(2)}(z_0) = g^{-1}\,. \tag{3.33}$$

From the null covectors and their duals, we can calculate the pseudoinverse of $J(z_0)$ by means of condition (3.16):

$$T(z_0) = \left(\frac{1}{g^2}\right)\begin{bmatrix}
0 & -l_0 & 0 & 0 & g & 0 \\
l_0 & 0 & 0 & -g & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & g & 0 & 0 & 0 & 0 \\
-g & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{3.34}$$

By substituting (3.25), (3.32), and (3.34) into equation (3.17), we obtain the coefficients of the transformation (3.12) and its inverse (3.13). These transformations are:

$$
\begin{aligned}
\eta^1 &= z^1 - \frac{\epsilon}{2g} z^1 z^3 \\
\eta^2 &= z^2 - \frac{\epsilon}{2g} z^2 z^3 \\
\eta^3 &= z^3 + \frac{\epsilon}{2g}\left[(z^1)^2 + (z^2)^2 + \frac{2}{3}(z^3)^2\right] \\
\eta^4 &= z^4 - \frac{\epsilon}{2g^2}(g z^1 z^6 + g z^3 z^4 - l_0 z^1 z^3) \\
\eta^5 &= z^5 - \frac{\epsilon}{2g^2}(g z^2 z^6 + g z^3 z^5 - l_0 z^2 z^3) \\
\eta^6 &= z^6 + \frac{\epsilon}{2g^2}\left[g\left(2z^1 z^4 + 2z^2 z^5 + \frac{4}{3} z^3 z^6\right) - l_0\left((z^1)^2 + (z^2)^2 + \frac{2}{3}(z^3)^2\right)\right] ,
\end{aligned}
\tag{3.35}
$$

and

$$
\begin{aligned}
z^1 &= \eta^1 + \frac{\epsilon}{2g}\eta^1\eta^3 + \mathcal{O}(\epsilon^2) \\
z^2 &= \eta^2 + \frac{\epsilon}{2g}\eta^2\eta^3 + \mathcal{O}(\epsilon^2) \\
z^3 &= \eta^3 - \frac{\epsilon}{2g}\left[(\eta^1)^2 + (\eta^2)^2 + \frac{2}{3}(\eta^3)^2\right] + \mathcal{O}(\epsilon^2) \\
z^4 &= \eta^4 + \frac{\epsilon}{2g^2}(g\eta^1\eta^6 + g\eta^3\eta^4 - l_0\eta^1\eta^3) + \mathcal{O}(\epsilon^2) \\
z^5 &= \eta^5 + \frac{\epsilon}{2g^2}(g\eta^2\eta^6 + g\eta^3\eta^5 - l_0\eta^2\eta^3) + \mathcal{O}(\epsilon^2) \\
z^6 &= \eta^6 - \frac{\epsilon}{2g^2}\left[g\left(2\eta^1\eta^4 + 2\eta^2\eta^5 + \frac{4}{3}\eta^3\eta^6\right) - l_0\left((\eta^1)^2 + (\eta^2)^2 + \frac{2}{3}(\eta^3)^2\right)\right] \\
&\quad + \mathcal{O}(\epsilon^2) .
\end{aligned}
\tag{3.36}
$$

Then, to find the Hamiltonian which generates the weakly nonlinear heavy top equations via the constant bracket given in equation (3.30), we only need to substitute (3.36) into (3.29). This yields

$$
F(\eta) = \frac{\epsilon^2}{2}\left\{\left[\left(\frac{l_0^2 - gI_3}{g^2 I_3}\right)((\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2)\right.\right.
$$

$$
\begin{aligned}
&+ \quad \frac{(\eta^4)^2}{I_1} + \frac{(\eta^5)^2}{I_2} + \frac{(\eta^6)^2}{I_3} - \left(\frac{2l_0}{gI_3}\right)(\eta^1\eta^4 + \eta^2\eta^5 + \eta^3\eta^6)\Bigg] \\
&+ \quad \epsilon\left[\frac{l_0}{g^2}\left(\frac{1}{I_3}(\eta^1)^2\eta^6 + \frac{1}{I_3}(\eta^2)^2\eta^6 + \frac{8}{3I_3}(\eta^3)^2\eta^6 + \frac{I_1 - I_3}{I_1 I_3}\eta^1\eta^3\eta^4\right.\right. \\
&+ \quad \left.\frac{I_2 - I_3}{I_2 I_3}\eta^2\eta^3\eta^5\right) + \frac{2l_0^2 - gI_3}{g^3 I_3}(\eta^3)^3 + \frac{1}{g}\left(\frac{1}{I_1}\eta^3(\eta^4)^2 + \frac{1}{I_2}\eta^3(\eta^5)^2\right. \\
&- \quad \left.\left.\frac{4}{3I_3}\eta^3(\eta^6)^2 + \frac{I_3 - 2I_1}{I_1 I_3}\eta^1\eta^4\eta^6 + \frac{I_3 - 2I_2}{I_2 I_3}\eta^2\eta^5\eta^6\right)\right]\Bigg\} \\
&+ \quad \frac{1}{2}\frac{gI_3 - l_0^2}{I_3} + \mathcal{O}(\epsilon^4)\,. \qquad (3.37)
\end{aligned}
$$

Notice that the Free Energy is exactly quadratic when written in terms of z (equation (3.29)), and cubic to $\mathcal{O}(\epsilon^2)$ when written in terms of η. In a sense, by transforming from z to η, we have taken the nonlinearity of the weakly nonlinear equations out of the $\mathcal{O}(\epsilon)$ term in the z-bracket (3.30) and put it in the Hamiltonian.

Before we derive the equations of motion, we will make a few observations that will simplify them considerably. First, we note that both the third and the sixth rows in the matrix in (3.30) are zeros. Therefore, the coordinates η^3 and η^6 are local Casimirs, that is Casimirs of the bracket defined by (3.30), and thus constants of motion to $\mathcal{O}(\epsilon^2)$. Their value for any given problem is determined by the initial perturbation. Second, we note the remarkable fact that every cubic term in the weakly nonlinear Free Energy (3.37) involves at least one power of either η^3 or η^6. But because these variables are constants of motion, our Hamiltonian is effectively quadratic, and thus leads to a *linear* set of equations. Also, conveniently, these linear equations have the same form as the linearized equations (3.31); only the constant coefficients are different, modified by a small correction.

So, setting $\eta^3 = K_1$ and $\eta^6 = K_2$, the nontrivial perturbation equations to $\mathcal{O}(\epsilon)$ are

$$
\begin{aligned}
\dot{\eta}^1 &= \left\{ \frac{l_0}{I_3} + \epsilon \left[\frac{K_1 l_0}{2g} \left(\frac{1}{I_2} - \frac{1}{I_3} \right) + \frac{K_2}{2} \left(\frac{2}{I_3} - \frac{1}{I_2} \right) \right] \right\} \eta^2 - \frac{g + \epsilon K_1}{I_2} \eta^5 \\
\dot{\eta}^2 &= -\left\{ \frac{l_0}{I_3} + \epsilon \left[\frac{K_1 l_0}{2g} \left(\frac{1}{I_1} - \frac{1}{I_3} \right) + \frac{K_2}{2} \left(\frac{2}{I_3} - \frac{1}{I_1} \right) \right] \right\} \eta^1 + \frac{g + \epsilon K_1}{I_1} \eta^4 \\
\dot{\eta}^4 &= \left\{ \frac{l_0 (I_2 - I_3)}{I_2 I_3} - \epsilon \left[\frac{K_1 l_0}{2g} \left(\frac{1}{I_2} + \frac{1}{I_3} \right) + \frac{K_2}{2} \left(\frac{1}{I_2} - \frac{2}{I_3} \right) \right] \right\} \eta^5 \\
&+ \left\{ 1 + \epsilon \left[\frac{K_1 l_0^2}{2g^2} \left(\frac{1}{I_2} - \frac{1}{I_3} \right) + \frac{K_2 l_0}{2g I_2} \right] \right\} \eta^2 \qquad (3.38) \\
\dot{\eta}^5 &= \left\{ \frac{l_0 (I_3 - I_1)}{I_2 I_3} + \epsilon \left[\frac{K_1 l_0}{2g} \left(\frac{1}{I_3} + \frac{1}{I_1} \right) + \frac{K_2}{2} \left(\frac{1}{I_1} - \frac{2}{I_3} \right) \right] \right\} \eta^4 \\
&- \left\{ 1 - \epsilon \left[\frac{K_1 l_0^2}{2g^2} \left(\frac{1}{I_3} - \frac{1}{I_1} \right) + \frac{K_2 l_0}{2g I_1} \right] \right\} \eta^1 .
\end{aligned}
$$

3.2 Noncanonical Perturbation Theory - Infinite Degrees of Freedom

The procedure just outlined for finding the Hamiltonian structure of a truncated system of equations must be generalized to infinite dimensional systems to be applicable to the Vlasov-Poisson system.

We begin with the Vlasov-Poisson system linearized about a *linearly stable*, homogeneous equilibrium denoted $f_0(v)$. So the distribution function can be written $f(x,v,t) = f_0(v) + \epsilon \hat{f}(x,v,t)$, where ϵ is a small parameter. Substituting this expression for the distribution function into equations (2.1) and (2.2) while assuming the background charge density neutralizes the equilibrium, yields the equations for the evolution of the disturbance $\hat{f}$:

$$
\frac{\partial \hat{f}}{\partial t} + v \frac{\partial \hat{f}}{\partial x} + \frac{e}{m} \frac{\partial f_0}{\partial v} \hat{E}(x,t) + \epsilon \frac{e}{m} \hat{E}(x,t) \frac{\partial \hat{f}}{\partial v} = 0 \qquad (3.39)
$$

$$
\frac{\partial \hat{E}}{\partial x} = 4\pi e \left(\int_{-\infty}^{\infty} \mathrm{d}v \, \hat{f}(x,v,t) \right) . \qquad (3.40)
$$

In analogy to the finite degree-of-freedom case, we first find a Free Energy for equation (3.39), and then determine the change of variables that flattens bracket (2.15) to $\mathcal{O}(\epsilon^2)$.

3.2.1 Vlasov-Poisson Free Energy

Given an equilibrium $f_0(v)$, a suitable Free Energy for equation (3.39) is constructed by adding (2.19) to (2.14), taking the variation, and setting it equal to zero. The resulting Free Energy is then

$$\mathcal{F}[\hat{f}] = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x \, mv^2 \hat{f} + C(\hat{f}) + \frac{1}{8\pi}\int_{-L}^{L} \mathrm{d}x \, E^2 \,, \tag{3.41}$$

where C satisfies

$$\left.\frac{\mathrm{d}C}{\mathrm{d}f}\right|_{f=f_0} = -\frac{1}{2}mv^2 + \phi_0 \,. \tag{3.42}$$

The RHS of (3.42) is to be understood as a function of f_0 (and energy, via the implicit function theorem), and ϕ_0 denotes the electrostatic potential generated by the equilibrium distribution. Since we will ultimately use the free energy defined by equations (3.41) and (3.42) to generate the weakly nonlinear Vlasov-Poisson system, it is convenient here to expand F through $\mathcal{O}(\epsilon)$ about f_0 :

$$\mathcal{F}[\hat{f}] = \frac{1}{8\pi}\int_{-L}^{L} \mathrm{d}x \, E^2 - m\int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x \left[\frac{v}{2f_0'} f^2 + \frac{\epsilon}{6f_0'^2}\left(1 - \frac{vf_0''}{f_0'}\right) f^3\right] + \mathcal{O}(\epsilon^2) \,. \tag{3.43}$$

At lowest order, expression (3.43) is that discovered by Kruskal and Oberman [47].

3.2.2 Flattening the Vlasov Bracket

Determining the variable transformation that flattens the zeroth-order bracket is difficult for general infinite-dimensional brackets, and is not solved here. However, extending the zeroth-order bracket of the form (2.15) is relatively simple.

The cosymplectic operator for the bracket (2.15) is given in equation (2.17). Near a spatially homogeneous equilibrium f_0, we write the distribution function as a sum of the equilibrium and a small perturbation of that equilibrium: $f = f_0 + \epsilon\hat{f}$. In terms of the perturbation $\hat{f}$, the bracket takes the form

$$\mathcal{J}[\hat{f}](\cdot) = -[f_0, \cdot] - \epsilon[\hat{f}, \cdot]\,. \tag{3.44}$$

So, we wish to find a new variable η such that

$$\mathcal{J}[\eta](\cdot) = -[f_0, \cdot] + \mathcal{O}(\epsilon^2)\,. \tag{3.45}$$

By the transformation law of cosymplectic operators,

$$\mathcal{J}[\eta](\cdot) = \frac{\delta\eta}{\delta\hat{f}}\mathcal{J}[\hat{f}]\frac{\delta\eta}{\delta\hat{f}}^{\dagger}\,. \tag{3.46}$$

To find the variable transformation, we set the RHS of (3.45) equal to the RHS of (3.46) and solve for η.

The procedure outlined above would be difficult if we had no idea of the relationship of η to $\hat{f}$, but in analogy to the finite dimensional case, we assume that the transformation is near identity. So, letting $\mathcal{D}$ be a linear operator, we assume

$$\eta = \hat{f} + \frac{\epsilon}{2}\mathcal{D}(\hat{f}^2)\,. \tag{3.47}$$

Linearizing equation (3.47) gives us the operators needed to transform the cosymplectic operator in (3.46):

$$\frac{\delta\eta}{\delta\hat{f}}(\cdot) = 1 + \epsilon\mathcal{D}(\hat{f}\cdot)\,, \qquad \frac{\delta\eta}{\delta\hat{f}}^{\dagger}(\cdot) = 1 + \epsilon\hat{f}\mathcal{D}^{\dagger}(\cdot)\,. \tag{3.48}$$

Then the equation that results from setting the RHS of (3.45) equal to the RHS of (3.46), and using the spatial homogeneity of f_0 is

$$f_0'\frac{\partial}{\partial x}(\cdot) = f_0'\frac{\partial}{\partial x}(\cdot) + \epsilon\left[f_0'\frac{\partial}{\partial x}\left(\hat{f}\mathcal{D}^{\dagger}\cdot\right) - [\hat{f}, \cdot]\right]\,. \tag{3.49}$$

Clearly, equation (3.49) will be satisfied if the coefficient of ϵ on the RHS vanishes. A small amount of algebra reveals that this occurs if we choose

$$\mathcal{D}(\cdot) = -\frac{\partial}{\partial v}\left(\frac{1}{f_0'}\cdot\right) . \tag{3.50}$$

Thus, the transformation to η is given by

$$\eta = \hat{f} - \epsilon\frac{\partial}{\partial v}\left(\frac{\hat{f}^2}{2f_0'}\right) , \tag{3.51}$$

and the inverse transformation by

$$\hat{f} = \eta + \epsilon\frac{\partial}{\partial v}\left(\frac{\eta^2}{2f_0'}\right) . \tag{3.52}$$

In terms of η, the Poisson bracket now valid to $\mathcal{O}(\epsilon)$ is given by substituting the cosymplectic operator (3.45) into the general form (2.13):

$$\{\mathcal{F},\mathcal{G}\}[\eta] = \int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x\, \frac{\delta F}{\delta\eta}\frac{f_0'}{m}\frac{\partial}{\partial x}\frac{\delta G}{\delta\eta} + \mathcal{O}(\epsilon^2) . \tag{3.53}$$

In the same variables, the truncated free energy for Vlasov-Poisson is

$$\mathcal{F}[\eta] = \frac{e}{8\pi}\int_{-L}^{L} \mathrm{d}x\, E^2 - \frac{m}{2}\int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x \left(\frac{v}{f_0'}\eta^2 + \frac{\epsilon}{3f_0'^2}\eta^3\right) + \mathcal{O}(\epsilon^2) , \tag{3.54}$$

where E is now understood as a functional of η and still determined by Poisson's equation

$$\frac{\partial E}{\partial x} = 4\pi e \int_{-\infty}^{\infty} \mathrm{d}v\, \eta . \tag{3.55}$$

From (3.53) and (3.54), we can finally derive the weakly nonlinear equation:

$$\frac{\partial\eta}{\partial t} = -v\frac{\partial\eta}{\partial x} - \frac{e}{m}f_0'E + \frac{\epsilon}{f_0'}\eta\frac{\partial\eta}{\partial x} . \tag{3.56}$$

Equation (3.56) differs from the linearized system only by the term proportional to ϵ. The remainder of the dissertation is concerned with analyzing the effects of that term.

One important result we can see almost immediately is that equation (3.56) has an integral of motion in addition to its Hamiltonian (3.54). The additional integral, P, is given by:

$$P := \int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x \, \frac{\eta^2}{f_0'} \, . \tag{3.57}$$

To see that (3.57) is indeed a constant of motion, we calculate the Poisson bracket (3.53) between it and the Hamiltonian. We find

$$\{P, F\}[\eta] = \int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x \left[\frac{e}{m} \eta E - \frac{1}{f_0'} \eta \frac{\partial \eta}{\partial x} - \epsilon \frac{2}{(f_0')^2} \eta^2 \frac{\partial \eta}{\partial x} \right] . \tag{3.58}$$

The second two terms in (3.58) are both exact x-derivatives, and so vanish. We can use Poisson's equation (3.55) rewrite the first term as

$$\{P, F\}[\eta] = \frac{e}{m} \int_{-L}^{L} \mathrm{d}x \, E \int_{-\infty}^{\infty} \mathrm{d}v \, \eta = \frac{4\pi e^2}{m} \int_{-L}^{L} \mathrm{d}x \, E \frac{\partial E}{\partial x} \, . \tag{3.59}$$

Clearly, the integrand of (3.59) is also an exact x-derivative, so $\{P, F\} = 0$.

Now, P as defined in (3.57) is proportional to a quantity that we can identify as the momentum of the system (3.56). Indeed, if we multiply it by m, it has the units of momentum. To find the precise proportionality factor, we argue as follows. Consider a one degree of freedom system with a "standard" Hamiltonian written in canonical variables: $H(q,p) = p^2/2m + V(q)$. If we perform a Galilean boost on this system, $q' = q - vt; p' = p - mv$ into a frame moving with speed v relative to the first frame, we find that the transformed Hamiltonian has the form

$$H(q', p', t) = \frac{(p')^2}{2m} + vp' + \frac{mv^2}{2} + V(q', t) \, . \tag{3.60}$$

We see that the functional form of (3.60) differs from that of the original Hamiltonian only by the new term vp' (and a constant frame energy which can be ignored). This fact suggests a rule of thumb for writing down a Hamiltonian in a frame boosted by a velocity v: change all the old variables in the Hamiltonian to new ones, and add the scalar product of v and the new momentum.

Since (3.54) is the physical energy of the weakly nonlinear Vlasov-Poisson system, we expect the same rule of thumb to apply. Applying the Galilean boost $x' = x - ut; v' = v - u$ to the Hamiltonian, we find

$$\mathcal{F}[\eta] = \frac{e}{8\pi}\int_{-L}^{L} \mathrm{d}x'\, E^2 - \frac{m}{2}\int_{-\infty}^{\infty} \mathrm{d}v' \int_{-L}^{L} \mathrm{d}x \left(\frac{v'+u}{f_0'}\eta^2 + \frac{\epsilon}{3f_0'^2}\eta^3\right) + \mathcal{O}(\epsilon^2)\,, \quad (3.61)$$

where f_0 is now a function of v' and the derivative on it is understood to be with respect to v'. The new term that appears is simply

$$-\frac{mu}{2}\int_{-\infty}^{\infty} \mathrm{d}v \int_{-L}^{L} \mathrm{d}x\, \frac{\eta^2}{f_0'}\,. \quad (3.62)$$

This is none other than the constant of motion P given by (3.57), multiplied by $-mu/2$. Hence, the momentum of the weakly nonlinear Vlasov-Poisson system is given by $mP/2$.

Before we conclude this chapter, there are a couple of important observations about bracket (3.53) that we must make. First, this bracket does not have the same Casimirs as (2.15). However, if the perturbed system is a valid approximation to the full system, the constraints imposed by the invariance of the Casimirs must still be obeyed. Indeed, we ensure these constraints by using the free energy as Hamiltonian [48],[13].

This is not to say that bracket (3.53) does not constrain the evolution of η at all. Since the mode η_0 does not appear in (3.53), η_0 remains constant during the evolution. Also, from the forms of bracket (3.53) and equation (2.3), we see that if the initial perturbation $\eta_k(v,0)$ satisfies $\eta_k(v,0) \propto f_0'(v)$, then the perturbation satisfies

$$\eta_k(v,t) \propto f_0'(v) \tag{3.63}$$

for all time and for $k \neq 0$. Physically speaking, relation (3.63) and the constancy of the $k = 0$ mode together comprise the perturbative expression of the fact that a distribution function is merely rearranged as it evolves under the Vlasov equation.

Of course, the initial perturbation need not be a rearrangement of an equilibrium distribution. The initial conditions that *are* rearrangements of the equilibrium distribution functions are called *dynamically accessible* by virtue of the fact that they could have been generated by some Hamiltonian using the Poisson bracket (2.15). For the sake of simplicity, we restrict our attention in this problem to dynamically accessible perturbations. Specifically, we require that $\eta_0(v,t) = 0$, and that $\eta_k(v,0) \propto f_0'(v)$. These requirements guarantee that the coordinate changes given by equations (3.51) and (3.52) suffer no singularities because of the presence of extrema of $f_0(v)$. These conditions are also used in deriving the stability result in Chapter 5.

Restricting to dynamically accessible perturbations is not as restrictive as it may seem; a perturbation that is not dynamically accessible from one equilibrium may be a perturbation accessible from another. In such a case, the evolution of the non-accessible perturbation could be studied by considering the equivalent dynamics of an accessible perturbation around the new equilibrium.

However, we must sound a note of caution about the new variable η. Unlike the finite dimensional flattening transformation in equations (3.12) and (3.13), the range of validity of the infinite dimensional transformation in (3.51) and (3.52) does not depend only on the magnitude of $\hat{f}$. Inspecting equation (3.51) shows that, at the very least, we have a condition on the derivative of $\hat{f}$ (and a similar one on η):

$$\epsilon \left| \frac{\partial \hat{f}}{\partial v} \right| << |f_0'(v)| \Rightarrow \epsilon \left| \frac{\partial \eta}{\partial v} \right| << |f_0'(v)| \, . \tag{3.64}$$

Condition (3.64) puts a strong restriction on the range of validity of any system which uses this variable transformation. As we shall see, it will amount to limiting the time range for which the weakly nonlinear Vlasov-Poisson equations are applicable. Still, the equations will be valid for times close to the initial time, and the nonlinear behavior of transients can be investigated.

3.3 Discussion

The technique of adding a Casimir to the Hamiltonian to construct a new Hamiltonian with a critical point at an equilibrium is the first step in the so-called energy-Casimir stability method [46]. This technique can be extended to include solutions that are equilibria in moving frames by adding constants of motion (momenta, for instance) that depend on the symmetries of the Hamiltonian, and not the Poisson bracket alone. This point, of course, is related to the interpretation of (3.62) as a momentum.

Another point we should make is that the restriction (3.64) on the validity of the bracket-flattening transformation may be inherent in the problem, rather than a mere limitation to the coordinate transformation. Holloway and Dorning argue correctly, in [49], that the assumption of a small perturbation does not in itself justify the neglect of the v-derivative of the perturbation. However, the further argument (see, for example, [50]), that the Van Kampen modes do not account for trapped particles has a more unclear status. Against it, one might bring the point that in the original BGK paper, the BGK modes were shown to limit into Van Kampen modes [18]. To further complicate the situation, it was noted in [51] that in a limit in which the background distribution has an inflection point, a BGK mode goes into a neutral oscillation. Restriction (3.64) may in fact be a symptom of the lack of particle trapping in Van Kampen modes; however, because of this restriction, the present weakly nonlinear theory cannot be used to decide the issue one way or another.

Chapter 4

Linear Theory and Three Mode Interactions

Now that we have flattened the Poisson bracket to $\mathcal{O}(\epsilon)$, we can easily find canonical variables for the bracket correct to $\mathcal{O}(\epsilon)$. Keeping in mind the goal of studying the system in the framework of canonical perturbation theory, though, the canonical variables we desire are action-angle variables for the unperturbed Hamiltonian. Since our unperturbed Hamiltonian is quadratic, we can get most of the way to action-angle variables by first diagonalizing it.

4.1 Diagonalization of Linearized Vlasov-Poisson, and Transformation to Action-Angle Variables

The problem of finding canonical variables that diagonalize the quadratic part of the Free Energy (3.54) was solved in [13]-[15]. The solution amounts to following Van

Kampen's approach to solving the linearized Vlasov-Poisson system, but with an important change of viewpoint. Van Kampen used an ansatz to solve the linearized equation; the authors of [13]-[15] made a transformation of variables and then determined the form the solutions take by actually solving the equation. The latter point of view makes it easier to explore the perturbed system for which a suitable ansatz may not be available.

4.1.1 A Singular Integral Transform

Diagonalizing the linearized Vlasov-Poisson system can be seen as one instance of a class of problems in which the linear operator involves a Hilbert transform, henceforth denoted by an overbar:

$$\bar{g}(v) := \frac{\mathcal{P}}{\pi} \int \frac{g(u)}{u - v} \mathrm{d}u \,. \tag{4.1}$$

The approach to solving these problems developed in (M & S, S) has the following basic outline. We assume that the diagonalizing variables are related to the original variables by a (family of) singular integral transform(s), of the form

$$G[g] := \alpha \bar{g} + \beta g \,, \tag{4.2}$$

with an inverse of the form

$$\hat{G}[g] := \zeta \bar{g} + \chi g \,. \tag{4.3}$$

In (4.2) and (4.3), we have introduced four unknown functions α, β, ζ, and ξ. These unknown functions are then determined by the requirements that (4.2) and (4.3) are inverses of each other, and that (4.2) actually diagonalizes the Hamiltonian. These requirements, of course, provide only three equations for the four unknowns.

A fourth condition that is convenient for the solution of the diagonalization of linearized Vlasov-Poisson is to assert

$$\beta = \bar{\alpha} + C \,, \tag{4.4}$$

where C is a constant.

Using well-known properties of the Hilbert transform (specifically $\bar{\bar{g}} = -g$ and $\overline{gh} = g\bar{h} + \bar{g}h + \overline{\bar{g}\bar{h}}$)), we find that the invertibility requirements yield

$$\zeta = -\frac{\alpha}{\alpha^2 + \beta^2}; \qquad \chi = \frac{\beta}{\alpha^2 + \beta^2} \,. \tag{4.5}$$

Obviously, equations (4.5) can only hold if

$$\alpha^2 + \beta^2 \neq 0 \,. \tag{4.6}$$

So, the only problem that remains is to find α, and that function is determined by the diagonalization requirement. But before we state the results of applying this algorithm to the linear Vlasov-Poisson system, we note two more identities that can be proved in this general framework, and will be useful to us later [15]:

$$\int \mathrm{d}v \, \frac{v}{\alpha} G[g]G[h] = -\int \mathrm{d}v \, \frac{v}{\zeta} gh - \frac{C}{\pi} \int \mathrm{d}v \, g(v) \int \mathrm{d}v' \, h(v') \,, \tag{4.7}$$

where C is the same constant in (4.4), and

$$\int \mathrm{d}v \, \frac{1}{\alpha} G[g]G[h] = -\int \mathrm{d}v \, \frac{1}{\zeta} gh \,. \tag{4.8}$$

4.1.2 Diagonalizing Linearized Vlasov-Poisson

Proceeding to the linearized Vlasov-Poisson system, we begin by expanding η in a Fourier series:

$$\eta(x, v, t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \eta_k(v, t) e^{ikx} \,. \tag{4.9}$$

The Fourier components of the electric field then are found from Poisson's equation to be

$$E_k(t) = \frac{4\pi e}{ik} \int \mathrm{d}v \, \eta_k(v,t) \,. \tag{4.10}$$

In terms of η_k, the Poisson bracket (3.53) takes the nearly canonical form

$$\{F,G\} = \frac{4i}{m} \sum_{k=1}^{\infty} k \int \mathrm{d}v \, f_0'(v) \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \eta_{-k}} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \eta_{-k}} \right) \,. \tag{4.11}$$

And, the unperturbed Hamiltonian can be written entirely in terms of η_k thanks to (4.10):

$$\mathcal{F}^{(2)}[\eta_k] = \frac{1}{4} \sum_{k=1}^{\infty} \left[\frac{(4\pi e)^2}{8\pi k^2} \int \mathrm{d}v \, \eta_k(v) \int \mathrm{d}v' \, \eta_{-k}(v') - \frac{m}{2} \int \mathrm{d}v \, v \frac{\eta_k \eta_{-k}}{f_0'} \right] \,. \tag{4.12}$$

Now, we attempt to diagonalize (4.12) by performing the coordinate transformation

$$\eta_k(v,t) = \frac{ik}{4\pi e} G[E_k(u,t)] \,, \tag{4.13}$$

using (4.2), and where we recognize that α (and thus β) may depend on k. Notice that the new coordinate $E_k(u,t)$ takes another argument besides t, and thus is *not* the same as the kth Fourier component of the electric field. Appealing to identity (4.7), we find

$$\alpha = -\pi \frac{\omega_p^2}{k^2} f_0'(u) \,, \tag{4.14}$$

diagonalizes the linearized Hamiltonian provided we set the constant in (4.4) to unity: $C = 1$. Thus we have

$$\beta = 1 + \bar{\alpha} \,. \tag{4.15}$$

The unknown functions in the inverse transformation are then found using equation (4.5). In terms of all these unknown functions, the diagonalized Hamiltonian is

$$\mathcal{F}^{(2)} = \frac{1}{32} \sum_{k=-\infty}^{\infty} \int \mathrm{d}u \, \frac{u}{\zeta} E_k(u,t) E_{-k}(u,t) \,. \tag{4.16}$$

Now, the Hamiltonian (4.16) has a compact form when written in terms of ζ, but may be a little obscure physically. We can put it in more physically accessible terms with a couple of observations.

Consider the plasma dielectric function for longitudinal waves. This is given by [15]:

$$\begin{aligned} \varepsilon(k, kv) &= 1 - \frac{\omega_p^2}{k^2}\frac{1}{v}\lim_{\mu\to 0^+}\int \mathrm{d}u \, \frac{u}{u - v - i\mu} f_0'(v) \\ &= 1 - \mathcal{P}\int \mathrm{d}u \, \frac{\omega_p^2}{k^2}\frac{f_0'(u)}{u - v} - i\pi\frac{\omega_p^2}{k^2} f_0'(v)\,. \end{aligned} \tag{4.17}$$

It is convenient to define the real and imaginary parts of expression (4.17) respectively as

$$\varepsilon_R(k, kv) := 1 - \mathcal{P}\int \mathrm{d}u \, \frac{\omega_p^2}{k^2}\frac{f_0'(u)}{u - v}\,, \tag{4.18}$$

and

$$\varepsilon_I(k, kv) := -\pi\frac{\omega_p^2}{k^2} f_0'(v)\,. \tag{4.19}$$

But comparing equations (4.18) and (4.19) to (4.14) and (4.15) we see that

$$\alpha \equiv \varepsilon_I(k, ku); \qquad \beta \equiv \varepsilon_R(k, ku)\,. \tag{4.20}$$

Substituting (4.20) into (4.5), we find the coefficients of the inverse transformation are

$$\zeta \equiv -\frac{\varepsilon_I(k, ku)}{|\varepsilon(k, ku)|^2}; \qquad \chi \equiv \frac{\varepsilon_R}{|\varepsilon(k, ku)|^2}\,. \tag{4.21}$$

Also, we can easily interpret the condition (4.6):

$$\alpha^2 + \beta^2 \equiv |\varepsilon(k, kv)|^2 \neq 0\,. \tag{4.22}$$

This tells us that $\varepsilon(k, kv) \neq 0$, or simply that the equilibrium is linearly stable, supporting no true wave solutions.

So, having seen the relationship between the plasma dielectric function and the functions in the integral transform (4.2), we can rewrite the unperturbed Hamiltonian as

$$\mathcal{F}^{(2)} = \frac{1}{16}\sum_{k=1}^{\infty}\int \mathrm{d}u\, u\frac{|\varepsilon(k,ku)|^2}{\varepsilon_I(k,ku)}E_k(u,t)E_{-k}(u,t)\,. \tag{4.23}$$

To find the form of the Poisson bracket, we note that $E_k = \left(\frac{4\pi e}{ik}\right)\hat{G}[\eta_k]$ implies

$$\frac{\delta F}{\delta\eta_k} = \left(\frac{4\pi e}{ik}\right)\hat{G}^\dagger\left[\frac{\delta F}{\delta E_k}\right]\,. \tag{4.24}$$

And substituting (4.24) into (4.11) gives us the Poisson bracket in terms of E_k:

$$\begin{aligned}\{F,G\} &= -16i\sum_{k=1}^{\infty}k\int \mathrm{d}u\,\frac{\varepsilon_I(k,ku)}{|\varepsilon(k,ku)|^2}\\ &\quad\times\left(\frac{\delta F}{\delta E_k}\frac{\delta G}{\delta E_{-k}} - \frac{\delta G}{\delta E_k}\frac{\delta F}{\delta E_{-k}}\right)\,,\end{aligned} \tag{4.25}$$

From (4.23) and (4.25) follow the equations of motion for all k :

$$\dot{E}_k(u,t) = -ikuE_k(u,t)\,. \tag{4.26}$$

And so, $E_k(u,t) = E_{k0}\exp(-ikut)$, which has the form of Van Kampen's ansatz [10].

While we are considering the form taken by the Hamiltonian in the variables $E_k(u,t)$, we should do the same for the momentum integral P given in (3.57). Rewriting the actual momentum $M = (m/2)P$ in Fourier components, we have

$$M = \frac{m}{8}\sum_{k=-\infty}^{\infty}\int \mathrm{d}v\,\frac{\eta_k\eta_{-k}}{f_0'}\,. \tag{4.27}$$

Using equations (4.13) and (4.14), we rewrite (4.27) as

$$M = \frac{m\omega_p^2}{128\pi e^2}\sum_{k=-\infty}^{\infty}\int \mathrm{d}v\,\frac{G[E_k]G[E_{-k}]}{\alpha}\,. \tag{4.28}$$

And here is where our other identity involving G becomes useful to us: applying (4.8) to the integral in (4.28), we find

$$M = -\frac{m\omega_p^2}{128\pi e^2} \sum_{k=-\infty}^{\infty} \int \mathrm{d}v \, \frac{E_k E_{-k}}{\zeta}\,, \tag{4.29}$$

where ζ is given in equation (4.21). Using the definitions of the plasma frequency and ζ, and the parity in k, we can rewrite (4.29) as

$$H = \frac{1}{16} \sum_{k=1}^{\infty} \int \mathrm{d}v \, \frac{|\varepsilon(k,ku)|^2}{\varepsilon_I(k,ku)} E_k E_{-k}\,. \tag{4.30}$$

We will revisit the momentum (4.30) in chapter 6 (as well as write it in terms of the action-angle variables for the unperturbed system below).

We end our discussion of the diagonalization of the linear Vlasov-Poisson system with a comment about the range of validity of the $E_k(v,t)$. (3.64) on the validity of the transformation from $f_k(v,t)$ to $\eta_k(v,t)$, implies a condition on the u-derivatives of $E_k(u,t)$. To derive this, we simply take the v-derivative of $\eta_k(v,t) = G[E_k](v,t)$. This gives us

$$\begin{aligned} \frac{\partial \eta_k}{\partial v} = \left(\frac{ik}{4\pi e}\right) \Big[\bar{E}_k(v,t)\frac{\partial}{\partial v}\varepsilon_I(k,kv) + \varepsilon_I(k,kv)\frac{\partial \bar{E}_k}{\partial v}(v,t) \\ + E_k(v,t)\frac{\partial}{\partial v}\varepsilon_R(k,kv) + \varepsilon_R(k,kv)\frac{\partial E_k}{\partial v}(v,t)\Big]\,. \end{aligned} \tag{4.31}$$

Substituting in the consequent of (3.64), we find

$$\begin{aligned} \epsilon k\Big|\bar{E}_k(v,t)\frac{\partial}{\partial v}\varepsilon_I(k,kv) + \varepsilon_I(k,kv)\frac{\partial \bar{E}_k}{\partial v}(v,t) \\ + E_k(v,t)\frac{\partial}{\partial v}\varepsilon_R(k,kv) + \varepsilon_R(k,kv)\frac{\partial E_k}{\partial v}(v,t)\Big| << |f_0'(v)|\,. \end{aligned} \tag{4.32}$$

Thus, for $t \sim 1/\epsilon$, the linearized equations (4.26) predict that $\partial E_k/\partial v \sim k/\epsilon$. This certainly violates (4.32) for $k \sim \mathcal{O}(1)$.

4.1.3 Action-angle Variables for Linearized Vlasov-Poisson

Now that we have a diagonal Hamiltonian (4.23), we can easily transform to action-angle variables. The transformation is given by

$$E_k(u) = \sqrt{\frac{16|k||\varepsilon_I(k,ku)|}{|\varepsilon(k,ku)|^2}J_{|k|}(u)}\,\exp[-i\mathrm{sgn}(k\varepsilon_I(k,ku))\theta_{|k|}(u)]\,. \tag{4.33}$$

Applying (4.33) to (4.25), we arrive at the canonical bracket:

$$\{F,G\} = \sum_{k=1}^{\infty}\int \mathrm{d}u\left(\frac{\delta F}{\delta\theta_k}\frac{\delta G}{\delta J_k} - \frac{\delta G}{\delta\theta_k}\frac{\delta F}{\delta J_k}\right)\,. \tag{4.34}$$

And substituting (4.33) into (4.23) gives us

$$\mathcal{F}^{(2)} = \sum_{k=1}^{\infty}\int \mathrm{d}u\,\mathrm{sgn}(\varepsilon_I(k,ku))kuJ_k(u)\,; \tag{4.35}$$

the unperturbed Hamiltonian thus has the same form in action-angle variables as the Hamiltonian for a set of uncoupled oscillators parameterized by a discrete index k and a continuous index u. It also closely resembles the form of the momentum (4.30) in action-angle variables:

$$M = \sum_{k=1}^{\infty}\int \mathrm{d}u\,\mathrm{sgn}(\varepsilon_I(k,ku))kJ_k(u)\,. \tag{4.36}$$

Having demonstrated with (4.35) that (4.33) is indeed a transformation to action-angle variables, we are finally in a position to begin applying the techniques of canonical perturbation theory to the weakly nonlinear Vlasov-Poisson equation. The first step is to write the perturbation in terms of the action-angle variables of the unperturbed system.

4.2 Vlasov-Poisson Three Mode Interactions

The cubic perturbative piece of the Vlasov–Poisson Hamiltonian in terms of the $E_k(v)$ is obtained by inserting $\eta_k(v,t) = G[E_k(u,t)]$ into the coefficient of ϵ in the

Fourier expansion of (3.54). The result is a sum of the energies of every relevant (at this order) interaction between three Van Kampen modes. It takes the form of a sum of iterated principal value integrals:

$$
\begin{aligned}
\mathcal{F}^{(3)} &= \frac{m}{48}\left(\frac{i}{4\pi e}\right)\sum_{k_a+k_b+k_c=0}(k_a k_b k_c) \\
&\times \left[\int \mathrm{d}v\, \frac{1}{(f_0')^2}\varepsilon_I(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_I(k_c,k_c v)\right. && (4.37)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_a\,\frac{E_{k_a}(u_a,t)}{u_a-v}\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_b\,\frac{E_{k_b}(u_b,t)}{u_b-v}\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_c\,\frac{E_{k_c}(u_c,t)}{u_c-v}\right) \\
&+ \int \mathrm{d}v\,\frac{\varepsilon_I(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_R(k_c,k_c v)}{(f_0')^2}E_{k_c}(v,t) && (4.38)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_a\,\frac{E_{k_a}(u_a,t)}{u_a-v}\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_b\,\frac{E_{k_b}(u_b,t)}{u_b-v}\right) \\
&+ \int \mathrm{d}v\,\frac{\varepsilon_I(k_a,k_a v)\varepsilon_R(k_b,k_b v)\varepsilon_I(k_c,k_c v)}{(f_0')^2}E_{k_b}(v,t) && (4.39)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_a\,\frac{E_{k_a}(u_a,t)}{u_a-v}\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_c\,\frac{E_{k_c}(u_c,t)}{u_c-v}\right) \\
&+ \int \mathrm{d}v\,\frac{\varepsilon_R(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_I(k_c,k_c v)}{(f_0')^2}E_{k_a}(v,t) && (4.40)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_b\,\frac{E_{k_b}(u_b,t)}{u_b-v}\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_c\,\frac{E_{k_c}(u_c,t)}{u_c-v}\right) \\
&+ \int \mathrm{d}v\,\frac{\varepsilon_I(k_a,k_a v)\varepsilon_R(k_b,k_b v)\varepsilon_R(k_c,k_c v)}{(f_0')^2}E_{k_b}(v,t)E_{k_c}(v,t) && (4.41)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_a\,\frac{E_{k_a}(u_a,t)}{u_a-v}\right) \\
&+ \int \mathrm{d}v\,\frac{\varepsilon_R(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_R(k_c,k_c v)}{(f_0')^2}E_{k_c}(v,t)E_{k_a}(v,t) && (4.42)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_b\,\frac{E_{k_b}(u_b,t)}{u_b-v}\right) \\
&+ \int \mathrm{d}v\,\frac{\varepsilon_R(k_a,k_a v)\varepsilon_R(k_b,k_b v)\varepsilon_I(k_c,k_c v)}{(f_0')^2}E_{k_a}(v,t)E_{k_b}(v,t) && (4.43)\\
&\quad \left(\frac{\mathcal{P}}{\pi}\int \mathrm{d}u_c\,\frac{E_{k_c}(u_c,t)}{u_c-v}\right) \\
&+ \left.\int \mathrm{d}v\,\frac{\varepsilon_R(k_a,k_a v)\varepsilon_R(k_b,k_b v)\varepsilon_R(k_c,k_c v)}{(f_0')^2}E_{k_a}(v,t)E_{k_b}(v,t)E_{k_c}(v,t)\right]. \\
& && (4.44)
\end{aligned}
$$

We exchange orders of integration (see Appendix B) to put all the singularities in the innermost integral. This yields (suppressing time dependence):

$$\begin{aligned}
\mathcal{F}^{(3)} &= \sum_{k_a+k_b+k_c=0} \left(\frac{m}{48}\right)\left(\frac{i}{4\pi e}\right)^3 k_a k_b k_c \\
&\times \Big\{ \frac{1}{\pi^3}\int \mathrm{d}u_a\,\mathrm{d}u_b\,\mathrm{d}u_c\, E_{k_a}(u_a)E_{k_b}(u_b)E_{k_c}(u_c) \\
&\qquad \left(\mathcal{P}\int \mathrm{d}v\, \frac{H_4(k_a,k_b,k_c,v)}{(u_a-v)(u_b-v)(u_c-v)}\right) && (4.45)\\
&+ \frac{1}{\pi^2}\int \mathrm{d}u_a\,\mathrm{d}u_b\, E_{k_a}(u_a)E_{k_b}(u_b)\left(\mathcal{P}\int \mathrm{d}v \frac{H_3(k_a,k_b,k_c,v)}{(u_a-v)(u_b-v)} E_{k_c}(v)\right) && (4.46)\\
&+ \frac{1}{\pi^2}\int \mathrm{d}u_c\,\mathrm{d}u_a\, E_{k_c}(u_c)E_{k_a}(u_a)\left(\mathcal{P}\int \mathrm{d}v \frac{H_3(k_c,k_a,k_b,v)}{(u_c-v)(u_a-v)} E_{k_b}(v)\right) && (4.47)\\
&+ \frac{1}{\pi^2}\int \mathrm{d}u_b\,\mathrm{d}u_c\, E_{k_b}(u_b)E_{k_c}(u_c)\left(\mathcal{P}\int \mathrm{d}v \frac{H_3(k_b,k_c,k_a,v)}{(u_b-v)(u_c-v)} E_{k_a}(v)\right) && (4.48)\\
&+ \frac{1}{\pi}\int \mathrm{d}u_a\, E_{k_a}(u_a)\left(\mathcal{P}\int \mathrm{d}v\, \frac{H_2(k_a,k_b,k_c,v)}{(u_a-v)} E_{k_b}(v)E_{k_c}(v)\right) && (4.49)\\
&+ \frac{1}{\pi}\int \mathrm{d}u_b\, E_{k_b}(u_b)\left(\mathcal{P}\int \mathrm{d}v\, \frac{H_2(k_b,k_c,k_a,v)}{(u_b-v)} E_{k_a}(v)E_{k_c}(v)\right) && (4.50)\\
&+ \frac{1}{\pi}\int \mathrm{d}u_c\, E_{k_c}(u_c)\left(\mathcal{P}\int \mathrm{d}v\, \frac{H_2(k_c,k_a,k_b,v)}{(u_c-v)} E_{k_a}(v)E_{k_b}(v)\right) && (4.51)\\
&+ \int \mathrm{d}v\, H_1(k_a,k_b,k_c;v)E_{k_a}(v)E_{k_b}(v)E_{k_c}(v)\Big\}\,. && (4.52)
\end{aligned}$$

where the functions $H_1 \ldots H_4$ are defined by

$$\begin{aligned}
H_4(k_a,k_b,k_c,v) &= (1/f_0'(v))^2 \varepsilon_I(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_I(k_c,k_c v) \\
&= \frac{k_a^2 k_b^2}{\pi^2\omega_p^4}\varepsilon_I(k_c,k_c v)\,; && (4.53)
\end{aligned}$$

$$\begin{aligned}
H_3(k_a,k_b,k_c,v) &= (1/f_0'(v))^2 \varepsilon_I(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_R(k_c,k_c v) \\
&= \frac{k_a^2 k_b^2}{\pi^2\omega_p^4}\varepsilon_R(k_c,k_c v)\,; && (4.54)
\end{aligned}$$

$$\begin{aligned}
H_2(k_a,k_b,k_c,v) &= (1/f_0'(v))^2 \left[\varepsilon_R(k_b,k_b v)\varepsilon_R(k_c,k_c v) + \varepsilon_I(k_b,k_b v)\varepsilon_I(k_c,k_c v)\right] \\
&\times\ \varepsilon_I(k_a,k_a v)\,; && (4.55)
\end{aligned}$$

$$
\begin{aligned}
H_1(k_a,k_b,k_c,v) \quad = \quad & (1/f_0'(v))^2\, [\, \varepsilon_R(k_a,k_a v)\varepsilon_R(k_b,k_b v)\varepsilon_R(k_c,k_c v) \\
& +\varepsilon_I(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_R(k_c,k_c v) \\
& +\varepsilon_I(k_a,k_a v)\varepsilon_R(k_b,k_b v)\varepsilon_I(k_c,k_c v) \\
& +\varepsilon_R(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_I(k_c,k_c v)\,]\ . \qquad (4.56)
\end{aligned}
$$

At this point, we can make some simplifications. Integral (4.45) can be broken up, through partial fractions, into a sum of three terms which respectively have the forms (4.46), (4.47), and (4.48), each with the function H_3 replaced by $-\bar{H}_4$. The resulting coefficient is a constant: $H_3 - \bar{H}_4 = k_a^2 k_b^2/\pi^2\omega_p^4$. Also, by suitably renaming the dummy u and k variables, and making use of the symmetry in k_a, k_b of $H_3 - \bar{H}_4$, we can formally combine the integrals (4.46)–(4.48). Similarly, using the symmetry in the last two discrete arguments displayed by H_2, we can combine (4.49)–(4.51). After all this is done, the nonlinear part of the Hamiltonian takes the simpler form:

$$
\begin{aligned}
\mathcal{F}^{(3)} \quad = \quad & \sum_{k_a+k_b+k_c=0} \left(\frac{m}{48}\right)\left(\frac{i}{4\pi e}\right)^3 k_a k_b k_c \\
\times \quad & \left\{ \frac{3}{\pi^2}\frac{k_a^2 k_b^2}{\pi^2\omega_p^4} \int \mathrm{d}u_a\, \mathrm{d}u_b \left(\mathcal{P}\int \mathrm{d}v\, \frac{E_{k_a}(u_a)E_{k_b}(u_b)E_{k_c}(v)}{(u_a-v)(u_b-v)}\right)\right. \qquad (4.57) \\
+ \quad & \frac{3}{\pi}\int \mathrm{d}u_a \left(\mathcal{P}\int \mathrm{d}v\, \frac{H_2(k_a,k_b,k_c,v)}{(u_a-v)} E_{k_a}(u_a)E_{k_b}(v)E_{k_c}(v)\right) \qquad (4.58) \\
+ \quad & \left. \int \mathrm{d}v\, H_1(k_a,k_b,k_c;v)E_{k_a}(v)E_{k_b}(v)E_{k_c}(v) \right\}\ . \qquad (4.59)
\end{aligned}
$$

Now, in analogy to finite degree of freedom canonical perturbation theory, we write the perturbation in terms of the action-angle variables of the unperturbed system. Inserting (4.33) into terms (4.57)–(4.59) yields

$$
\begin{aligned}
\mathcal{F}^{(3)} \quad = \quad & -\sum_{k_a+k_b+k_c=0} \left(\frac{m}{12}\right)\left(\frac{i}{4\pi e}\right)(k_a k_b k_c)\sqrt{|k_a k_b k_c|} \\
\times \quad & \left\{ \int \mathrm{d}u_a\, \mathrm{d}u_b \left[\mathcal{P}\int \mathrm{d}v\, \frac{F_3(k_a,k_b,k_c;u_a,u_b,v)}{(u_a-v)(u_b-v)} (J_{|k_a|}(u_a)J_{|k_b|}(u_b)J_{|k_c|}(v))^{1/2}\right.\right.
\end{aligned}
$$

$$\exp\left[-i\left(s_{k_a}(u_a)\theta_{|k_a|}(u_a)+s_{k_b}(u_b)\theta_{|k_b|}(u_b)+s_{k_c}(v)\theta_{|k_c|}(v)\right)\right]\Big] \quad (4.60)$$

$$+ \int \mathrm{d}u_a \left[\mathcal{P}\int \mathrm{d}v\,\frac{1}{(u_a-v)}F_2(k_a,k_b,k_c;u_a,v)(J_{|k_a|}(u_a)J_{|k_b|}(v)J_{|k_c|}(v))^{1/2}\right.$$

$$\left.\exp\left[-i\left(s_{k_a}(u_a)\theta_{|k_a|}(u_a)+s_{k_b}(v)\theta_{|k_b|}(v)+s_{k_c}(v)\theta_{|k_c|}(v)\right)\right]\right] \quad (4.61)$$

$$+ \int \mathrm{d}v\,F_1(k_a,k_b,k_c;v)(J_{|k_a|}(v)J_{|k_b|}(v)J_{|k_c|}(v))^{1/2}$$

$$\exp\left[-i\left(s_{k_a}(v)\theta_{|k_a|}(v)+s_{k_b}(v)\theta_{|k_b|}(v)+s_{k_c}(v)\theta_{|k_c|}(v)\right)\right]\ \Big\}. \quad (4.62)$$

In terms (4.60)–(4.62), we have defined a new function $s_k(v) := \mathrm{sgn}(k\varepsilon_I(k,kv))$, and three new coefficients, F_1, F_2, and, F_3, which have the expressions:

$$\begin{aligned}
F_3(k_a,k_b,k_c;u_a,u_b,v) &:= \frac{3k_a^2k_b^2}{\pi^4\omega_p^4}\sqrt{\frac{|\varepsilon_I(k_a,k_au_a)||\varepsilon_I(k_b,k_bu_b)||\varepsilon_I(k_c,k_cv)|}{|\varepsilon(k_a,k_au_a)|^2|\varepsilon(k_b,k_bu_b)|^2|\varepsilon(k_c,k_cv)|^2}}\,;\\
F_2(k_a,k_b,k_c;u_a,v) &:= \frac{3}{\pi}\sqrt{\frac{|\varepsilon_I(k_a,k_au_a)||\varepsilon_I(k_b,k_bv)||\varepsilon_I(k_c,k_cv)|}{|\varepsilon(k_a,k_au_a)|^2|\varepsilon(k_b,k_bv)|^2|\varepsilon(k_c,k_cv)|^2}}\\
&\quad\times H_2(k_a,k_b,k_c,v)\,; \qquad (4.63)\\
F_2(k_a,k_b,k_c;v) &:= \sqrt{\frac{|\varepsilon_I(k_a,k_av)||\varepsilon_I(k_b,k_bv)||\varepsilon_I(k_c,k_cv)|}{|\varepsilon(k_a,k_av)|^2|\varepsilon(k_b,k_bv)|^2|\varepsilon(k_c,k_cv)|^2}}\\
&\quad\times H_1(k_a,k_b,k_c,v)\,.
\end{aligned}$$

Chapter 5

Canonical Perturbation Theory

5.1 Finite Dimensional Perturbation Theory

Roughly speaking, the aim of canonical perturbation theory is to determine how the behavior of an integrable Hamiltonian system changes in the presence of an additional small (still Hamiltonian) effect. More precisely, we begin with an (n degree-of-freedom) integrable Hamiltonian, $H_0(J)$, expressed in terms of action-angle variables, J and θ. We form a new Hamiltonian $H(J,\theta)$ by adding to H_0 a perturbation of the form $\epsilon H_1(J,\theta)$, where we assume ϵ is small. We can decompose the perturbation into a Fourier series in the angles:

$$H(J,\theta) = H_0(J) + \epsilon \sum_m H_{1m}(J) \exp(i(m \cdot \theta)) \,. \tag{5.1}$$

This decomposition is convenient because we will need to consider the average of the perturbation over (at least some) of the angles.

5.1.1 One Degree of Freedom — Averaging

It is well known that any Hamiltonian system with only one degree of freedom can be integrated up to quadrature. Hence, perturbation theory in this context is somewhat of an academic exercise. Still, it can be instructive in a couple of ways. For one, it can provide better intuition about the nature of a system's solutions than an exact solution expressed in terms of esoteric functions. But more importantly for our purposes, it demonstrates concepts required to handle perturbations in larger systems. In particular, it brings out the importance of *averaging* [33].

In one degree of freedom, equation (5.1) becomes simply

$$H(J,\theta) = H_0(J) + \epsilon \sum_n H_{1n}(J) \exp(in\theta)\,. \tag{5.2}$$

The best we could hope for is that this Hamiltonian is itself integrable. This would be shown if we could find a canonical transformation from the variables (J,θ) to new variables $(\bar{J},\bar{\theta})$ in which the full Hamiltonian only depended on $\bar{J}$. Said another way, if we can somehow remove the angular dependence from the perturbed Hamiltonian, we can integrate the perturbed system.

One possible strategy for removing the angle quickly suggests itself. First, we observe that the angle in the unperturbed system evolves linearly in time: $\theta(t) = (\partial H_0/\partial J)t + \theta(0)$. And then we note that in the perturbation, θ appears only as the phase of oscillatory terms. We expect from these two facts that, in some coarse-grained sense, the oscillations in the perturbation will average out to give a zero effect on the evolution. So, we can simply suggest the $\theta-$averaged Hamiltonian as a candidate for the form of the Hamiltonian in the new variables:

$$H(\bar{J}) = H_0(\bar{J}) + \epsilon H_{10}(\bar{J})\,. \tag{5.3}$$

And, in fact, our instincts are confirmed in this case. Solving the Hamilton-Jacobi equation yields the generating function $S(\bar{J},\theta)$ needed to transform to these new variables:

$$S(\bar{J},\theta) = \bar{J}\theta + \text{const} - \sum_{n\neq 0} \frac{H_{1n}(\bar{J})}{in\partial H_0(\bar{J})/\partial\bar{J}} \exp(in\bar{\theta})\,. \tag{5.4}$$

(Of course, equation (5.4) is not valid when H_0 has critical points.)

There are two ways in which we can view the role of $\theta-$averaging in the above development. From one point of view, averaging plays only an auxiliary role in the computation. Conceivably, we could have arrived at the above (or perhaps another) transformation to action-angle variables by other means. But from a different point of view, the $\theta-$averaged Hamiltonian is an (admittedly extraordinary) approximation to the exact Hamiltonian. Unfortunately, it turns out that once we add only one more degree of freedom to the problem, we lose the luxury of the first point of view.

5.1.2 Multiple Degrees of Freedom — Resonances

We would like to attempt to integrate a perturbed integrable system with n degrees of freedom in a similar way to that described in Section 5.1.1. In this case, equation (5.1) takes the form

$$H(J_1,\ldots,J_n,\theta^1,\ldots,\theta^n) = H_0(J_1,\ldots,J_n) + \epsilon \sum_{m\epsilon\mathbf{Z}^n} H_{1m}(J_1,\ldots,J_n)\exp(im_i\theta^i)\,. \tag{5.5}$$

Again we note that in the unperturbed system that $\theta^i(t) = \omega^i(J_1,\ldots,J_n)t + \theta^i(0)$, where

$$\omega^i = \frac{\partial H_0}{\partial J_i}(J_1,\ldots,J_n)\,. \tag{5.6}$$

Since each θ_i appears only in phases, we may naively think averaging will again give us a good candidate for the Hamiltonian in the new variables $(\bar{J}_1, \ldots, \bar{J}_n, \bar{\theta}^1, \ldots, \bar{\theta}^n)$. But when solving the Hamilton-Jacobi equation for the generating function that would give us this transformation, we find it necessary that

$$S(\bar{J}_1, \ldots, \bar{J}_n, \theta^1, \ldots, \theta^n) = \bar{J}_i \theta^i + \text{const} - \sum_{m \neq 0} \frac{H_{1m}(\bar{J}_1, \ldots, \bar{J}_n)}{m_i \omega^i(\bar{J})} \exp(i m_i \theta^i) \,. \quad (5.7)$$

It is clear from equation (5.7) that if the relation

$$m_i \omega^i = 0 \quad (5.8)$$

ever holds, the θ-averaged perturbation does not provide a good candidate for the Hamiltonian in the new variables. The reason our naive intuition fails in this case is simple. Unlike in the one degree-of-freedom case, the θ_i do not, in general, *individually* appear as phases. Instead, the phases of the Fourier modes are linear combinations of the θ_i. Whenever equation (5.8) holds— whenever the angles are in *resonance*— the phase given by $m_i \theta^i$ will be approximately stationary in time. Hence the m_i Fourier mode will not, over time, average to zero.

All is not lost, though. Perhaps we cannot integrate the system generated by Hamiltonian (5.5) using the method of averaging. But, reverting to the second point of view mentioned at the end of Section 5.1.1, we can use averaging to find a simpler, approximate Hamiltonian. It is the business of so-called secular perturbation theory, [52] (but see [53] for a clear introduction outside the Hamiltonian framework) to do just this.

5.1.3 Secular Perturbation Theory

A Motivational Toy

To motivate how the method of averaging can be partially salvaged, we consider a simple two degree-of-freedom system generated by the Hamiltonian

$$H(J_1, J_2, \theta^1, \theta^2) = \omega_1 J_1 + \omega_2 J_2 + \epsilon \left[\cos(\theta^1 + \theta^2) + \cos(\theta^1 - \theta^2)\right] . \qquad (5.9)$$

A θ-average of Hamiltonian (5.9) would nullify both terms in the perturbation ($\mathcal{O}(\epsilon)$) term. As we will shortly see, such an average is sometimes a good approximation, and sometimes not.

The equations of motion generated by Hamiltonian (5.9) are

$$\dot{\theta}^i = \omega_i \qquad i = 1, 2 , \qquad (5.10)$$

and

$$\begin{aligned} \dot{J}_1 &= \epsilon \left[\sin(\theta^1 + \theta^2) + \sin(\theta^1 - \theta^2)\right] \\ \dot{J}_2 &= \epsilon \left[\sin(\theta^1 + \theta^2) - \sin(\theta^1 - \theta^2)\right] \end{aligned} \qquad (5.11)$$

Equations (5.10) are easily integrated to give

$$\begin{aligned} \theta^1(t) &= \omega_1 t + \theta^1(0) \qquad (5.12) \\ \theta^2(t) &= \omega_2 t + \theta^2(0) . \end{aligned}$$

Provided $\omega_1 \neq \pm\omega_2$, the solutions to equations (5.11) are thus

$$\begin{aligned} J_1(t) &= -\frac{\epsilon}{(\omega_1 + \omega_2)} \cos((\omega_1 + \omega_2)t + \theta^1(0) + \theta^2(0)) \qquad (5.13) \\ &\quad -\frac{\epsilon}{(\omega_1 - \omega_2)} \cos((\omega_1 - \omega_2)t + \theta^1(0) - \theta^2(0)) + J_1(0) \\ J_2(t) &= -\frac{\epsilon}{(\omega_1 + \omega_2)} \cos((\omega_1 + \omega_2)t + \theta^1(0) + \theta^2(0)) \\ &\quad +\frac{\epsilon}{(\omega_1 - \omega_2)} \cos((\omega_1 - \omega_2)t + \theta^1(0) - \theta^2(0)) + J_2(0) . \end{aligned}$$

Inspection of the denominators in equations (5.13) reveals that the nonresonance condition $\omega_1 \neq \pm\omega_2$ is crucial for the validity of these expressions as solutions of (5.10) and (5.11). The size of those denominators also determine when taking a θ-average of Hamiltonian (5.9) is a good approximation. If $\omega_1 \pm \omega_2$ are both $\mathcal{O}(1)$, the perturbation modifies the evolution of J_1 and J_2 at only $\mathcal{O}(\epsilon)$. However, if at least one of $\omega_1 \pm \omega_2$ is $\mathcal{O}(\epsilon)$, the perturbation modifies the evolution of J_1 and J_2 at $\mathcal{O}(1)$; in this case, there is no sense in which the θ-averaged Hamiltonian is a good approximate Hamiltonian.

To emphasize the trouble with neglecting the perturbation just because it looks oscillatory (and to complete the solution to equations (5.10) and (5.11)), we consider the resonant case. Suppose $\omega_1 = \omega_2$. The solutions to (5.11) take the form

$$\begin{aligned} J_1(t) &= -\frac{\epsilon}{(\omega_1+\omega_2)}\cos((\omega_1+\omega_2)t+\theta^1(0)+\theta^2(0)) \\ &\quad -\epsilon t\cos(\theta^1(0)-\theta^2(0)) + J_1(0) \\ J_2(t) &= -\frac{\epsilon}{(\omega_1+\omega_2)}\cos((\omega_1+\omega_2)t+\theta^1(0)+\theta^2(0)) \\ &\quad +\epsilon t\cos(\theta^1(0)-\theta^2(0)) + J_2(0)\,. \end{aligned} \tag{5.14}$$

In contrast to the solutions (5.13) of the nonresonant problem, a linear growth in t appears in (5.14). In fact, after a time $t \sim 1/\epsilon$, this secularity contributes to the solution at $\mathcal{O}(1)$. On the other hand, the nonresonant term of (5.9), which gives rise to the oscillatory terms in (5.14). contributes only at $\mathcal{O}(\epsilon)$. In some sense, then, we could approximate (5.9) by neglecting the nonresonant term.

Partial Averaging

The procedure of systematic neglecting nonresonant oscillatory terms in a Hamiltonian of the form (5.5) is called *partial averaging* [33]. The basic idea is to keep in the perturbation only those Fourier modes m_i that satisfy equation (5.8). We make this more explicit below, but we must first call attention to an important distinction.

Notice that for any given mode, condition (5.8) can be satisfied in two distinct ways. It could be that (5.8) holds only for some values of J_i. In this case, the resonance is said to be an *accidental* resonance. Hamiltonians with this type of resonance were treated extensively in [32]. In this case, the Hamiltonian in which the resonant modes are kept is only a valid approximation near the resonant values of J_i.

In contrast, (5.8) may hold for all values of J_i. In this case, the resonance is said to be *intrinsic*, and this is the case of interest in this dissertation. Systems for which this holds are considered in [35] and later in [31]. Physically relevant examples of intrinsically resonant systems include any weakly-nonlinear system with at least two commensurate frequencies, as well as the Kepler problem.

We take as our starting point the Hamiltonian (5.5), supposing condition (5.8) holds for l distinct n-tuples $m^{(1)}, \ldots, m^{(l)}$, independently of the values of $J_1, \ldots, J_n$.

To find an approximate Hamiltonian to an intrinsically resonant system, we essentially follow two steps. First, (in the language of (L&L,1982)), we "remove the resonances." This is nothing more than making a canonical transformation to variables $(I_1, \ldots, I_n, \psi^1, \ldots, \psi^n)$ so that the resonant combinations of angles are among our new variables. Then our new angles would be

$$\psi^1 = m_i^{(1)}\theta^i, \ldots, \psi^l = m_i^{(l)}\theta^i, \psi^{l+1} = \theta^{l+1}, \ldots, \psi^n = \theta^n \, . \tag{5.15}$$

A canonical transformation to these coordinates is generated by

$$F(\theta^1, \ldots, \theta^n, I_1, \ldots, I_n) = (m_i^{(1)}\theta^i)I_1 + \ldots + (m_i^{(l)}\theta^i)I_l + \theta^{l+1}I_{l+1} + \ldots + \theta^n I_n \, . \tag{5.16}$$

In these variables, the Hamiltonian takes the form

$$\bar{H}(I_1, \ldots, I_n, \psi^1, \ldots, \psi^n) = \bar{H}_0(I_{l+1}, \ldots, I_n) + \epsilon \sum_{m \epsilon \mathbf{Z}^n} \bar{H}_{1m}(I_1, \ldots, I_n) \exp(im_i \psi^i) . \tag{5.17}$$

The importance of the new variables is manifest in the fact that $\bar{H}_0$ in (5.17) does not depend on $I_1, \ldots, I_l$. Hence, in the unperturbed system, the resonant angles $\psi^1, \ldots, \psi^l$ are stationary in time, and the nonresonant angles $\psi^{l+1}, \ldots, \psi^n$ are not. We can then argue that on the time scale $\tau = \epsilon t$, the Hamiltonian (5.17) is well-approximated by its average over all values of the "fast variables" $\psi^{l+1}, \ldots, \psi^n$.

So, the second step in finding an approximate Hamiltonian is to average over the nonresonant angles. After taking this average, we are left with the most general "Resonance Hamiltonian"

$$\bar{H}(I_1, \ldots, I_n, \psi^1, \ldots, \psi^l) = \bar{H}_0(I_{l+1}, \ldots, I_n) + \epsilon \sum_{p \epsilon \mathbf{Z}^l} \bar{H}_{1p}(I_1, \ldots, I_n) \exp(ip_i \psi^i) . \tag{5.18}$$

Before we proceed to a couple of examples, and then determining a Resonance Hamiltonian for a system with a continuum of degrees of freedom, we need to make one observation about the role of the canonical transformation given by the generating function (5.16). The new resonant variables are particularly convenient when the perturbation has not been expanded in a Fourier series: in this case, it makes sense to actually compute the average over the nonresonant angles. But if the perturbation is already in the form of a Fourier series, we know immediately that the average of any nonresonant mode vanishes. Mere inspection is enough to determine which modes will not contribute on the long time scale τ.. So, the step of "removing the resonances" is essentially a bookkeeping step in the approximation, and can be ignored if it is convenient, as it will be in the infinite-dimensional case.

Two Examples of Partial Averaging

To make concrete the process of partial averaging, we present two examples, one in two degrees-of-freedom and the other in many degrees-of-freedom.

For our two degree-of-freedom system we consider two harmonic oscillators coupled by a small, but otherwise arbitrary cubic nonlinearity. The Hamiltonian of such a system is

$$H = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \frac{\omega_2}{2}(q_2^2 + p_2^2) + \epsilon G(q_1, p_1, q_2, p_2)\,, \tag{5.19}$$

where $G(q_1, p_1, q_2, p_2)$ is a homogeneous cubic polynomial in q_1, p_1, q_2, p_2. Also, we assume that the oscillators are in $2:1$ resonance:

$$\omega_2 = 2\omega_1\,. \tag{5.20}$$

The unperturbed system is simply a pair of uncoupled harmonic oscillators, so the action-angle variables for the unperturbed system are simply given by two sets of harmonic oscillator action-angle variables:

$$q_i = \sqrt{2J_i}\sin(\theta_i), \qquad p_1 = \sqrt{2J_i}\cos(\theta_i)\,. \tag{5.21}$$

To prepare (5.19) for partial averaging, we apply the transformation (5.21) to it, and expand $G(q_1, p_1, q_2, p_2)$ in a Fourier series in θ_1 and θ_2:

$$\begin{aligned} & H = \omega_1 J_1 + \omega_2 J_2 \\ + \quad & \epsilon \Big\{ J_1^{3/2}[A_1\sin(\theta_1) + B_1\cos(\theta_1) + C_1\sin(3\theta_1) + D_1\cos(3\theta_1)] \\ + \quad & J_2^{3/2}[A_2\sin(\theta_2) + B_2\cos(\theta_2) + C_2\sin(3\theta_2) + D_2\cos(3\theta_2)] \\ + \quad & J_1\sqrt{J_2}[A_{12}\sin(\theta_2) + B_{12}\cos(\theta_2) + C_{12}\sin(2\theta_1 + \theta_2) \\ & \quad + D_{12}\cos(2\theta_1 + \theta_2) + E_{12}\sin(2\theta_1 - \theta_2) + F_{12}\cos(2\theta_1 - \theta_2)] \end{aligned} \tag{5.22}$$

$$+ \quad J_2\sqrt{J_1}[A_{21}\sin(\theta_1) + B_{21}\cos(\theta_1) + C_{21}\sin(2\theta_2+\theta_1)$$
$$+D_{21}\cos(2\theta_2+\theta_1) + E_{21}\sin(2\theta_2-\theta_1) + F_{21}\cos(2\theta_2-\theta_1)]\Big\} \ ,$$

where all the coefficients of the Fourier series depend on the coefficients in the polynomial $G(q_1, p_1, q_2, p_2)$, and so can be considered arbitrary.

To determine which of the terms in the Fourier series in (5.22) we can average away, we look at the unperturbed evolution of the angles:

$$\theta_1(t) = \omega_1 t + \theta_{10}, \qquad \theta_2(t) = \omega_2 t + \theta_{20}\,. \tag{5.23}$$

Substituting (5.23) into each Fourier mode represented in (5.22), we find that only those depending on the phase combination $2\theta_1 - \theta_2$ remain stationary at lowest order. Hence, we argue that all the modes, except those with coefficients E_{12} and F_{12} average away, yielding the resonance Hamiltonian:

$$H_r = \omega_1 J_1 + \omega_2 J_2 + \epsilon J_1\sqrt{J_2}[E_{12}\sin(2\theta_1-\theta_2) + F_{12}\cos(2\theta_1-\theta_2)]\,. \tag{5.24}$$

The equations of motion generated by the resonance Hamiltonian (5.24) are

$$\dot{\theta}_1 = \omega_1 + \epsilon\sqrt{J_2}[E_{12}\sin(2\theta_1-\theta_2) + F_{12}\cos(2\theta_1-\theta_2)] \tag{5.25}$$
$$\dot{\theta}_2 = \omega_2 + \frac{\epsilon}{2}\frac{J_1}{\sqrt{J_2}}[E_{12}\sin(2\theta_1-\theta_2) + F_{12}\cos(2\theta_1-\theta_2)] \tag{5.26}$$
$$\dot{J}_1 = -2\epsilon J_1\sqrt{J_2}[E_{12}\cos(2\theta_1-\theta_2) - F_{12}\sin(2\theta_1-\theta_2)] \tag{5.27}$$
$$\dot{J}_2 = \epsilon J_1\sqrt{J_2}[E_{12}\cos(2\theta_1-\theta_2) - F_{12}\sin(2\theta_1-\theta_2)]\,. \tag{5.28}$$

As will be seen in chapter 6), equations (5.25)–(5.28) (with $F_{12} = 0$) are relevant to weakly nonlinear Vlasov-Poisson dynamics, and will be considered in greater detail there. For now it suffices to note that these equations have a second constant of motion, $I = J_1 + 2J_2$, as can be easily seen by adding equation (5.27) to twice (5.28).

The second example we consider illustrates how to approximate a many degree-of-freedom Hamiltonian with many resonance and near-resonance conditions. This example is a discreet analog of an infinite degree-of-freedom system with a continuous spectrum, and so provides a segue to consideration of the weakly nonlinear Vlasov-Poisson system. The Hamiltonian for this example is given by:

$$\begin{aligned} H &= \sum_{i=0}^{N} \omega_1 J_1 \\ &+ \epsilon \sum_{j,k,l=0}^{N} \sqrt{J_j J_k J_l} \cos(\theta_j + \theta_k - \theta_l) \,, \end{aligned} \tag{5.29}$$

where $N \sim 1/\epsilon$. The linear frequencies are defined as follows:

$$\begin{aligned} \omega_0 &= \omega_l \\ \omega_N &= \omega_u \\ \omega_i &= \omega_l + \frac{i}{N}(\omega_u - \omega_l) \,. \end{aligned} \tag{5.30}$$

For simplicity in evaluating the resonance conditions, we require that $\Delta\omega := \omega_u - \omega_l = \mathcal{O}(1)$, but $\omega_l >> 1$.

At lowest order, the angles θ_i evolve according to Hamiltonian (5.29) as

$$\theta_i(t) = (\omega_l + \frac{i}{N}(\Delta\omega))t + \theta_i(0) + \mathcal{O}(1/N) \qquad n = 1, 2 \,. \tag{5.31}$$

Substituting these expressions for θ_i into the arguments $\phi_{jkl} := \theta_j + \theta_k - \theta_l$ of the Fourier modes in the perturbation term of (5.29), we find that the phases evolve approximately as

$$\phi_{jkl}(t) = \frac{j + k - l}{N}(\Delta\omega)t + \phi_{jkl}(0) + \mathcal{O}(\epsilon) \qquad j, k, l = 0 \ldots N \,. \tag{5.32}$$

Clearly, then, ϕ_{jkl} is stationary when $l = j + k$. The resonance Hamiltonian we find after partial averaging is thus:

$$\begin{aligned} H_r &= \sum_{i=0}^{N} \omega_i J_i \\ &+ \epsilon \sum_{j,k=0}^{N} \sqrt{J_j J_k J_l} \cos(\theta_j + \theta_k - \theta_{j+k}) \,, \end{aligned} \tag{5.33}$$

(For convenience in writing the sums, we define $\theta_M := 0$ for $M > N$.)

However, in neglecting all but the exactly resonant modes, we neglect many, nearly resonant modes that vary slowly, perhaps not completing even a single oscillation in the time interval of interest. Such modes, in the present situation, are characterized by $\dot{\phi}_{jkl} = \mathcal{O}(\epsilon)$. So the near resonance condition is given by

$$j + k - 2l \sim \mathcal{O}(1) \,. \tag{5.34}$$

Hence, a more correct resonance Hamiltonian would be given by

$$\begin{aligned} H_r &= \sum_{i=0}^{N} \omega_i J_i \\ &+ \epsilon \left[\sum_{j,k=0}^{N} \sum_{m=-n}^{n} \sqrt{J_j J_k J_{j+k}} \cos(\theta_j + \theta_k - \theta_{j+k+m}) \right] \,, \end{aligned} \tag{5.35}$$

where $n \sim \mathcal{O}(1)$. (Again, for notational convenience, we define additional angles $\theta_M :=$ for $M < 0$ or $M > N$.)

5.2 Infinite Dimensional Perturbation Theory

Our starting point for applying the method of averaging to the Vlasov-Poisson system is the weakly nonlinear Hamiltonian written in terms of the action-angle variables of the linearized system. This is given at the end of chapter (4) in expressions (4.60)–(4.62).

We essentially proceed as in finite dimensional resonant perturbation theory: first we identify the resonances, and then we neglect the nonresonant Fourier modes of the perturbation. However, the continuity of the linear spectrum raises a difficulty in neglecting all but the exactly resonant terms. A solution to this problem is found in the concept of a *resonant layer*, and this is discussed first. Following that, the resonant layers are isolated from the integrals, finally yielding the Vlasov-Poisson Resonance Hamiltonian at the end of this section.

5.2.1 Identifying Resonances - The Resonant Layer

Following the analogy to the finite degree of freedom case, we would now transform to a new set of coordinates that separate the resonant angles from the nonresonant angles, and then average over the nonresonant angles. But, transforming to the new variables is unnecessary here. Our perturbation is conveniently already in the form of an infinite series of Fourier integrals.

Thus we can argue, in analogy with the finite-dimensional case, that terms (4.60)–(4.62) contribute significantly on the long time scale only when the arguments of the exponentials have a sufficiently slow unperturbed evolution. This obviously occurs near the points where the exponentials have stationary phase. We will refer to these points as resonances (for clarity, we will not use this word to refer to points in the integral where the denominator vanishes.) We identify the relevant resonances by substituting the unperturbed evolution,

$$\theta_{|k|}(u,t) = |k|u\mathrm{sgn}(\varepsilon_I(k,ku))t\,, \tag{5.36}$$

into the perturbation. For instance, term (4.60) is resonant when

$$k_a u_a + k_b u_b + k_c v = 0\,. \tag{5.37}$$

From this point on though, a strict analogy cannot be kept with the finite-dimensional approximation procedure. For one thing, u_a, u_b, and v are integrated over the whole range of velocities; condition (5.37) actually defines a *plane* of resonances in the domain of integration in term (4.60). Thus, a continuum of resonant interaction "terms" must be kept. While this much is a natural generalization of the discrete case, another effect comes into play. Near the plane of resonances, there are a layer of points for which (5.37) almost holds. This layer corresponds to interaction "terms" in which the phase of the exponential varies slowly; our long time scale approximation does not justify neglecting them, either. And so, this resonant layer should also be included in the approximate Hamiltonian.

However, for evolution up to time $t \sim \epsilon^{-1}$, it is reasonable to assume that the width of the resonant layer is itself small. Indeed, our test for neglecting a nonresonant term is to check whether it oscillates many times over the time interval of interest. Hence, we can neglect those modes for which the zeroth order phase obeys:

$$k_a u_a + k_b u_b + k_c v >> \epsilon \,. \tag{5.38}$$

But since we require $\epsilon << 1$, it is possible to find another small parameter, δ, such that

$$\epsilon << \delta << 1 \,. \tag{5.39}$$

The smallness of the parameter δ makes computation of the Vlasov-Poisson Resonance Hamiltonian tractable.

With these considerations in mind, we now outline the procedure for computing the Resonant Hamiltonian for the Vlasov–Poisson system. First, for each term (4.60)–(4.62), we determine the resonant subset of the range of integration by substituting in the unperturbed solutions. Then, setting the argument of the exponentials to zero yields the resonance condition. Second, we restrict the ranges of integration to a layer of half-width δ around the resonance condition. (In general, δ could depend on the values of k_a and k_b, but we will ignore this for the time being.). Finally, we expand the integrals in small δ, keeping terms through $\mathcal{O}(\delta)$.

5.2.2 Calculating the Resonance Hamiltonian

We will now give details of the calculation of the Vlasov–Poisson Resonance Hamiltonian. To keep the computation somewhat compact, we prefer to deal with terms (4.57)–(4.59), using (4.60)–(4.62) only to obtain the resonance conditions. Since the independent variables remain unchanged in the transformation (4.33), there is no computational difference between terms of either form.

Among (4.57)–(4.59), we can classify terms by the number of poles in the integrand. Each term requires a different computation, but the calculation is similar for all terms with the same number of poles. We begin with the most complicated terms.

5.2.3 Keeping the resonant layer - two pole case

Keeping the resonant layer in term (4.57) is a somewhat delicate, and extremely tedious calculation that is reproduced in detail in appendix C. However, the basic procedure is similar to that described immediately below. It is in the two-pole calculation that the rearranging of orders of integration in chapter 4 really pays off. Not only do we find that it makes the computation a little easier, we also find that it has the result of removing all the important resonance terms from the two pole integrals. In other words, we ultimately find that we can neglect term (4.57) in the first approximation.

Keeping the resonant layer - one pole case

We now turn our attention to isolating the resonant part of term (4.58). We will denote this term by I_1.

To find the resonance condition for (4.58), we consider the action-angle variable form of this term, (4.61). Substituting (5.36) into the exponential in (4.61) yields a line of resonances in the domain of integration:

$$k_a u_a + (k_b + k_c) v = 0\,. \tag{5.40}$$

But since $k_a + k_b + k_c = 0$, (5.40) simplifies to

$$v = u_a\,. \tag{5.41}$$

We define the resonant variable $w := v - u_a$, and restrict the limits of the innermost integral in (4.58) to a layer of half-width δ around $w = 0$. Thus (for the moment suppressing k dependence in H_2.),

$$I_1 \approx -\frac{3}{\pi} \int_{-\infty}^{\infty} \mathrm{d}u_a\, E_{k_a}(u_a) \left(\mathcal{P} \int_{-\delta}^{\delta} \frac{\mathrm{d}w}{w} H_2(u_a + w) E_{k_b}(u_a + w) E_{k_c}(u_a + w) \right)\,. \tag{5.42}$$

Again, we wish to keep terms only through $\mathcal{O}(\delta)$; to make this easier, we make the variable change $w = \delta w'$, yielding

$$I_1 \approx -\frac{3}{\pi} \int_{-\infty}^{\infty} \mathrm{d}u_a\, E_{k_a}(u_a) \left(\mathcal{P} \int_{-1}^{1} \frac{\mathrm{d}w'}{w'} H_2(u_a + \delta w') E_{k_b}(u_a + \delta w') E_{k_c}(u_a + \delta w') \right)\,. \tag{5.43}$$

Now we can easily expand (5.43) in δ. This gives us,

$$\begin{aligned} I_1 &\approx -\frac{3}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u_a\, E_{k_a}(u_a)\Bigg\{ \\ & H_2(u_a)E_{k_b}(u_a)E_{k_c}(u_a)\left(\mathcal{P}\int_{-1}^{1}\frac{\mathrm{d}w'}{w'}\right) \qquad (5.44) \\ &+ \delta\frac{\partial}{\partial u_a}(H_2(u_a)E_{k_b}(u_a)E_{k_c}(u_a))\int_{-1}^{1}\mathrm{d}w'\Bigg\} \qquad (5.45) \\ &+ \mathcal{O}(\delta^2)\,. \end{aligned}$$

Since $(1/w)$ is odd, the integral in (5.44) vanishes. And after a substantial amount of algebra, we can show that the integral in (5.45) has the value 2. Hence,

$$I_1 \approx -\frac{6\delta}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u_a\, E_{k_a}(u_a)\frac{\partial}{\partial u_a}(H_2(u_a)E_{k_b}(u_a)E_{k_c}(u_a)) + \mathcal{O}(\delta^2)\,. \qquad (5.46)$$

We can make one further simplification to (5.46) by integrating by parts (and assuming the boundary terms vanish):

$$I_1 \approx \frac{6\delta}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u_a\, H_2(u_a)\frac{\partial E_{k_a}(u_a)}{\partial u_a}E_{k_b}(u_a)E_{k_c}(u_a) + \mathcal{O}(\delta^2)\,. \qquad (5.47)$$

Finally, for clarity, we rewrite (5.47), changing the integration variable to v, and putting the k's back into H_2.

$$I_1 \approx \frac{6\delta}{\pi}\int_{-\infty}^{\infty} \mathrm{d}v\, H_2(k_a,k_b,k_c,v)\frac{\partial E_{k_a}(v)}{\partial v}E_{k_b}(v)E_{k_c}(v) + \mathcal{O}(\delta^2)\,. \qquad (5.48)$$

Keeping the resonant layer - no pole case

Finally, we must isolate the resonant part of term (4.59). However, substituting (5.36) into (4.62) shows us that the exponential is always stationary with respect to the unperturbed motion. In other words, (4.59) is always resonant. No simplification can be made to it.

The Resonance Hamiltonian itself

Adding term (4.59) to the appropriate permutations of (5.48), we arrive at the cubic term of the Vlasov–Poisson Resonance Hamiltonian. Adding the quadratic term yields the full Hamiltonian. To explicitly show the symmetry in the k dependence, we split the coefficient of δ into three parts:

$$
\begin{aligned}
\mathcal{F}_r &= \frac{1}{16}\sum_{k=1}^{\infty}\int \mathrm{d}v\,\frac{|\varepsilon(k,kv)|^2}{\varepsilon_I(k,kv)} v E_k(v) E_{-k}(v) \\
&+ \epsilon \sum_{k_a+k_b+k_c=0}\left(\frac{m}{48}\right)\left(\frac{i}{4\pi e}\right)^3 k_a k_b k_c \Big\{ \\
&+ \int \mathrm{d}v\, H_1(k_a,k_b,k_c;v) E_{k_a}(v) E_{k_b}(v) E_{k_c}(v) \\
&+ \frac{2\delta}{\pi}\Big[\int \mathrm{d}v\, H_2(k_a,k_b,k_c;v)\frac{\partial E_{k_a}(v)}{\partial v} E_{k_b}(v) E_{k_c}(v) \\
&+ \int \mathrm{d}v\, H_2(k_b,k_c,k_a;v)\frac{\partial E_{k_b}(v)}{\partial v} E_{k_c}(v) E_{k_a}(v) \\
&+ \int \mathrm{d}v\, H_2(k_c,k_a,k_b;v)\frac{\partial E_{k_c}(v)}{\partial v} E_{k_a}(v) E_{k_b}(v)\Big]\Big\}\,. \qquad (5.49)
\end{aligned}
$$

Chapter 6

Averaged Weakly Nonlinear Vlasov-Poisson

In this chapter we explore the resonance Hamiltonian of the Vlasov-Poisson system. We begin by displaying its equations of motion.

$$
\begin{aligned}
\dot{E}_k &= -ikE_k - m\left(\frac{1}{4\pi e}\right)^3 \sum_{k_b+k_c=k} k^2 k_b k_c s(k,v) \\
&\quad \times \left\{ H_1(k,k_b,k_c)E_{k_b}E_{k_c} + \frac{2\delta}{\pi}\left[H_2(k_b,k_c,k,v)\frac{\partial E_{k_b}}{\partial v}E_{k_c} \right.\right. \\
&\quad \left.\left. +\ H_2(k_c,k,k_b,v)E_{k_b}\frac{\partial E_{k_c}}{\partial v} - \frac{\partial}{\partial v}\left(H_2(k,k_b,k_c,v)E_{k_b}E_{k_c}\right)\right]\right\}\ , \qquad (6.1)
\end{aligned}
$$

where

$$
s(k,v) := \frac{\varepsilon_I(kk_0, kk_0 v)}{|\varepsilon(kk_0, kk_0 v)|^2}\,. \qquad (6.2)
$$

The function $s(k,v)$ is so named because its sign is the energy signature of the (k,v) normal mode.

6.1 Stability of Vlasov-Poisson Solutions Under Resonant Interaction

One of the main results that we can derive from the Weakly Nonlinear Vlasov-Poisson resonance Hamiltonian (5.49) is the existence of a positive definite integral of motion:

$$I = \frac{1}{32} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v \, \frac{1}{|s(k,v)|} E_{-k} E_k \,. \tag{6.3}$$

To show that (6.3) is in fact an integral of motion, we first must demonstrate that the related integral

$$M = \frac{1}{32} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v \, \frac{1}{s(k,v)} E_{-k} E_k \tag{6.4}$$

is itself a constant of motion. This constant is none other than the momentum given in (4.30), which is conserved by the *unaveraged* system. The invariance of (6.4) under the partially averaged system then follows immediately from the fact that partial averaging does not disrupt the $k_a + k_b + k_c = 0$ constraint on the sum of interaction terms. For the purpose of proving the invariance of (6.3), though, we find it valuable to prove explicitly the invariance of (6.4) under (6.1). To do this, we derive the flux-entropy conservation law for M. In other words, we show that the time derivative of its integrand, $\mathcal{M}$, is an exact v-derivative.

The time derivative of the integrand of (6.4) is

$$\dot{\mathcal{M}} = \sum_{k=-\infty}^{\infty} \frac{1}{s(k,v)} E_{-k} \dot{E}_k \,. \tag{6.5}$$

Upon substituting the RHS of equation (6.1) for $\dot{E}_k$ in (6.5), some index manipulation reveals that the only terms remaining in the sum are exact v-derivatives. In particular, we find that

$$\begin{aligned}
\dot{\mathcal{M}} &= -2m\left(\frac{1}{4\pi e}\right)^3 \Bigg[\sum_{(k_a,k_b,k_c)} k_a k_b k_c \frac{\partial}{\partial v}\left((k_a H_2(k_a,k_b,k_c,v) + k_b H_2(k_a,k_b,k_c,v)\right. \\
&\quad \left. + k_c H_2(k_a,k_b,k_c,v))E_{k_a}E_{k_b}E_{k_c}\right) \\
&+ \sum_{k=-\infty}^{\infty} 2k^4 \frac{\partial}{\partial v}\left((H_2(2k,k,k,v) - H_2(k,k,2k,v))E_{-2k}E_k^2\right)\Bigg] . \qquad (6.6)
\end{aligned}$$

Since we assume $E_{k_a} \to 0$ as $v \to \pm\infty$, equation (6.6) implies that (6.4) is a constant of motion. However, the form of the "flux" in equation (6.6) gives us more information. We recall from chapter 3 that dynamical accessibility implies that $E_k(v) \propto s(k,v)$. Now, suppose the equilibrium distribution has extrema at $v = v_1$ and $v = v_2$. And so, for all k, $s(k,v_1) = s(k,v_2) = 0$. Therefore equation (6.6) implies

$$\dot{M}_{v_1}^{v_2} = \int_{v_1}^{v_2} \mathrm{d}v\, \dot{\mathcal{I}} = 0 . \qquad (6.7)$$

Going further, we can imagine that f_0 has n extrema at $v = v_1, \ldots, v_n$. Then, in the notation of equation (6.7) the following is also a constant of motion:

$$I^+ = M_{-\infty}^{v_1} - M_{v_1}^{v_2} + \cdots + M_{v_n}^{\infty} . \qquad (6.8)$$

Since $s(k,v) \propto -f_0'$, we have $I^+ = I$, where I is defined in equation (6.3). Thus we have proved that (6.3) does in fact define a constant of motion.

So we see that the extrema of the background distribution divide the velocity axis into a series of dynamically isolated regions: in particular, there is no momentum transfer between these regions. As explained above, this is a consequence of dynamical accessibility, and so physically is a manifestation of the fact that extrema are obstacles to rearrangement.

6.2 A Truncated System

The full system obtained from equations (5.49) and (2.15) are still fairly horrible. But we can gain a small amount of insight into the behavior of its solutions by considering the case when the initial perturbation is limited to long wavelengths with respect to the spatial period of the plasma. The simplest nontrivial case includes only the fundamental and first harmonic, or $|k| = k_0, 2k_0$. These harmonics, of course, interact with higher ones, so the truncation below is only valid while the higher harmonics remain negligible.

Even this truncation admits further simplifications for certain situations. So we begin by considering the unsimplified truncation, and then later explore the results of further simplification.

6.2.1 Resonant Interactions With Nearly Resonant Effects

To derive the restricted dynamics described above, we need to truncate both the Poisson bracket and the Hamiltonian. Truncating the bracket (4.25) involves merely setting the upper limit of the sum to $k = 2$. In other words,

$$\{F, G\} = -16ik_0 \int_{-\infty}^{\infty} \mathrm{d}u \left[\frac{\varepsilon_I(k_0, k_0 u)}{|\varepsilon(k_0, k_0 u)|^2} \left(\frac{\delta F}{\delta E_1} \frac{\delta G}{\delta E_{-1}} - \frac{\delta G}{\delta E_1} \frac{\delta F}{\delta E_{-1}} \right) \right.$$
$$\left. + 2 \frac{\varepsilon_I(2k_0, 2k_0 u)}{|\varepsilon(2k_0, 2k_0 u)|^2} \left(\frac{\delta F}{\delta E_2} \frac{\delta G}{\delta E_{-2}} - \frac{\delta G}{\delta E_2} \frac{\delta F}{\delta E_{-2}} \right) \right] .$$

Truncating the Hamiltonian (5.49) is somewhat more tedious, but still straightforward. The truncated Hamiltonian can be put into a compact form by making use of the symmetries of the functions H_1 and H_2. There are three useful properties to note. First, both H_1 and H_2 depend on their k arguments only as k^2, so minus signs can be dropped in those arguments. Second, H_1 is symmetric under all permutations of the k arguments. And finally, H_2 is symmetric under permutation of its final two k arguments. Keeping all these in mind, the truncated Hamiltonian can be written:

$$\begin{aligned}
\mathcal{F}_r &= \frac{1}{16}\int \mathrm{d}v\, v \left(\frac{|\varepsilon(k_0,k_0v)|^2}{\varepsilon_I(k_0,k_0v)}E_1(v)E_{-1}(v) + \frac{|\varepsilon(2k_0,2k_0v)|^2}{\varepsilon_I(2k_0,2k_0v)}E_2(v)E_{-2}(v)\right) \\
&+ \epsilon\frac{m}{8}\left(\frac{i}{4\pi e}\right)^3 k_0^3 \left\{\int \mathrm{d}v\, H_1(1,1,2;v)\left((E_{-1})^2E_2 - (E_1)^2E_{-2}\right)\right. \\
&+ \frac{2\delta}{\pi}\int \mathrm{d}v \left[2H_2(1,1,2;v)\left(\frac{\partial E_{-1}}{\partial v}E_{-1}E_2 - \frac{\partial E_1}{\partial v}E_1E_{-2}\right)\right. \\
&+ \left.\left. H_2(2,1,1;v)\left(\frac{\partial E_2}{\partial v}(E_{-1})^2 - \frac{\partial E_{-2}}{\partial v}(E_1)^2\right)\right]\right\}. \qquad (6.9)
\end{aligned}$$

Noting that derivatives of $(E_{\pm 1}^2$ appear in the first integral containing gradients, we can arrive at a slightly cleaner Hamiltonian by integrating that term by parts:

$$\begin{aligned}
\mathcal{F}_r &= \frac{1}{16}\int \mathrm{d}v\, v \left(\frac{|\varepsilon(k_0,k_0v)|^2}{\varepsilon_I(k_0,k_0v)}E_1(v)E_{-1}(v) + \frac{|\varepsilon(2k_0,2k_0v)|^2}{\varepsilon_I(2k_0,2k_0v)}E_2(v)E_{-2}(v)\right) \\
&+ \epsilon\frac{m}{8}\left(\frac{i}{4\pi e}\right)^3 k_0^3 \left\{\int \mathrm{d}v \left(H_1(1,1,2;v) - \frac{2\delta}{\pi}\frac{\partial}{\partial v}H_2(1,1,2;v)\right)\right. \\
&\quad \times\left((E_{-1})^2E_2 - (E_1)^2E_{-2}\right) \qquad (6.10) \\
&+ \left.\frac{2\delta}{\pi}\int \mathrm{d}v\, (H_2(2,1,1;v) - H_2(1,1,2;v))\left(\frac{\partial E_2}{\partial v}(E_{-1})^2 - \frac{\partial E_{-2}}{\partial v}(E_1)^2\right)\right\}.
\end{aligned}$$

The equations of motion generated by (6.10) are

$$\begin{aligned}
\frac{\partial E_1}{\partial t} &= -ik_0vE_1 - \epsilon\alpha_1(v)E_{-1}E_2 + \epsilon\delta\beta_1(v)E_{-1}E_2 - \epsilon\delta\gamma_1(v)E_{-1}\frac{\partial E_2}{\partial v} \\
\frac{\partial E_2}{\partial t} &= -i2k_0vE_2 + \epsilon\alpha_2(v)(E_1)^2 - \epsilon\delta\beta_2(v)(E_1)^2 - \epsilon\delta\gamma_2(v)E_1\frac{\partial E_1}{\partial v} \qquad (6.11) \\
\frac{\partial E_{-1}}{\partial t} &= ik_0vE_{-1} - \epsilon\alpha_1(v)E_1E_{-2} + \epsilon\delta\beta_1(v)E_{-1}E_{-2} - \epsilon\delta\gamma_1(v)E_1\frac{\partial E_{-2}}{\partial v} \\
\frac{\partial E_{-2}}{\partial t} &= i2k_0vE_{-2} + \epsilon\alpha_2(v)(E_{-1})^2 - \epsilon\delta\beta_2(v)(E_{-1})^2 - \epsilon\delta\gamma_2(v)E_{-1}\frac{\partial E_{-1}}{\partial v}
\end{aligned}$$

where the coefficient functions are

$$\alpha_1(v) = \frac{4mk_0^4}{(4\pi e)^3}\frac{\varepsilon_I(k_0,k_0v)}{|\varepsilon(k_0,k_0v)|^2}H_1(1,1,2,v)$$

$$\begin{aligned}
\beta_1(v) &= \frac{8mk_0^4}{\pi(4\pi e)^3}\frac{\varepsilon_I(k_0,k_0v)}{|\varepsilon(k_0,k_0v)|^2}\frac{\partial}{\partial v}H_2(1,1,2,v) \\
\gamma_1(v) &= \frac{8mk_0^4}{\pi(4\pi e)^3}\frac{\varepsilon_I(k_0,k_0v)}{|\varepsilon(k_0,k_0v)|^2}(H_2(2,1,1;v)-H_2(1,1,2,v)) \\
\alpha_2(v) &= \frac{4mk_0^4}{(4\pi e)^3}\frac{\varepsilon_I(2k_0,2k_0v)}{|\varepsilon(2k_0,2k_0v)|^2}H_1(1,1,2;v) \qquad (6.12)\\
\beta_2(v) &= \frac{8mk_0^4}{\pi(4\pi e)^3}\frac{\varepsilon_I(2k_0,2k_0v)}{|\varepsilon(2k_0,2k_0v)|^2}\frac{\partial}{\partial v}H_2(2,1,1,v) \\
\gamma_2(v) &= \frac{16mk_0^4}{\pi(4\pi e)^3}\frac{\varepsilon_I(2k_0,2k_0v)}{|\varepsilon(2k_0,2k_0v)|^2}(H_2(2,1,1,v)-H_2(1,1,2,v)) .
\end{aligned}$$

The functions H_1 and H_2 are already defined in equations (4.56) and (4.55) for general values of k, but it is here useful to explicitly evaluate them. We have

$$\begin{aligned}
H_1(1,1,2,v) &= \left(\frac{1}{f_0'(v)}\right)^2\Big[\varepsilon_R(2k_0,2k_0v)|\varepsilon(k_0,k_0v)|^2 \\
&\quad +2\varepsilon_I(k_0,k_0v)\varepsilon_I(2k_0,2k_0v)\varepsilon_R(k_0,k_0v)\Big] \\
&= \left(\frac{1}{f_0'(v)}\right)^2\Big[\frac{3}{4}|\varepsilon(k_0,k_0v)|^2 \qquad (6.13)\\
&\quad +\frac{1}{4}|\varepsilon(k_0,k_0v)|^2\varepsilon_R(k_0,k_0v)+\frac{1}{2}(\varepsilon_I(k_0,k_0v))^2\varepsilon_R(k_0,k_0v)\Big] ; \\
H_2(1,1,2,v) &= \left(\frac{1}{f_0'(v)}\right)^2\Big[\varepsilon_I(k_0,k_0v)\varepsilon_R(k_0,k_0v)\varepsilon_R(2k_0,2k_0v) \\
&\quad +(\varepsilon_I(k_0,k_0v))^2\varepsilon_I(2k_0,2k_0v)\Big] \qquad (6.14)\\
&= \left(\frac{\pi^2\omega_p^4}{k_0^4}\right)\left(\frac{1}{\varepsilon_I(k_0,k_0v)}\right)\Big[\frac{3}{4}\varepsilon_R(k_0,k_0v)+\frac{1}{4}|\varepsilon(k_0,k_0v)|^2\Big] ; \\
H_2(2,1,1,v) &= \left(\frac{1}{f_0'(v)}\right)^2\Big[\varepsilon_I(2k_0,2k_0v)(\varepsilon_R(k_0,k_0v))^2 \\
&\quad +\varepsilon_I(2k_0,2k_0v)(\varepsilon_I(k_0,k_0v))^2\Big] \qquad (6.15)\\
&= \left(\frac{\pi^2\omega_p^4}{4k_0^4}\right)\left(\frac{1}{\varepsilon_I(k_0,k_0v)}\right)|\varepsilon(k_0,k_0v)|^2 .
\end{aligned}$$

The simplified expressions (in which all instances of $k=2$ have been eliminated) after the second equal sign in each of equations (6.13)–(6.15) were derived using the definitions (4.18) and (4.19), as well as the following consequences of those definitions:

$$\varepsilon_I(2k_0, 2k_0v) = \frac{1}{4}\varepsilon_I(k_0, k_0v), \qquad \varepsilon_R(2k_0, 2k_0v) = \frac{3}{4} + \frac{1}{4}\varepsilon_R(k_0, k_0v)\,. \tag{6.16}$$

It is worthwhile to note that subtracting (6.14) from (6.15) yields the difference which appears in the coefficients $\gamma_1(v)$ and $\gamma_2(v)$, and is especially simple:

$$H_2(2,1,1;v) - H_2(1,1,2;v) = -\frac{3}{4}\left(\frac{\pi^2\omega_p^4}{k_0^4}\right)\left(\frac{\varepsilon_R(k_0,k_0v)}{\varepsilon_I(k_0,k_0v)}\right)\,. \tag{6.17}$$

Equations (6.11) have so far proven to be intractable because of the presence of the terms proportional to δ. While we are tempted to simply neglect these terms since $\epsilon\delta << \epsilon$, we are not justified in doing so. The v-derivatives of the E_i may grow in magnitude to $\mathcal{O}(\delta^{-1})$ without violating either our ordering assumptions or the conditions of validity of the bracket flattening transformation (4.32). Indeed, the linear theory predicts that the v-derivatives grow to this size in a time $t \sim \mathcal{O}(\delta^{-1})$.

However, it turns out that we can still gain some information about the interaction of transients from equations (6.11). One way would be to truncate the equations even further. Provided the v-derivatives of the initial condition are comparable in magnitude to $f_0'(v)$, we can neglect the terms proportional to δ, at least at first. The price we pay for such a truncation is that we must restrict our attention only to times $t << \delta^{-1}$, which is a considerably shorter interval than we would have desired from the method of averaging.

A more appealing route to find information about the interaction of transients is to consider the limit of small k_0. Indeed, the transient time scale is set by the time for the onset of Landau damping. This is given by $t \sim (k_0b)^{-1}$, where b is the minimum of two characteristic velocity variation scales, that of the background distribution, and that of the perturbation [11]. We can see that the dependence of the transient time on k_0 is at least reasonable by noting that the linear terms in (6.11) are the ones responsible for the phase mixing that leads to Landau damping, and that those terms are all proportional to k_0.

6.2.2 Dynamics of Transients in the Small k_0 Limit

We begin our consideration of the small k_0 limit by deriving the equations of motion in that limit. These equations are ordinary differential equations, and so have a finite-dimensional Hamiltonian structure. We give this structure, then comment on the integrability of the system. We then give a rough physical interpretation of the equations, and follow that by solving them.

The Small k_0 limit

In taking the small k_0 limit, we first have to see how the $E_k(v)$ must scale in that limit to preserve the small amplitude ordering. We will be aided in this by first deriving the asymptotic forms of ε_R and ε_I.

Since ε_I has only one term, its asymptotic form is identical to its exact expression:

$$\varepsilon_I(kk_0, kk_0 v) \sim -\pi \frac{\omega_p^2}{k^2 k_0^2} f_0'(v)\,. \tag{6.18}$$

The expression for ε_R turns out to be the Hilbert transform of (6.18):

$$\varepsilon_R(kk_0, kk_0 v) \sim -\mathcal{P}\int \mathrm{d}u \frac{\omega_p^2}{k^2 k_0^2} \frac{f_0'(u)}{u-v} = -\pi \frac{\omega_p^2}{k^2 k_0^2} \bar{f}_0'(v)\,. \tag{6.19}$$

So, from equations (4.13), (4.14), and (4.15), we find

$$\eta_k(v) = -\frac{i\omega_p^2}{4\pi e k k_0}\left[f_0'(v)\bar{E}_k(v) + \bar{f}_0'(v) E_k(v)\right]. \tag{6.20}$$

Since we plan on considering arbitrarily small k_0, we see that we must rescale $E_k(v)$, to allow η_k to remain $\mathcal{O}(1)$, and preserve the small amplitude ordering. Hence we define a new variable $\tilde{E}_k(v)$:

$$E_k(v) =: k_0 \tilde{E}_k(v)\,. \tag{6.21}$$

Moving on to the asymptotic form of the Poisson bracket, we see we need the small-k_0 behavior of $|\varepsilon|^2$. To find it, we merely add up the squares of (6.18) and (6.19):

$$|\varepsilon(kk_0, kk_0 v)|^2 \sim \pi^2 \frac{\omega_p^4}{k^4 k_0^4}[(f_0'(v))^2 + (\bar{f}_0'(v))^2]\,. \tag{6.22}$$

Writing the Poisson bracket in terms of $\tilde{E}_k$, and substituting in (6.18) and (6.22), we obtain the small-k_0 Poisson bracket:

$$\begin{aligned} \{F, G\} &= \frac{16ik_0}{\pi\omega_p^2} \int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{f_0'(v)}{[(f_0'(v))^2 + (\bar{f}_0'(v))^2]} \right) \\ &\times \left[\left(\frac{\delta F}{\delta \tilde{E}_1} \frac{\delta G}{\delta \tilde{E}_{-1}} - \frac{\delta G}{\delta \tilde{E}_1} \frac{\delta F}{\delta \tilde{E}_{-1}} \right) + 8 \left(\frac{\delta F}{\delta \tilde{E}_2} \frac{\delta G}{\delta \tilde{E}_{-2}} - \frac{\delta G}{\delta \tilde{E}_2} \frac{\delta F}{\delta \tilde{E}_{-2}} \right) \right]. \end{aligned} \tag{6.23}$$

To find the asymptotic form of the Hamiltonian, we should first determine the small-k_0 behavior of the functions H_1 and H_2 defined in (6.13)–(6.15). The small-k_0 forms turn out to be

$$H_1(1,1,2,v) \sim -\left(\frac{\pi^3 \omega_p^6}{k_0^6} \right) \left(\frac{1}{f_0'(v)} \right)^2 \left[\frac{3}{4} (f_0'(v))^2 \bar{f}_0'(v) + \frac{1}{4} (\bar{f}_0'(v))^3 \right], \tag{6.24}$$

and

$$H_2(1,1,2,v) \sim H_2(2,1,1,v) \sim -\left(\frac{\pi^3 \omega_p^6}{4k_0^6} \right) \left(\frac{1}{f_0'(v)} \right) \left[(f_0'(v))^2 + (\bar{f}_0'(v))^2 \right]. \tag{6.25}$$

Because of the equality of the asymptotic forms of $H_2(1,1,2,v)$ and $H_2(2,1,1,v)$, their difference does not contribute at the dominant order in k_0. Therefore, in this limit, we can neglect the terms containing the v-derivatives of E_k in the Hamiltonian. And so, (noting the fact that the Hilbert transform and derivative commute), we can finally write the small-k_0 Hamiltonian:

$$
\begin{aligned}
\mathcal{F}_r \;=\; & -\frac{\pi\omega_p^2}{16}\int \mathrm{d}v\, v\left(\frac{(f_0'(v))^2+(\bar{f}_0'(v))^2}{f_0'(v)}\tilde{E}_1(v)\tilde{E}_{-1}(v)\right. \\
& \left. + \frac{(f_0'(v))^2+(\bar{f}_0'(v))^2}{4f_0'(v)}\tilde{E}_2(v)\tilde{E}_{-2}(v)\right) \\
& - \epsilon i^3\frac{\pi^2\omega_p^2}{128}\left(\frac{e}{m}\right)\int \mathrm{d}v\left(\frac{1}{f_0'(v)}\right)^2\Big[3(f_0'(v))^2\bar{f}_0'(v)+(\bar{f}_0'(v))^3 \\
& -\frac{2\delta}{\pi}(2f_0'(v)\bar{f}_0'(v)\bar{f}_0''(v)+(f_0'(v))^2 f_0''(v)-(\bar{f}_0'(v))^2 f_0''(v))\Big] \\
& \times\left((\tilde{E}_{-1})^2\tilde{E}_2-(\tilde{E}_1)^2\tilde{E}_{-2}\right)
\end{aligned}
\tag{6.26}
$$

Plugging (6.26) into bracket (6.23), we arrive at the small-k_0 equations of motion:

$$
\frac{\partial\tilde{E}_1}{\partial t} = -ik_0v\tilde{E}_1-\epsilon k_0\frac{e}{m}\left(\alpha_1^0(v)-\frac{2\delta}{\pi}\beta_1^0(v)\right)\tilde{E}_{-1}\tilde{E}_2 \tag{6.27}
$$

$$
\frac{\partial\tilde{E}_2}{\partial t} = -i2k_0v\tilde{E}_2+\epsilon k_0\frac{e}{m}\left(4\alpha_1^0(v)-\frac{2\delta}{\pi}4\beta_1^0(v)\right)(\tilde{E}_1)^2 \tag{6.28}
$$

$$
\frac{\partial\tilde{E}_{-1}}{\partial t} = ik_0v\tilde{E}_{-1}-\epsilon k_0\frac{e}{m}\left(\alpha_1^0(v)-\frac{2\delta}{\pi}\beta_1^0(v)\right)\tilde{E}_1\tilde{E}_{-2} \tag{6.29}
$$

$$
\frac{\partial\tilde{E}_{-2}}{\partial t} = i2k_0v\tilde{E}_{-2}+\epsilon k_0\frac{e}{m}\left(4\alpha_1^0(v)-\frac{2\delta}{\pi}4\beta_1^0(v)\right)(\tilde{E}_{-1})^2\,. \tag{6.30}
$$

In (6.27)–(6.30), we have introduced the small-k_0 coefficients, which are defined by:

$$
\begin{aligned}
\alpha_1^0(v) &:= \left(\frac{\pi}{4}\right)\frac{3(f_0'(v))^2\bar{f}_0'(v)+(\bar{f}_0'(v))^3}{f_0'(v)((f_0'(v))^2+(\bar{f}_0'(v))^2)}\,; \\
\beta_1^0(v) &:= \left(\frac{\pi}{4}\right)\frac{2f_0'(v)\bar{f}_0'(v)\bar{f}_0''(v)+(f_0'(v))^2 f_0''(v)-(\bar{f}_0'(v))^2 f_0''(v)}{4f_0'(v)((f_0'(v))^2+(\bar{f}_0'(v))^2)}\,.
\end{aligned}
\tag{6.31}
$$

Like the equations of motion of the linearized system, the small-k_0 system is a family of ordinary differential equations parameterized by the continuous parameter v. The form of these equations is standard, often arising when the two-wave limit of three-wave coupling is considered (for instance, in the context of cold counterstreaming ion beams [54]) and was at least known to Cherry as long ago as 1925 [55]. Of course, since the entire family of equations is relevant to any one given initial condition, analysis of them is more complicated than analysis of the usual two-wave problem; the system can be treated as a two degree-of-freedom system, but the results for all values of v must be considered.

Finite-Dimensional Hamiltonian Structure of the Small-k_0 System

As ordinary differential equations, (6.27)-(6.30) have a finite dimensional Hamiltonian structure. Formally, this follows from the fact that a functional derivative of an integral reduces to a derivative of the integrand when there are no (relevant) derivatives in the integrand. Hence, the finite dimensional Poisson bracket for the present system is given by

$$\begin{aligned} \{F,G\} &= \frac{16ik_0}{\pi\omega_p^2}\left(\frac{f_0'(v)}{[(f_0'(v))^2+(\bar{f}_0'(v))^2]}\right) \\ &\times \left[\left(\frac{\partial F}{\partial \tilde{E}_1}\frac{\partial G}{\partial \tilde{E}_{-1}} - \frac{\partial G}{\partial \tilde{E}_1}\frac{\partial F}{\partial \tilde{E}_{-1}}\right) + 8\left(\frac{\partial F}{\partial \tilde{E}_2}\frac{\partial G}{\partial \tilde{E}_{-2}} - \frac{\partial G}{\partial \tilde{E}_2}\frac{\partial F}{\partial \tilde{E}_{-2}}\right)\right] . \end{aligned} \tag{6.32}$$

where F and G are now simply functions of v. And the finite dimensional Hamiltonian for this system is simply the integrand of equation (6.26):

$$\begin{aligned} \mathcal{F}_r &= -\left(\frac{\pi\omega_p^2}{16}\right)\left(\frac{(f_0'(v))^2+(\bar{f}_0'(v))^2}{f_0'(v)}\right) v \left[\tilde{E}_1(v)\tilde{E}_{-1}(v) + \frac{1}{4}\tilde{E}_2(v)\tilde{E}_{-2}(v)\right] \\ &- \epsilon i^3\frac{\pi^2\omega_p^2}{128}\left(\frac{e}{m}\right)\left(\frac{1}{f_0'(v)}\right)^2\Big[3(f_0'(v))^2\bar{f}_0'(v) + (\bar{f}_0'(v))^3 \\ &\qquad -\frac{2\delta}{\pi}(2f_0'(v)\bar{f}_0'(v)\bar{f}_0''(v) + (f_0'(v))^2 f_0''(v) - (\bar{f}_0'(v))^2 f_0''(v))\Big] \\ &\qquad \times \left((\tilde{E}_{-1})^2\tilde{E}_2 - (\tilde{E}_1)^2\tilde{E}_{-2}\right) \end{aligned} \tag{6.33}$$

Recognizing the finite dimensional Hamiltonian structure of equations (6.27)-(6.30) gives us more of a notational than a conceptual advantage in the present context. But even this advantage is keenly felt when we recognize that the total momentum, given in equation (6.4), actually provides us an integral of motion for each value of v of the small-k_0 system. After truncating the sum to run between $k = -2$ and $k = 2$, stripping off the integral sign, and passing to small k_0, we find that

$$M = -\left(\frac{\pi\omega_p^2}{16}\right)\left(\frac{(f_0'(v))^2 + (\bar{f}_0'(v))^2}{f_0'(v)}\right)\left[\tilde{E}_1(v)\tilde{E}_{-1}(v) + \frac{1}{4}\tilde{E}_2(v)\tilde{E}_{-2}(v)\right] . \qquad (6.34)$$

Since (6.34) is independent of the Hamiltonian (6.33), and is certainly in involution with the Hamiltonian (else it would not be an integral), the nonlinear transient system is integrable in the Liouville sense.

The existence of a second constant of motion is especially easy to see when the Hamiltonian (6.33) is written in the small-k_0 action-angle variables of the linearized system. The transformation to these variables is given by the asymptotic form of (4.33):

$$\tilde{E}_k(u) = \sqrt{\left[\frac{16k^3k_0}{\pi\omega_p^2}\left(\frac{|f_0'(v)|}{[(f_0'(v))^2 + (\bar{f}_0'(v))^2]}\right)J_{|k|}(u)\right]}\exp[i\operatorname{sgn}(-kf_0'(v))\theta_{|k|}(u)] . \qquad (6.35)$$

In terms of these variables, of course, the Poisson bracket is canonical:

$$\{F, G\} = \left(\frac{\partial F}{\partial\theta_1}\frac{\partial G}{\partial J_1} - \frac{\partial G}{\partial\theta_1}\frac{\partial F}{\partial J_1}\right) + \left(\frac{\partial F}{\partial\theta_2}\frac{\partial G}{\partial J_2} - \frac{\partial G}{\partial\theta_2}\frac{\partial F}{\partial J_2}\right) . \qquad (6.36)$$

In the same variables, the Hamiltonian has the form

$$H = vs(v)k_0J_1 + 2vs(v)k_0J_2 + \epsilon A(v)J_1\sqrt{J_2}\sin(2\theta_1 - \theta_2) , \qquad (6.37)$$

where $s(v) := \mathrm{sgn}(-f_0'(v))$, and the coefficient $A(v)$ is defined as

$$A(v) := -\frac{e}{m}\left(\frac{8k_0^3}{\pi\omega_p^2}\right)^{1/2} s(v)\left(\frac{1}{[(f_0'(v))^2 + (\bar{f}_0'(v))^2]}\right)^{3/2}\left(\frac{1}{|f_0'(v)|}\right)^{1/2} \tag{6.38}$$
$$\times\Big[3(f_0'(v))^2\bar{f}_0'(v) + (\bar{f}_0'(v))^3 - \frac{2\delta}{\pi}\Big(2f_0'(v)\bar{f}_0'(v)\bar{f}_0''(v)$$
$$+ (f_0'(v))^2 f_0''(v) - (\bar{f}_0'(v))^2 f_0''(v)\Big)\Big]\,.$$

The resulting equations of motion are:

$$\begin{aligned}
\dot{\theta}_1 &= s(v)k_0 v + \epsilon A(v)\sqrt{J_2}\sin(2\theta_1 - \theta_2) \\
\dot{\theta}_2 &= 2s(v)k_0 v + \epsilon A(v)\frac{J_1}{2\sqrt{J_2}}\sin(2\theta_1 - \theta_2) \\
\dot{J}_1 &= -2\epsilon A(v)J_1\sqrt{J_2}\cos(2\theta_1 - \theta_2) \\
\dot{J}_2 &= \epsilon A(v)J_1\sqrt{J_2}\cos(2\theta_1 - \theta_2)\,.
\end{aligned} \tag{6.39}$$

As is expected with Resonance Hamiltonians with a single resonance, only one combination of angles appears in the Hamiltonian (6.37). Thus, a canonical transformation can be made to a new coordinate system for which only one angle appears; the action conjugate to the *other* angle is therefore a constant of motion. This constant is, of course, the momentum (6.34). In the linear system's action-angle variables it is simply

$$M = s(v)k_0 J_1 + 2s(v)k_0 J_2\,. \tag{6.40}$$

So we immediately obtain a physical interpretation for the actions: $kk_0 J_k(v)$ gives the amount of momentum carried by the (k, v) Van Kampen mode. The details of the above line of argument will be explicitly followed through below. Before we take on this task, a few observations about the integrability of this system are in order.

Having an integrable system presents us with an opportunity too rare to pass up: we can find general solutions to it, at least up to quadrature. So, in the following section, we show how the nonlinear transient system can be integrated up to quadrature. While this has undoubtedly been done somewhere in the literature, it is useful to see it done in the variables used in this dissertation. After integrating the system up to quadrature, we integrate again in a different way: finding action-angle variables. These variables, to our knowledge, have not been presented in the literature. We present them here for reference purposes, since they may be useful in a higher order perturbation theory.

Also, it might be argued that this integrability is not a surprise; as we point out in our discussion of resonance Hamiltonians in chapter 5, retaining only one resonant interaction term unavoidably gives rise to an integrable system. And since we have truncated to only two spatial Fourier modes (and we have a cubic nonlinearity), we expect to only have one resonant interaction term to keep. However, the present situation is not so simple. In reality, we are keeping an infinite number of *normal modes* of the system, and as a result, an infinite number of resonant interaction terms. In this light, it seems that the integrability of even this extremely simplified system is a nontrivial result.

Physical interpretation

Equations (6.39) are almost in a convenient form for physical interpretation of the exactly resonant interaction of transients. As it stands, $A(v)$ is apparently singular for any v which extremizes f_0. This is a removable singularity, however, as can be seen from the dynamical accessibility condition $E_k(v) \propto s(k,v)$. This implies $\tilde{E}_k(v) \propto f_0'(v)$. From this and the coordinate transformation (6.35) follows a condition on $J_k(v)$:

$$J_k \propto |f_0'(v)| \,. \tag{6.41}$$

For the explicit integration of this system, it is convenient to use variables in which the singularity has been removed. With that in mind, we define the new (noncanonical) coordinates $\tilde{J}_k$ via

$$J_k =: |f_0'(v)| \tilde{J}_k \,. \tag{6.42}$$

After transforming to J_k', equations (6.39) become

$$\begin{aligned}
\dot{\theta}_1 &= s(v)k_0 v + \epsilon \tilde{A}(v)\sqrt{\tilde{J}_2}\sin(2\theta_1 - \theta_2) \\
\dot{\theta}_2 &= 2s(v)k_0 v + \epsilon \tilde{A}(v)\frac{\tilde{J}_1}{2\sqrt{\tilde{J}_2}}\sin(2\theta_1 - \theta_2) \\
\dot{\tilde{J}}_1 &= -2\epsilon \tilde{A}(v)\tilde{J}_1\sqrt{\tilde{J}_2}\cos(2\theta_1 - \theta_2) \\
\dot{\tilde{J}}_2 &= \epsilon \tilde{A}(v)\tilde{J}_1\sqrt{\tilde{J}_2}\cos(2\theta_1 - \theta_2) \,,
\end{aligned} \tag{6.43}$$

where we have introduced the new coefficient function $\tilde{A}(v)$. We obtain an expression for $\tilde{A}(v)$ by multiplying equation (6.38) by $\sqrt{|f_0'(v)|}$:

$$\begin{aligned}
\tilde{A}(v) :&= -\frac{e}{m}\left(\frac{8k_0^3}{\pi\omega_p^2}\right)^{1/2} s(v)\left(\frac{1}{[(f_0'(v))^2 + (\bar{f}_0'(v))^2]}\right)^{3/2} \\
&\times \Big[3(f_0'(v))^2\bar{f}_0'(v) + (\bar{f}_0'(v))^3 - \frac{2\delta}{\pi}\Big(2f_0'(v)\bar{f}_0'(v)\bar{f}_0''(v) \\
&\qquad + (f_0'(v))^2 f_0''(v) - (\bar{f}_0'(v))^2 f_0''(v)\Big)\Big] \,.
\end{aligned} \tag{6.44}$$

Now, a Van Kampen mode has the *spatial* structure of a traveling wave, and the variable $\theta_i(v)$ measures the phase of the "wave" with wavenumber kk_0 and phase velocity v. As we saw above, the variables $J_k(v)$ are proportional to the momentum carried by the same mode.

In the limit $\epsilon \to 0$, we recover the solutions of the linearized system:

$$\begin{aligned} \theta_{1L}(t) &= s(v)k_0vt + \theta_{10} \\ \theta_{2L}(t) &= 2s(v)k_0vt + \theta_{20} \\ \tilde{J}_{1L} &= \tilde{J}_{10} \\ \tilde{J}_{2L} &= \tilde{J}_{20}\,. \end{aligned} \tag{6.45}$$

(The subscript L stands for "linear.") The fact that these two modes are in resonance is reflected in the fact that the resonant phase,

$$\psi_L := 2\theta_{1L} - \theta_{2L} = 2\theta_{10} - \theta_{20}\,, \tag{6.46}$$

is a constant.

When we include the nonlinear terms, we see that the $k = 1$ and $k = 2$ modes simply exchange momentum. This, of course, is a transparent proof of the statement that the total momentum, given in (6.40), is conserved. To estimate the rate of momentum exchange, we can substitute the solutions of the linearized system (6.45) into the RHS of one of the momentum equations. We choose the $k = 1$ equation:

$$\dot{\tilde{J}}_1 = -2\epsilon\tilde{A}(v)\tilde{J}_{10}\sqrt{\tilde{J}_{20}}\cos(\psi_L)\,. \tag{6.47}$$

And so, the rate of momentum transfer is slow (proportional to ϵ), and dependent more strongly on the magnitude of the momentum contained in the $k = 1$ mode than the $k = 2$ mode. It also depends on the background equilibrium through $\tilde{A}(v)$.

The sign of the RHS of (6.47) also gives us information about the direction of momentum transfer. First, we note that to get the actual sign of momentum exchange, we must multiply through by $s(v)$, as follows from (6.40). Since the values of $\tilde{J}_{10}$ and $\tilde{J}_{20}$ are necessarily positive, the overall sign is thus given by the product of the signs of $s(v)\tilde{A}(v)$ and $\cos(\psi_L)$.

We consider the sign of $s(v)\tilde{A}(v)$ first. From equation (6.44), we immediately see that the factor $s(v)\tilde{A}(v)$ is negative if

$$3(f_0'(v))^2\bar{f}_0'(v) + (\bar{f}_0'(v))^3 - \frac{2\delta}{\pi}\Big(2f_0'(v)\bar{f}_0'(v)\bar{f}_0''(v) + (f_0'(v))^2 f_0''(v) - (\bar{f}_0'(v))^2 f_0''(v)\Big) > 0\,. \tag{6.48}$$

Though (6.48) is a complicated condition, we can make a couple general observations. First, if $\bar{f}_0'(v) \sim \mathcal{O}(1)$, the first two terms dominate, and the condition reduces to $\bar{f}_0'(v) > 0$. The other is that the continuity of the spectrum (embodied in δ) becomes important to the direction of momentum transfer for values of v at which $\bar{f}_0'(v) \sim \mathcal{O}(\delta)$.

The other factor that influences the direction of momentum transfer is the sign of $\cos(\psi_L)$. This is negative when $\frac{\pi}{2} < \psi_L < \frac{3\pi}{2}$, or

$$\frac{\pi}{4} + \frac{\theta_{20}}{2} < \theta_{10} < \frac{3\pi}{4} + \frac{\theta_{20}}{2}\,. \tag{6.49}$$

Thus, momentum will flow from the $k = 2$ mode to the $k = 1$ mode (for any given v violating (6.48)) if the modes are initially nearly in phase, and vice versa if the two modes begin mostly out of phase.

We see a similar dependence on the relative initial phase of the two modes when we consider the nonlinear effects on the frequencies of the modes. In fact, the lowest order shifts in frequency, $\Delta\omega_k$ are given by

$$\begin{aligned}\Delta\omega_1 &= \epsilon\tilde{A}(v)\sqrt{\tilde{J}_{20}}\sin(\psi_L) \\ \Delta\omega_2 &= \epsilon\tilde{A}(v)\frac{\tilde{J}_{10}}{2\sqrt{\tilde{J}_{20}}}\sin(\psi_L)\,.\end{aligned} \tag{6.50}$$

The small change in frequencies shown in equations (6.50) depend on both the initial actions *and* the initial relative phase of the modes. The latter dependence, at this order, is simply a reflection of the resonance between the two modes.

Also, equations (6.50) imply no frequency shift for $\psi_L = 0, \pi$. It turns out that this is exactly true for the nonlinear system; the resulting completely phase-locked solution corresponds to evolution along a separatrix, as will be seen below.

Integration of the nonlinear transient system

Having gotten a rough feeling for the physics described in equations (6.39), we now show how they can be integrated. The natural way to begin is with a canonical transformation to the resonant variables mentioned in chapter 5.

Denoting the new momenta by I, J, we define a generating function

$$S(\theta_1, \theta_2; I, J) = (\theta_2 - 2\theta_1)J + \theta_1 I\,. \tag{6.51}$$

Then, the new position coordinates conjugate to J and I are respectively given by

$$\psi = \frac{\partial S}{\partial J} = \theta_2 - 2\theta_1, \qquad \theta = \frac{\partial S}{\partial I} = \theta_1\,. \tag{6.52}$$

And using the facts that $J_1 = \partial S/\partial\theta_1$, and $J_2 = \partial S/\partial\theta_2$, we find expressions for the new momenta in terms of the old:

$$J = J_2, \qquad I = J_1 + 2J_2\,. \tag{6.53}$$

The Hamiltonian in the new variables is given by

$$H = s(v)k_0 v I + \epsilon A(v)(2J - I)\sqrt{J}\sin(\psi)\,. \tag{6.54}$$

And the new equations generated by (6.54) are

$$\begin{aligned} \dot{\theta} &= s(v)k_0 v - \epsilon A(v)\sqrt{J}\sin(\psi) \\ \dot{\psi} &= \epsilon A(v)\left(3\sqrt{J} - \frac{I}{2\sqrt{J}}\right)\sin(\psi) \\ \dot{I} &= 0 \\ \dot{J} &= \epsilon A(v)(I\sqrt{J} - 2J^{3/2})\cos(\psi)\,. \end{aligned} \tag{6.55}$$

From equations (6.54), we immediately see that $I = const = J_{10} + 2J_{20}$ (our old friend the momentum.) This allows us to treat the (ψ, J) equations as a one degree-of-freedom system. Instead of using the whole Hamiltonian as an integral of the reduced system, we find it easier to recognize that the constancy of I implies that the nonlinear term in (6.54) is itself a constant of motion. We therefore define

$$I_2 := (I - 2J)\sqrt{J}\sin(\psi)\,. \tag{6.56}$$

At this point, we are in a position either to directly integrate equations (6.55) or to transform to action-angle variables for the nonlinear system. The former approach is more straightforward, so we consider that first. (To get sensible results from the following solutions, we should be careful to remove that singularity in the A here by factoring out a $|f_0'(v)|$ from I, J, and I_2 as above. In practice, though, when evaluating $J(t)$, all we have to make sure of doing is substituting $\tilde{A}(v)$ for $A(v)$ and then multiplying the result by $|f_0'(v)|$.)

We begin by considering the exceptional solutions, that is, the equilibria and separatrices, of the ψ, J system. There are four equilibria. Two of them,

$$\psi = \frac{\pi}{2}, \frac{3\pi}{2}; \qquad J = \frac{I}{6}\,, \tag{6.57}$$

are elliptic points. The other two are hyperbolic, and are found at

$$\psi = 0, \pi; \qquad J = \frac{I}{2} \,. \tag{6.58}$$

The fact that $I = 2J$ for the hyperbolic points shows them to be a limiting case of our truncated model in which $J_1 = 0$. Points in phase space for which $J > I/2$ require $I < 2J$, a condition incompatible with the positivity of the actions, and are therefore unphysical.

In keeping with this restriction on J, the circle (in the $J - \psi$ plane) defined by $J = I/2$ is a separatrix of the ψ, J equations. While it, in itself, is not a physical solution, it can be used to understand the orbits for which $J \simeq I/2$. The equation for evolution along this outer separatrix is

$$\dot{\psi} = \epsilon A(v)\sqrt{2I}\,\sin(\psi) \,. \tag{6.59}$$

Equation (6.59) has the solution

$$\psi(t) = 2\tan^{-1}\left[\tan\left(\frac{\psi(0)}{2}\right)\exp(\epsilon A(v)\sqrt{2I}t)\right] \,. \tag{6.60}$$

The solution (6.60) shows us (as we could have deduced directly from (6.59)) that the $J = I/2$ circle is the unstable manifold of $\psi = 0, J = I/2$, and the stable manifold of $\psi = \pi, J = I/2$. We will now show that (one component of) the respective stable and unstable manifolds of these points are more physically relevant.

We find these manifolds along the directions $\psi = 0, \pi$. The evolution equations along these curves are respectively

$$\dot{J} = \pm\epsilon A(v)(I\sqrt{J} - 2J^{3/2}) \,. \tag{6.61}$$

Integrating (6.61), we find

$$J(t) = \frac{I}{2}\sin^2\left[2\tan^{-1}\left(C\exp\left(\pm\sqrt{\epsilon A(v)\frac{I}{2}t}\right)\right) - \frac{\pi}{2}\right] \,, \tag{6.62}$$

where the constant C is given by

$$C = \tan\left[\frac{\pi}{4} + \frac{1}{2}\sin^{-1}\left(\sqrt{\frac{2J(0)}{I}}\right)\right] . \tag{6.63}$$

One interesting thing to note about the solution (6.62) is that if the initial condition is $J(0) = J_0, \psi = \pi$, the solution reaches $J = 0$ in a finite time:

$$t_{J=0} = \frac{1}{\epsilon A(v)}\sqrt{\frac{2}{I}}\log\left[\frac{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\sqrt{\frac{2J_0}{I}}\right)\right)}{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\sqrt{\frac{2J_0}{I}}\right)\right)}\right] . \tag{6.64}$$

So, even though there is an apparent singularity in equations (6.55) at $J = 0$, it is actually a perfectly accessible point on a separatrix.

The solution (6.62) is a physically relevant solution of our equations. It corresponds to a situation where a $k = 1$ Van Kampen mode continually shuffles off its momentum to a $k = 2$ mode with which it is completely phase-locked.

Now that we have considered the exceptional solutions of system (6.55), we move on to the generic solutions. Solving for $\sin(\psi)$ in terms of I_2, we find

$$\sin(\psi) = \frac{I_2}{(I - 2J)} . \tag{6.65}$$

From (6.65) it follows that

$$\cos(\psi) = \pm\sqrt{1 - \frac{I_2^2}{(I-2J)^2 J}} . \tag{6.66}$$

Since the case where $\sin(\psi) = 0$ is one of the exceptional solutions, we can safely write the equation of motion for J in terms of I_2 as $\dot{J} = \epsilon A(v) I_2 \cot(\psi)$, or

$$\dot{J} = \pm\epsilon A(v)\sqrt{(I-2J)^2 J - I_2^2} . \tag{6.67}$$

As follows from (6.66), the proper sign in equation (6.67) depends on the concurrent value of ψ: it should be positive in the first and fourth quadrants of the $J - \psi$ plane, and negative in the second and third.

We can integrate (6.67) by separation of variables:

$$\mathrm{d}t = \pm \frac{\mathrm{d}J}{\epsilon A(v)\sqrt{4J^3 - 4IJ^2 + I^2 J - I_2^2}} \,. \tag{6.68}$$

Integrating (6.68) from $t = t_0$ to an arbitrary time t yields

$$\pm\epsilon A(v)(t - t_0) = \int_{J(t_0)}^{J(t)} \frac{\mathrm{d}J'}{\sqrt{4J'^3 - 4IJ'^2 + I^2 J' - I_2^2}} \,. \tag{6.69}$$

At this point, we recognize the RHS of (6.68) as some sort of elliptic integral. In fact, it is the inverse of a Weierstrass $\wp$-function [56]. We see this by first transforming to a new dummy variable $y = J' - \frac{I}{3}$. This puts the integrand in the correct form for the inverse of the Weierstrass $\wp$-function with invariants

$$g_2 = \frac{I^2}{3}; \qquad g_3 = I_2^2 - \frac{1}{27} I^3 \,. \tag{6.70}$$

But to match the conventional definition, we need an upper limit of ∞. So we break up the integral:

$$\int_{J(t_0)-\frac{1}{3}I}^{J(t)-\frac{1}{3}I} \frac{\mathrm{d}y}{\sqrt{4y^3 - \frac{I^2}{2}y - (I_2^2 - \frac{I^3}{27})}} = \int_{J(t_0)-\frac{1}{3}I}^{\infty} \frac{\mathrm{d}y}{\sqrt{4y^3 - \frac{I^2}{2}y - (I_2^2 - \frac{I^3}{27})}}$$
$$- \int_{J(t)-\frac{1}{3}I}^{\infty} \frac{\mathrm{d}y}{\sqrt{4y^3 - \frac{I^2}{2}y - (I_2^2 - \frac{I^3}{27})}} \,. \tag{6.71}$$

The first integral on the RHS of (6.71) is a constant which we will denote by ϕ. Specifically,

$$\phi := \wp^{-1}\left(J(t_0) - \frac{I}{3}, g_2, g_3\right) \,. \tag{6.72}$$

And so, we find an expression for the evolution of J to be

$$J(t) = \wp(\pm\epsilon A(t - t_0) - \phi, g_2, g_3) + I/3 \,. \tag{6.73}$$

Of course, the presence of the $\pm$ means that this is not the most explicit expression we can find for the solution. To find that, we need to determine the times at which the solution passes from the right side to the left side of the $J-\psi$ plane or vice-versa. This happens when $\sin(\psi) = 1, -1$. For concreteness, we will suppose ψ starts in the first quadrant. Then (6.65) implies that the first crossing time t_1, must satisfy:

$$\wp(\epsilon A(t_1 - t_0) - \phi, g_2, g_3) = \frac{I}{6} - \frac{I_2}{2}. \tag{6.74}$$

The second crossing time would then be given by

$$\wp(-\epsilon A(t_2 - t_1) - \phi_1, g_2, g_3) = \frac{I}{6} - \frac{I_2}{2}, \tag{6.75}$$

where $\phi_1 = \wp^{-1}(J(t_1), g_2, g_3)$. The later crossing times can be found by alternating between (6.74) and (6.75). We will abbreviate this procedure with expression (6.73). Furthermore, to reduce clutter, we will suppress the dependence on the invariants g_2 and g_3.

Plugging $J(t)$ into (6.65) allows us to solve for $\psi(t)$:

$$\psi(t) = \sin^{-1}\left(\frac{I_2}{(\frac{1}{3}I - \wp(\pm\epsilon A(t-t_0) - \phi)\sqrt{\wp(\pm\epsilon A(t-t_0) - \phi) + 1/3I}}\right). \tag{6.76}$$

Since we already know that $I = J_{10} + 2J_{20}$, we have a complete solution once we integrate the equation for θ in (6.55). The result can be expressed entirely in terms of functions related to the Weierstrass $\wp$-function. Substituting (6.76) into this equation gives us

$$\dot{\theta} = s(v)k_0 v - \epsilon A(v)\frac{I_2}{2}\frac{1}{\frac{I}{6} - \wp(\pm\epsilon A(v)(t-t_0) - \phi)}. \tag{6.77}$$

Though the first term in (6.77) is easy to integrate, the second term looks hopeless. However, by defining $a := \wp^{-1}(\frac{I}{6})$, and $z := \epsilon A(v) \mp \phi$, we find that integrating (6.77) with respect to time yields

$$\theta(t) = s(v)k_0 v(t-t_0) + \frac{I_2}{2}\int_{\mp\phi}^{\epsilon A(v)(t-t_0)\mp\phi}\frac{\mathrm{d}z}{\wp(z) - \wp(a)}. \tag{6.78}$$

The integral in (6.78) is in a standard form, and can be written in terms of the Weierstrass σ- and ζ-functions, and the derivative of the $\wp$-function. Thus,

$$\begin{aligned}\theta(t) &= s(v)k_0v(t-t_0)+\frac{I_2}{2\wp'(a)}\Big[2\epsilon A(v)\zeta(a)(t-t_0) \\ &\quad \log\left(\frac{\sigma(\epsilon A(v)(t-t_0)\mp\phi-a)\sigma(\epsilon A(v)(t-t_0)\mp\phi+a)}{\sigma(\mp\phi-a)\sigma(\mp\phi+a)}\right)\Big]\,.\end{aligned} \tag{6.79}$$

Another approach to integrating equations (6.55) is to find a transformation from the variables (θ, ψ, I, J) to action-angle variables $(\phi_1, \phi_2, I_1, I_2)$. The natural choices for the actions are the two constants of motion, $I_1 = I$ and I_2 as defined in (6.56). In terms of these new actions, the Hamiltonian would have the form

$$H = s(v)vk_0I_1 + \epsilon A(v)I_2\,. \tag{6.80}$$

The angles would then evolve according to

$$\dot{\phi}_1 = s(v)vk_0, \qquad \dot{\phi}_2 = \epsilon A(v)\,. \tag{6.81}$$

In a sense, then, these action-angle variables completely separate the fast, linearized motion from the slow nonlinear effects. To reach such a simple form for the Hamiltonian, though, we must use a rather complicated canonical transformation.

Knowing the form of the actions allows us to construct the appropriate canonical transformation via the Poincaré generating function [57]:

$$\begin{aligned}S(I_1,I_2,\theta,\psi) &= \int_{\theta_0}^{\theta} I(I_1,I_2,\theta',\psi)\mathrm{d}\theta' + \int_{\psi_0}^{\psi} J(I_1,I_2,\theta,\psi')\mathrm{d}\psi' \\ &= (\theta-\theta_0)I_1 + \int_{\psi_0}^{\psi} J(I_1,I_2,\psi')\mathrm{d}\psi'\,,\end{aligned} \tag{6.82}$$

where the integrands are typically multiply branched functions, and the lower limit on the integrals specifies which branch is to be used. Clearly, I_1 is not an example of a multiply-branched function, so we can safely set $\theta_0 = 0$. However, the functional form of J is implicitly given in (6.56), and so has multiple branches. These will be treated below.

The canonical transformation generated by (6.82) is then given by

$$I = \frac{\partial S}{\partial \theta} \qquad J = \frac{\partial S}{\partial \psi} \qquad (6.83)$$
$$\phi_1 = \frac{\partial S}{\partial I_1} \qquad \phi_2 = \frac{\partial S}{\partial I_2} \quad .$$

Dealing with the first integral in (6.82) is trivial; handling the second is more difficult. The first step is to solve equation (6.56) for J. We begin by squaring (6.56) to arrive at a cubic equation:

$$J^3 - I_1 J^2 + \frac{I_1^2}{4} J - \frac{I_2^2}{4\sin^2(\psi)} = 0\,. \qquad (6.84)$$

When factoring a cubic, it is useful to calculate two auxiliary quantities, q and r, that depend on the coefficients [56]. For the case of (6.84), these quantities are

$$q = -\frac{I_1^2}{36}\,, \qquad (6.85)$$

and

$$r = \frac{I_2^2}{8\sin^2(\psi)} - \frac{I_1^3}{216}\,. \qquad (6.86)$$

From them, we can calculate the discriminant $q^3 + r^2$:

$$D := q^3 + r^2 = \frac{I_2^4}{64\sin^4(\psi)} - \frac{I_1^3 I_2^2}{864\sin^2(\psi)}\,. \qquad (6.87)$$

When its discriminant is negative, (6.84) has three solutions, but only two of them are relevant to our problem. Which solutions should be kept can be seen by considering our discussion of the exceptional solutions of (6.55) above. We recall that there exist elliptic fixed points at $J = I_1/6, \psi = \pm\pi/2$. On the other hand, when the discriminant vanishes, at least two of the roots are equal. We therefore expect the two physical roots of (6.84) will converge on the value $J = I_1/6$ when we set the discriminant to zero.

Using the above test, we find the two physical solutions to be

$$\begin{aligned} J &= \frac{I_1}{3} - (r^2 + D)^{1/6} \left\{ \cos\left[\frac{1}{3}\tan^{-1}\left(\frac{\sqrt{-D}}{r}\right)\right] \right. \\ &\left. \pm\sqrt{3}\sin\left[\frac{1}{3}\tan^{-1}\left(\frac{\sqrt{-(D)}}{r}\right)\right]\right\} . \end{aligned} \tag{6.88}$$

All that is left to do to complete the integration of (6.55) is to write out the transformation to the angles ϕ_1, and ϕ_2. (The new actions were given above.) As indicated in (6.83), we need only take derivatives of J with respect to I_1 and I_2.

Chapter 7

Conclusions

In this dissertation, we have considered the effect of the lowest-order nonlinear corrections to the linearized Vlasov-Poisson system using the techniques of canonical perturbation theory. To accomplish this, we had to overcome two technical obstacles. First, we had to show how the Vlasov-Poisson system, naturally a noncanonical Hamiltonian system, could be canonized to the order of interest. Second, we had to adapt the method of partial averaging, normally used to analyze resonant perturbations in finite-dimensional Hamiltonian systems, to an infinite-dimensional system, which furthermore has a continuous spectrum. Thus, the work necessary to even formulate the weakly nonlinear equations forms a good portion of this dissertation.

This work bore fruit in two ways, displayed in chapter 6. One is the demonstration of (nonlinear) stability of the weakly nonlinear system, in spite of the presence of negative energy modes. (It is possible, of course, that a higher-order interaction between modes of different signature could produce instability, but that is a question to be addressed by future work.) The other fruit is the derivation of a system that models the nonlinear behavior of transients. This system turns out to be integrable, a fact which immeasurably aids us in its analysis. This analysis shows a slow transfer of momentum (second harmonic generation), and slow shift in frequency (nonlinear dispersion) between the linear normal modes. Both the exact solution and the transformation to action-angle variables for this system are given.

The analysis contained in this dissertation only begins to explore the weakly nonlinear Vlasov-Poisson system. We hope, however, that it will form a solid foundation for deeper explorations.

Appendix A

Details of calculating the flattening transformation

The content of this appendix is taken almost entirely from [44].

Our goal is to find a transformation from coordinates z^i to coordinates η^i such that the Poisson bracket written in terms of η^i has the form (3.10). Hence, we set the RHS of equation (3.10) equal to the RHS of equation (3.11):

$$J^{ij}(z_0) + \mathcal{O}(\epsilon^2) = \frac{\partial \eta^i}{\partial z^k} J^{kl}(z_0 + \epsilon z) \frac{\partial \eta^j}{\partial z^l}\,, \tag{A.1}$$

and then solve for η. We can safely assume that the transformation is near-identity, and so introduce the ansatz

$$\eta^i_{kl} = z^i + \frac{\epsilon}{2} D^i_{kl} z^k z^l + \mathcal{O}(\epsilon^2)\,, \tag{A.2}$$

where D^i_{kl} are the components of a third-rank tensor symmetric in kl. Then substituting this ansatz into (A.1), we arrive at an equation for D^i_{kl}:

$$J^{ik}(z_0) D^j_{kl} + J^{kj}(z_0) D^i_{kl} + \frac{\partial J^{ij}}{\partial z^l}(z_0) = \mathcal{O}(\epsilon^2)\,. \tag{A.3}$$

Because of the skew-symmetry of $J^{ij}(z_0)$, equation (A.3) is underdetermined. (For N dimensions, (A.3) has at most $N^2(N-1)/2$ independent equations, while D^i_{kl} has $N^2(N+1)/2$ components.)

To solve (A.3), we decompose D^i_{kl} as follows:

$$D^i_{kl} = \frac{\partial J^{im}}{\partial z^k}(z_0) S_{ml} + \frac{\partial J^{im}}{\partial z^l}(z_0) S_{mk} + \hat{D}^i_{kl} \,, \tag{A.4}$$

where S_{ml} is skew-symmetric, and D^i_{kl} is symmetric in kl. Inserting (A.4) into (A.3) yields

$$\begin{aligned} & \left(J^{ik}(z_0) \frac{\partial J^{jm}}{\partial z^k}(z_0) + J^{kj}(z_0) \frac{\partial J^{im}}{\partial z^k}(z_0) \right) S_{ml} \\ + & \left(J^{ik}(z_0) \frac{\partial J^{jm}}{\partial z^l}(z_0) + J^{kj}(z_0) \frac{\partial J^{im}}{\partial z^l}(z_0) \right) S_{mk} \\ + & \; J^{ik}(z_0) \hat{D}^j_{kl} + J^{kj}(z_0) \hat{D}^i_{kl} + \frac{\partial J^{ij}}{\partial z^l}(z_0) = 0 \,. \end{aligned} \tag{A.5}$$

We can put (A.5) into a more convenient form by applying the Jacobi identity (2.7) to the first two terms, switching the order of several pairs of indices (using skew-symmetry of $J^{ij}(z_0)$ and S_{ml}), and splitting the lone derivative into three parts. This gives us

$$\begin{aligned} & \frac{\partial J^{ij}}{\partial z^k}(z_0) \left(J^{km}(z_0) S_{ml} + \frac{1}{3}\delta^k_l \right) + \frac{\partial J^{im}}{\partial z^l}(z_0) \left(J^{jk}(z_0) S_{km} + \frac{1}{3}\delta^j_m \right) \\ + & \; \frac{\partial J^{mj}}{\partial z^l}(z_0) \left(J^{ik}(z_0) S_{km} + \frac{1}{3}\delta^i_m \right) + J^{ik}(z_0) \hat{D}^j_{kl} + J^{kj}(z_0) \hat{D}^i_{kl} = 0 \,. \end{aligned} \tag{A.6}$$

The fact that the quantities in the three sets of parentheses in (A.6) all have the same form suggests the following solution:

$$\hat{D}^i_{kl} = 0 \tag{A.7}$$

$$S_{ml} = -\frac{1}{3} J^{-1}_{mk}(z_0) \delta^k_l \,. \tag{A.8}$$

Substituting (A.7) and (A.8) into (A.4) yields the solution given in (3.14). Of course, this solution is only valid when $J^{ij}(z_0)$ is invertible.

When $J^{ij}(z_0)$ is singular, a product of the form $J^{km}(z_0)S_{ml}$ cannot completely cancel a term of the form δ^k_l. Still, we can cancel most of the Kronecker delta by choosing S_{ml} to be ($-1/3$ of) the Moore-Penrose psuedoinverse (defined in (3.16)) of $J^{ij}(z_0)$:

$$J^{ij}(z_0)S_{jk} = -\frac{1}{3}\delta^i_k + \frac{1}{3}\chi^i_{(\alpha)}\chi^{(\alpha)}_k\,, \tag{A.9}$$

where $\chi_{(\alpha)}$ and $\chi^{(\alpha)}$ are the null covectors and their duals defined in (2.9) and (2.10). With this choice of S_{ml}, equation (A.6) takes the form

$$\begin{aligned}&\frac{1}{3}\left[\frac{\partial J^{ij}}{\partial z^k}(z_0)\chi^{(\alpha)}_l\chi^k_{(\alpha)} + \frac{\partial J^{ij}}{\partial z^k}(z_0)\chi^{(\alpha)}_l\chi^k_{(\alpha)} + \frac{\partial J^{ij}}{\partial z^k}(z_0)\chi^{(\alpha)}_l\chi^k_{(\alpha)}\right]\\ &+J^{ik}(z_0)\hat{D}^j_{kl} + J^{kj}(z_0)\hat{D}^i_{kl} = 0\,.\end{aligned} \tag{A.10}$$

Observing that taking the derivative of definition (2.9) implies

$$\frac{\partial J^{im}}{\partial z^l}(z_0)\chi^{(\alpha)}_m = -J^{im}(z_0)\frac{\partial \chi^{(\alpha)}_m}{\partial z^l}(z_0)\,, \tag{A.11}$$

we see that we can cancel the last two terms in the square brackets in (A.10) by choosing

$$\hat{D}^i_{kl} = \frac{1}{3}\frac{\partial \chi^{(\alpha)}_l}{\partial z^k}(z_0)\chi^i_{(\alpha)} + A^i_{(\alpha)k}\chi^{(\alpha)}_l + A^i_{(\alpha)l}\chi^{(\alpha)}_k\,, \tag{A.12}$$

where now $A^i_{(\alpha)k}$ will have to be chosen in such a way to cancel the first term in the square brackets. But before we determine how this can be done, we note that (A.12) is symmetric in kl, as required. Indeed, the symmetry of the first term in kl follows from equation (2.11), from which we can infer

$$\frac{\partial \chi^{(\alpha)}_l}{\partial z^k}(z_0) = \frac{\partial^2 C^{(\alpha)}}{\partial z^k z^l}(z_0) = \frac{\partial \chi^{(\alpha)}_k}{\partial z^l}(z_0)\,. \tag{A.13}$$

Now, to find $A^i_{(\alpha)k}$, we substitute (A.12) into (A.10) and find that only three terms remain in the equation:

$$\chi_l^{(\alpha)} \left(J^{ik}(z_0) A^j_{(\alpha)k} + J^{kj}(z_0) A^i_{(\alpha)k} + \frac{1}{3} \frac{\partial J^{ij}}{\partial z^k}(z_0) \chi^k_{(\alpha)} \right) = 0 \, . \tag{A.14}$$

Again making use of (A.9) and (A.11), we find that we can make the factor in parenthesis in (A.14) vanish by defining $A^i_{(\alpha)k}$ by

$$A^i_{(\alpha)k} := \frac{1}{2} \frac{\partial J^{in}}{\partial z^m}(z_0) S_{nk} \chi^m_{(\alpha)} + \frac{1}{6} \chi^i_{(\beta)} \frac{\partial \chi^{(\beta)}_k}{\partial z^k}(z_0) \chi^m_{(\alpha)} \, . \tag{A.15}$$

And so, we arrive at an expression for D^i_{kl} applicable to singular Poisson brackets:

$$\begin{aligned} D^i_{jk} \quad = \quad & \frac{\partial J^{il}}{\partial \xi^m}(z_0) S_{lj} \left(\delta^m_k + \frac{1}{2} \chi^m_{(\alpha)} \chi^{(\alpha)}_k \right) + \frac{1}{6} \chi^i_{(\beta)} \frac{\partial \chi^{(\beta)}_j}{\partial \xi^m}(z_0) \left(\delta^m_k + \frac{1}{2} \chi^m_{(\alpha)} \chi^{(\alpha)}_k \right) \\ & \frac{\partial J^{il}}{\partial \xi^m}(z_0) S_{lk} \left(\delta^m_j + \frac{1}{2} \chi^m_{(\alpha)} \chi^{(\alpha)}_j \right) + \frac{1}{6} \chi^i_{(\beta)} \frac{\partial \chi^{(\beta)}_k}{\partial \xi^m}(z_0) \left(\delta^m_j + \frac{1}{2} \chi^m_{(\alpha)} \chi^{(\alpha)}_j \right) \, . \end{aligned} \tag{A.16}$$

We finally arrive at equation (3.17) by setting $S_{lj} = -1/3 T_{lj}$.

Appendix B

Changing Order of Integration

When we transform the Fourier components η_k of the distribution function to the new fields E_k, the cubic piece of the Hamiltonian becomes a sum of iterated principal value integrals (4.37)-(4.44). For analyzing the cubic term, we find it most convenient to collect the principal value singularities in the innermost integral of each term in the sum. The result of this process was given in chapter 4, and is reproduced here for the reader's convenience.

$$
\begin{aligned}
\mathcal{F}^{(3)} \quad = \quad & \sum_{k_a+k_b+k_c=0} \left(\frac{m}{48}\right)\left(\frac{i}{4\pi e}\right)^3 k_a k_b k_c \Bigg\{ \\
& \frac{1}{\pi^3}\int \mathrm{d}u_a\,\mathrm{d}u_b\,\mathrm{d}u_c\, E_{k_a}(u_a)E_{k_b}(u_b)E_{k_c}(u_c)\times \\
& \quad \left(\mathcal{P}\int \mathrm{d}v \frac{H_4(k_a,k_b,k_c,v)}{(u_a-v)(u_b-v)(u_c-v)}\right) && \text{(B.1)} \\
+ \quad & \frac{1}{\pi^2}\int \mathrm{d}u_a\,\mathrm{d}u_b\, E_{k_a}(u_a)E_{k_b}(u_b)\left(\mathcal{P}\int \mathrm{d}v \frac{H_3(k_a,k_b,k_c,v)}{(u_a-v)(u_b-v)}E_{k_c}(v)\right) && \text{(B.2)} \\
+ \quad & \frac{1}{\pi^2}\int \mathrm{d}u_c\,\mathrm{d}u_a\, E_{k_c}(u_c)E_{k_a}(u_a)\left(\mathcal{P}\int \mathrm{d}v \frac{H_3(k_c,k_a,k_b,v)}{(u_c-v)(u_a-v)}E_{k_b}(v)\right) && \text{(B.3)} \\
+ \quad & \frac{1}{\pi^2}\int \mathrm{d}u_b\,\mathrm{d}u_c\, E_{k_b}(u_b)E_{k_c}(u_c)\left(\mathcal{P}\int \mathrm{d}v \frac{H_3(k_b,k_c,k_a,v)}{(u_b-v)(u_c-v)}E_{k_a}(v)\right) && \text{(B.4)}
\end{aligned}
$$

$$+ \quad \frac{1}{\pi}\int \mathrm{d}u_a\, E_{k_a}(u_a)\left(\mathcal{P}\int \mathrm{d}v \frac{H_2(k_a,k_b,k_c,v)}{(u_a-v)} E_{k_b}(v)E_{k_c}(v)\right) \tag{B.5}$$

$$+ \quad \frac{1}{\pi}\int \mathrm{d}u_b\, E_{k_b}(u_b)\left(\mathcal{P}\int \mathrm{d}v \frac{H_2(k_b,k_c,k_a,v)}{(u_b-v)} E_{k_a}(v)E_{k_c}(v)\right) \tag{B.6}$$

$$+ \quad \frac{1}{\pi}\int \mathrm{d}u_c\, E_{k_c}(u_c)\left(\mathcal{P}\int \mathrm{d}v \frac{H_2(k_c,k_a,k_b,v)}{(u_c-v)} E_{k_a}(v)E_{k_b}(v)\right) \tag{B.7}$$

$$+ \quad \int \mathrm{d}v\, H_1(k_a,k_b,k_c;v)E_{k_a}(v)E_{k_b}(v)E_{k_c}(v) \Bigg\} \,. \tag{B.8}$$

Changing the order of integration of iterated principal value integrals requires using the Poincare-Bertrand lemma [58]:

$$\begin{aligned} \mathcal{P}\int \frac{\mathrm{d}v}{v-w}\left(\mathcal{P}\int \frac{f(u,v)\mathrm{d}u}{u-v}\right) \quad &= \\ \int \mathrm{d}u \left(\mathcal{P}\int \frac{f(u,v)\mathrm{d}v}{(v-w)(u-v)}\right) \quad &- \quad \pi^2 f(w,w)\,. \end{aligned} \tag{B.9}$$

We begin with the simplest terms. Since there is at most one singularity in terms (4.41)-(4.44), the v integral can be brought to the innermost spot immediately. These terms contribute to terms (B.5)-(B.8) in the reshuffled Hamiltonian.

The next most complicated terms in the Hamiltonian are the terms with two singularities, (4.38)-(4.40). We can easily apply equation (B.9) to them. For example, consider term (4.38). The outer v-integral slips right past the u_a integral giving

$$\begin{aligned} - \quad \int_{-\infty}^{\infty} \mathrm{d}u_a\, E_{k_a}(u_a,t) \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}v}{v-u_a} \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}u_b}{u_b-v} \\ \times \frac{\varepsilon_I(k_a,k_a v)\varepsilon_I(k_b,k_b v)\varepsilon_R(k_c,k_c v)}{(f_0'(v))^2} E_{k_b}(u_b,t)E_{k_c}(v,t)\,. \end{aligned} \tag{B.10}$$

Then, switching the order of integration of the v and u_b integrals by means of (B.9) gives two terms,

$$- \quad \int_{-\infty}^{\infty} \mathrm{d}u_a\, E_{k_a}(u_a,t) \int_{-\infty}^{\infty} \mathrm{d}u_b \tag{B.11}$$

$$\frac{\mathcal{P}}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{(v-u_a)(u_b-v)}\frac{\varepsilon_I(k_a,k_av)\varepsilon_I(k_b,k_bv)\varepsilon_R(k_c,k_cv)}{(f_0'(v))^2}E_{k_b}(u_b,t)E_{k_c}(v,t)$$

$$+\quad \pi^2\int_{-\infty}^{\infty}\mathrm{d}u_a\,E_{k_a}(u_a,t)\frac{\varepsilon_I(k_a,k_au_a)\varepsilon_I(k_b,k_bu_a)\varepsilon_R(k_c,k_cu_a)}{\pi^2(f_0'(u_a))^2}E_{k_b}(u_a,t)E_{k_c}(u_a,t)\,.$$

We can clean up expression (B.11) by dropping limits, abbreviating multiple (non-singular) integrals, rearranging factors, and renaming u_a to v in the second term:

$$\frac{1}{\pi^2}\int\mathrm{d}u_a\,\mathrm{d}u_b\,E_{k_a}(u_a,t)E_{k_b}(u_b,t) \tag{B.12}$$

$$\mathcal{P}\int\mathrm{d}v\frac{\varepsilon_I(k_a,k_av)\varepsilon_I(k_b,k_bv)\varepsilon_R(k_c,k_cv)}{(f_0'(v))^2(u_a-v)(u_b-v)}E_{k_c}(v,t)$$

$$+\quad\int\mathrm{d}v\frac{\varepsilon_I(k_a,k_av)\varepsilon_I(k_b,k_bv)\varepsilon_R(k_c,k_cv)}{(f_0'(v))^2}E_{k_a}(v,t)E_{k_b}(v,t)E_{k_c}(v,t)\,. \tag{B.13}$$

Term (B.12) is none other than term (B.2) in the Hamiltonian. The coefficient defined in (4.54), $H_3(k_a,k_b,k_c,v)$, is simply the coefficient of $E_{k_c}(v,t)$ in (B.12). The second term, (B.13), is one contribution to term (B.8) of the Hamiltonian; the coefficient of the E_k in the present integral is but one term in the function $H_4(k_a,k_b,k_c,v)$ defined in (4.53).

In an exactly similar way, exchanging the order of integration in terms (4.39) and (4.40) respectively gives rise completely to terms (B.3) and (B.4), and each contributes to term (B.8).

Finally, we come to term (4.37). We slip the v integral right past the u_a integral, and we can use equation (B.9) to exchange it with the u_b integral giving us

$$\pi^{-3}\int\mathrm{d}u_a\frac{E_{k_a}(u_a)E_{k_b}(u_a)}{(f_0'(u_a))^2}\varepsilon_I(k_a,k_au_a)\varepsilon_I(k_b,k_bu_a)\varepsilon_I(k_c,k_cu_a) \tag{B.14}$$

$$\times\,\mathcal{P}\int\mathrm{d}u_c\frac{E_{k_c}(u_c)}{u_c-u_a}$$

$$+\quad\int\mathrm{d}u_a\,\mathrm{d}u_b\,E_{k_a}(u_a)E_{k_b}(u_b)$$

$$\times\mathcal{P}\int\frac{\mathrm{d}v}{(u_a-v)(u_b-v)}\frac{\varepsilon_I(k_a,k_av)\varepsilon_I(k_b,k_bv)\varepsilon_I(k_c,k_cv)}{(f_0'(v))^2}\mathcal{P}\int\mathrm{d}u_c\frac{E_{k_c}(u_c)}{u_c-v}\,.$$

We consider the two terms of (B.14) separately.

In the first term, we can immediately switch the order of integration, and rename u_a to v. This yields

$$\pi^{-1} \int \mathrm{d}u_c \, E_{k_c}(u_c) \tag{B.15}$$
$$\mathcal{P} \int \frac{\mathrm{d}v}{u_c - v} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2} E_{k_a}(v) E_{k_b}(v) \, .$$

But now we see that (B.15) is of the same form as term (B.7), and the coefficient of the E's in the innermost integrand is simply one term of $H_2(k_a, k_b, k_c; v)$ (see definition (4.55)).

To make further progress in exchanging integrals in the second term of (B.14), we must expand the double pole in partial fractions:

$$\frac{1}{(u_a - v)(u_b - v)} = \frac{1}{u_a - u_b}\frac{1}{v - u_a} + \frac{1}{u_b - u_a}\frac{1}{v - u_b} \, . \tag{B.16}$$

This expansion lets us split up the innermost two integrals of (B.14) into a sum of two ordinary iterated principal value integrals:

$$\frac{1}{u_a - u_b} \mathcal{P} \int \frac{\mathrm{d}v}{(v - u_a)} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2} \tag{B.17}$$
$$\mathcal{P} \int \mathrm{d}u_c \frac{E_{k_c}(u_c)}{u_c - v}$$
$$+ \quad \frac{1}{u_b - u_a} \mathcal{P} \int \frac{\mathrm{d}v}{(v - u_b)} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2}$$
$$\mathcal{P} \int \mathrm{d}u_c \frac{E_{k_c}(u_c)}{u_c - v} \tag{B.18}$$

Now, we can apply (B.9) once again to exchange the integrals in both terms of (B.17). This yields four terms:

$$-\pi^2 \frac{1}{u_a - u_b} \frac{\varepsilon_I(k_a, k_a u_a)\varepsilon_I(k_b, k_b u_a)\varepsilon_I(k_c, k_c u_a)}{(f_0'(u_a))^2} E_{k_c}(u_a) \tag{B.19}$$

$$
\begin{aligned}
+ \quad & -\pi^2 \frac{1}{u_b - u_a} \frac{\varepsilon_I(k_a, k_a u_b)\varepsilon_I(k_b, k_b u_b)\varepsilon_I(k_c, k_c u_b)}{(f_0'(u_b))^2} E_{k_c}(u_b) \\
+ \quad & \frac{1}{u_a - u_b} \int \mathrm{d}u_c \, E_{k_c}(u_c) \\
& \mathcal{P} \int \frac{\mathrm{d}v}{(v - u_a)(u_c - v)} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2} \\
+ \quad & \frac{1}{u_b - u_a} \int \mathrm{d}u_c \, E_{k_c}(u_c) \\
& \mathcal{P} \int \frac{\mathrm{d}v}{(v - u_b)(u_c - v)} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2} \, .
\end{aligned}
$$

Again, we consider the terms of (B.19) individually. When we insert the first term back into the outer integrals in (B.14), and rename u_a to v, we obtain:

$$
\begin{aligned}
& \pi^{-1} \int \mathrm{d}u_b \, E_{k_c}(u_c) \\
& \mathcal{P} \int \frac{\mathrm{d}v}{u_b - v} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2} E_{k_a}(v) E_{k_c}(v) \, .
\end{aligned} \tag{B.20}
$$

In the same way that (B.15) contributes to term (B.7), (B.20) contributes to term (B.6). Likewise, the second term of (B.19) contributes to (B.5) upon renaming u_b to v.

Finally, we come to the last two terms of (B.19). These can be easily added together (which merely undoes the partial fraction expansion), and reinserted in the outer integrals to yield:

$$
\begin{aligned}
& \frac{1}{\pi^3} \int \mathrm{d}u_a \, \mathrm{d}u_b \, \mathrm{d}u_c \, E_{k_a}(u_a) E_{k_b}(u_b) E_{k_c}(u_c) \\
& \mathcal{P} \int \frac{\mathrm{d}v}{(u_a - v)(u_b - v)(u_c - v)} \frac{\varepsilon_I(k_a, k_a v)\varepsilon_I(k_b, k_b v)\varepsilon_I(k_c, k_c v)}{(f_0'(v))^2} \, .
\end{aligned} \tag{B.21}
$$

The nonsingular part of the innermost integrand is exactly $H_4(k_a, k_b, k_c, v)$ (see (4.53)). Therefore, term (B.21) gives rise completely to term (B.1). And in fact, this completes the derivation of the form of the cubic part of the Hamiltonian given by expression (4.45)-(4.52).

Appendix C

Isolating the resonant part of the two pole terms

We consider term (4.57), henceforth denoted I_2, surpressing dependence of k_a, k_b, and k_c in M. It will turn out that (4.57) is $\mathcal{O}(\delta^2)$, but we must follow a mildly torturous route to see this.

Identifying and Isolating the Resonant Layer

To begin, the demoninator can be split up by partial fractions:

$$\frac{1}{(u_a - v)(u_b - v)} = \frac{1}{(u_a - u_b)} \left(\frac{1}{v - u_a} - \frac{1}{v - u_b} \right) , \tag{C.1}$$

splitting the inner integral into a sum of two:

$$\begin{aligned} I_2 &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \mathrm{d}u_a \int_{-\infty}^{\infty} \mathrm{d}u_b \, E_{k_a}(u_a) E_{k_b}(u_b) \frac{1}{u_a - u_b} \\ & \left(\mathcal{P} \int_{-\infty}^{+\infty} \mathrm{d}v \frac{M(v) E_{k_c}(v)}{(v - u_a)} - \mathcal{P} \int_{-\infty}^{+\infty} \mathrm{d}v \frac{M(v) E_{k_c}(v)}{(v - u_b)} \right) \end{aligned} \tag{C.2}$$

Now, the resonance condition for (4.57) has already been given in (5.37). It is convenient to define some new variables here. We define the resonant quanitity

$x = -(1/k_c)(k_a u_a + k_b u_b)$, the deviation from the resonance (of the innermost integration variable) $w = v - v_r$, and the two variables $y = -(k_b/k_c)(u_a - u_b)$, and $z = (k_a/k_c)$. After restricting the domain of integration to the resonant layer, (C.2) becomes

$$\begin{aligned} I_2 \approx & \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}u_b\, E_{k_b}(u_b) \int_{-\infty}^{\infty} \mathrm{d}y\, E_{k_a}(u_b - (k_c/k_b)y) \\ & \mathcal{P}\int_{-\delta}^{\delta} \mathrm{d}w \frac{1}{y} \frac{M(w+x)E_{k_c}(w+x)}{w-y} \qquad \text{(C.3)} \\ - & \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}u_b\, E_{k_b}(u_b) \int_{-\infty}^{\infty} \mathrm{d}z\, E_{k_a}(u_b + (k_c/k_a)z) \\ & \mathcal{P}\int_{-\delta}^{\delta} \mathrm{d}w \frac{1}{z} \frac{M(w+x)E_{k_c}(w+x)}{w-z}\,. \qquad \text{(C.4)} \end{aligned}$$

The inner integrals in both terms have exactly the same form, so we will only evaluate the innermost integral in (C.3), and substitute z for y in this result to get the result for (C.4).

Doing the innermost integrals

We denote the innermost integral in (C.3) by I_{2w}. Since δ is small, w is also small, and we can profitably expand in powers of w the numerator in I_{2w}. To make the power series more compact, we will introduce the following notation for $n = 0, \ldots, \infty$:

$$a_n(x) := \frac{1}{n!}\frac{\partial^n}{\partial w^n}(M(x+w)E_{k_c}(x+w))\Big|_{w=0} \qquad \text{(C.5)}$$

Then,

$$I_{2w} = \mathcal{P}\int_{-\delta}^{\delta} \mathrm{d}w \frac{1}{y}\left(\frac{a_0(x)}{w-y} + \sum_{n=1}^{\infty} a_n(x)\frac{w^n}{w-y}\right)\,. \qquad \text{(C.6)}$$

Rewriting $w^n = w^n - y^n + y^n$, and recalling that $w^n - y^n = (w-y)(w^{n-1} + w^{n-2}y + \cdots + w^1 y^{n-2} + y^{n-1})$, we can compute the integral in (C.6) term by term. The result has a different form depending on whether n is even or odd, so we break it up accordingly. By symmetry, only even powers of w contribute, leaving

$$\begin{aligned}
I_{2w} &= a_0(x)\frac{1}{y}\log\left|\frac{\delta-y}{\delta+y}\right| \\
&+ \sum_{n=1}^{\infty} a_{2n-1}(x)\left[\left(\frac{2\delta^{2n-1}}{2n-1}+\frac{2\delta^{2n-3}y^2}{2n-3}+\cdots+\frac{2\delta^3y^{2n-4}}{3}+2\delta y^{2n-2}\right)\right. \\
&\left. \quad + y^{2n-1}\log\left|\frac{\delta-y}{\delta+y}\right|\right] \\
&+ \sum_{n=1}^{\infty} a_{2n}(x)\frac{1}{y}\left[\left(\frac{2\delta^{2n-1}y}{2n-1}+\frac{2\delta^{2n-3}y^3}{2n-3}+\cdots+\frac{2\delta^3y^{2n-3}}{3}+2\delta y^{2n-1}\right)\right. \\
&\left. \quad + y^{2n}\log\left|\frac{\delta-y}{\delta+y}\right|\right]
\end{aligned} \tag{C.7}$$

Because of the absolute values inside the logs, we have to consider (C.7) in two regions, $|y| \le \delta$ and $|y| > \delta$.

Neglecting terms of $\mathcal{O}(\delta^2)$ in $|y| > \delta$.

For the "outside" region, $|y| > \delta$, we can rewrite the log as

$$\log\left|\frac{\delta-y}{\delta+y}\right| = \log\left(\frac{y-\delta}{y+\delta}\right) = \log\left(\frac{1-\delta/y}{1+\delta/y}\right) . \tag{C.8}$$

It is further convenient to expand (C.8) in a power series in (δ/y) :

$$\log\left(\frac{1-\delta/y}{1+\delta/y}\right) = -\left(\frac{2\delta}{y}+\frac{2\delta^3}{3y^3}+\cdots+\frac{2\delta^{2n-1}}{(2n-1)y^{2n-1}}\right) - \left(\frac{2\delta^{2n+1}}{(2n+1)y^{2n+1}}+\cdots\right) . \tag{C.9}$$

At this point, we might be tempted to simply keep only the first term of this series. We would be mistaken, though, because y can take values of $\mathcal{O}(\delta)$ in the region $|y| > \delta$. So, we have no choice but to leave intact the log in the coefficient of $a_0(x)$ in (C.7). On the other hand, the same log also appears in the infnite series terms of the same expression, except accompanied by a factor of y^{2n-1} in the odd series, and

y^{2n} in the even series. Multiplying these factors through the expanded log gives us

$$y^{2n-1}\log\left(\frac{1-\delta/y}{1+\delta/y}\right) = \tag{C.10}$$
$$-\left(2\delta y^{2n-2}+\frac{2\delta^3 y^{2n-4}}{3}+\cdots+\frac{2\delta^{2n-1}}{(2n-1)}\right)-\left(\frac{2\delta^{2n+1}}{(2n+1)y^2}+\cdots\right),$$

and

$$y^{2n}\log\left(\frac{1-\delta/y}{1+\delta/y}\right) = \tag{C.11}$$
$$-\left(2\delta y^{2n-1}+\frac{2\delta^3 y^{2n-3}}{3}+\cdots+\frac{2\delta^{2n-1}y}{(2n-1)}\right)-\left(\frac{2\delta^{2n+1}}{(2n+1)y}+\cdots\right).$$

Notice that the contents of the first set of parentheses in both (C.10) and (C.11) exactly cancels the y-series in the corresponding term of the appropriate infinite series in (C.7). Multiplying through by the remaing factor of $1/y$ gives us the forms of the nth term of the odd and even series respectively as

$$-a_{2n-1}(x)\left(\frac{2\delta^{2n+1}}{(2n+1)y^3}+\cdots\right), \tag{C.12}$$

and

$$-a_{2n}(x)\left(\frac{2\delta^{2n+1}}{(2n+1)y^2}+\cdots\right). \tag{C.13}$$

Both (C.12) and (C.13) contribute up to $\mathcal{O}(\delta)$ only when $n=1$. We can write the two terms that do contribute more compactly by defining a function

$$\Upsilon(\xi) := -2\left(\frac{\xi^3}{3}+\frac{\xi^5}{5}+\cdots+\frac{\xi^j}{j}+\cdots\right). \tag{C.14}$$

Then terms (C.12) and (C.13) (at $n=1$) become respectively $a_1(x)\Upsilon(\delta/y)$, and $a_2(x)y\Upsilon(\delta/y)$.

And so, in the region $|y|>\delta$, (C.7) becomes

$$I_{2w}^{o} = a_0(x)\frac{1}{y}\log\left(\frac{1-\delta/y}{1+\delta/y}\right)$$

$$
\begin{aligned}
&+ \quad a_1(x)\Upsilon(\delta/y) \\
&+ \quad a_2(x)y\Upsilon(\delta/y) + \mathcal{O}(\delta^2)\,. && \text{(C.15)}
\end{aligned}
$$

(The superscript o denotes "outside" region.)

Neglecting terms of $\mathcal{O}(\delta^2)$ in $|y| \le \delta$.

Now we turn to the "inside" region $|y| \le \delta$. Although the log terms in (C.7) are singular at $|y| = \delta$, they are integrable on the interval $(-\delta, \delta)$. So, we can neglect the points $|y| = \delta$ in the following, and consider only $|y| < \delta$. As before, we want to remove the absolute value signs, so we write the log as

$$
\log\left|\frac{\delta - y}{\delta + y}\right| = \log\left(\frac{\delta - y}{\delta + y}\right) = \log\left(\frac{1 - y/\delta}{1 + y/\delta}\right)\,. \qquad \text{(C.16)}
$$

Here, since δ appears as a denominator, it would be fruitless to expand the log in a δ power series. However, the smallness of y lets us truncate with impunity the sums that appear in the infinite sums in (C.7). Doing this yields the form of I_{2w} appropriate to the "inside" region as

$$
\begin{aligned}
I^i_{2w} &= a_0(x)\frac{1}{y}\log\left(\frac{1 - y/\delta}{1 + y/\delta}\right) \\
&+ \sum_{n=1}^{\infty} a_n(x)\left[2\delta y^{n-2} + y^{n-1}\log\left(\frac{1 - y/\delta}{1 + y/\delta}\right)\right] \\
&+ \mathcal{O}(\delta^2)\,. && \text{(C.17)}
\end{aligned}
$$

(The superscript i denotes "inside.") Since we need only keep one term along with the log for any given n, there is no reason to distinguish between odd and even series.

Substituting back into (C.3) and (C.4) - Changing integration variables

We can now substitute the results in (C.15) and (C.17) for the innermost integrals in (C.3) and (C.4) (substituting z for y where necessary.) We will make things more compact by renaming z to y. But first, we must recognize that x has a different form when written in terms of y than it does when written in terms of z. These are:

$$x = u_b + (k_a/k_b)y\,; \tag{C.18}$$

$$x = u_b - z\,. \tag{C.19}$$

So, substituting (C.18) and (C.19) for x when plugging (C.15) and (C.17) into (C.3) and (C.4) respectively, we obtain the following integral (split into the "outside" and "inside" regions) after renaming z to y:

$$\begin{aligned}
I_2 \approx\ & \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}u_b\, E_{k_b}(u_b) \Bigg\{ \\
& \int_{-\infty}^{-\delta} \mathrm{d}y \Bigg[[E_{k_a}(u_b - (k_c/k_b)y) a_0(u_b + (k_a/k_b)y) \\
& -E_{k_a}(u_b + (k_c/k_a)y) a_0(u_b - y)] \frac{1}{y} \log\left(\frac{1-\delta/y}{1+\delta/y}\right) \\
& +[E_{k_a}(u_b - (k_c/k_b)y) a_1(u_b + (k_a/k_b)y) \\
& \qquad -E_{k_a}(u_b + (k_c/k_a)y) a_1(u_b - y)] \Upsilon(\delta/y) \\
& +[E_{k_a}(u_b - (k_c/k_b)y) a_2(u_b + (k_a/k_b)y) \\
& \qquad -E_{k_a}(u_b + (k_c/k_a)y) a_2(u_b - y)] y \Upsilon(\delta/y) \Bigg] \qquad\qquad (C.20) \\
+\ & \int_{\delta}^{\infty} \mathrm{d}y \Bigg[[E_{k_a}(u_b - (k_c/k_b)y) a_0(u_b + (k_a/k_b)y) \\
& -E_{k_a}(u_b + (k_c/k_a)y) a_0(u_b - y)] \frac{1}{y} \log\left(\frac{1-\delta/y}{1+\delta/y}\right) \\
& +[E_{k_a}(u_b - (k_c/k_b)y) a_1(u_b + (k_a/k_b)y) \\
& \qquad -E_{k_a}(u_b + (k_c/k_a)y) a_1(u_b - y)] \Upsilon(\delta/y) \\
& +[E_{k_a}(u_b - (k_c/k_b)y) a_2(u_b + (k_a/k_b)y)
\end{aligned}$$

$$
\begin{aligned}
& -E_{k_a}(u_b + (k_c/k_a)y)a_2(u_b - y)]y\Upsilon(\delta/y)\Big] && \text{(C.21)} \\
+ \quad & \mathcal{P}\int_{-\delta}^{\delta} \mathrm{d}y \Big[[E_{k_a}(u_b - (k_c/k_b)y)a_0(u_b + (k_a/k_b)y) \\
& -E_{k_a}(u_b + (k_c/k_a)y)a_0(u_b - y)]\frac{1}{y}\log\left(\frac{1 - y/\delta}{1 + y/\delta}\right) \\
& +E_{k_a}(u_b - (k_c/k_b)y)\sum_{n=1}^{\infty} a_n(u_b + (k_a/k_b)y) \\
& \qquad \times \left(2\delta y^{n-2} + y^{n-1}\log\left(\frac{1 - y/\delta}{1 + y/\delta}\right)\right) \\
& -E_{k_a}(u_b + (k_c/k_a)y)a_1(u_b - y)\sum_{n=1}^{\infty} a_n(u_b - y) \\
& \qquad \times \left(2\delta y^{n-2} + y^{n-1}\log\left(\frac{1 - y/\delta}{1 + y/\delta}\right)\right)\Big]\Big\} \\
+ \quad & \mathcal{O}(\delta^2)\,. && \text{(C.22)}
\end{aligned}
$$

Now, to get the δ dependence out of the limits and into the integrand, we make the variable changes $y = \delta/\xi$ in (C.20) and (C.21), and $y = \delta\xi$ in (C.22). With these changes of variable, the intervals of integration in the ξ integral are now $(-1, 0), (0, 1)$, and $(-1, 1)$ in (C.20), (C.21), and (C.22) respectively. Furthermore, these transformations remove the δ's from the logs, making it still easier to see when terms are small. In fact, we become immediately able to neglect a few more terms.

First, consider the factor $y\Upsilon(\delta/y)$, which appears in (C.20) and (C.21). Since $\mathrm{d}y = -(\delta/\xi^2)\mathrm{d}\xi$, this factor becomes $\delta^2\frac{1}{\xi^3}\Upsilon(\xi)$. We can thus neglect this term as long as $\frac{1}{\xi^3}\Upsilon(\xi)$ is integrable in the intervals $(-1, 0)$ and $(0, 1)$. But, comparing (C.14) to (C.9) (with $(\delta/y) = \xi$), we see that

$$
\left|\frac{\Upsilon(\xi)}{\xi^3}\right| < \left|\frac{1}{\xi}\log\left(\frac{1 - \xi}{1 + \xi}\right)\right|\,. \tag{C.23}
$$

Since the LHS in the inequality is absolutely integrable on the intervals in question, so is the RHS, and we can neglect this term.

Now we turn our attention to (C.22) as it appears after we set $y = \delta\xi$. Since $\mathrm{d}y = \delta \mathrm{d}\xi$, the only term in the infinite sum that contributes at $\mathcal{O}(\delta)$ is the $n = 1$ term. Neglecting the rest brings us to the simplified integral:

$$
\begin{aligned}
I_2 \quad \approx \quad & \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}u_b\, E_{k_b}(u_b) \Bigg\{ \\
& \int_{-1}^{0} \mathrm{d}\xi \Bigg[[E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))a_0(u_b + (k_a/k_b)(\delta/\xi)) \\
& -E_{k_a}(u_b + (k_c/k_a)(\delta/\xi))a_0(u_b - (\delta/\xi)]\frac{1}{\xi}\log\left(\frac{1-\xi}{1+\xi}\right) \\
& +[E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))a_1(u_b + (k_a/k_b)(\delta/\xi)) \\
& -\ E_{k_a}(u_b + (k_c/k_a)(\delta/\xi)a_1(u_b - (\delta/\xi))]\Upsilon(\xi)\ \Bigg] \qquad \text{(C.24)} \\
+ \quad & \int_{0}^{1} \mathrm{d}\xi \Bigg[[E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))a_0(u_b + (k_a/k_b)(\delta/\xi)) \\
& -E_{k_a}(u_b + (k_c/k_a)(\delta/\xi))a_0(u_b - (\delta/\xi)]\frac{1}{\xi}\log\left(\frac{1-\xi}{1+\xi}\right) \\
& +[E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))a_1(u_b + (k_a/k_b)(\delta/\xi)) \\
& -\ E_{k_a}(u_b + (k_c/k_a)(\delta/\xi)a_1(u_b - (\delta/\xi))]\Upsilon(\xi)\ \Bigg] \qquad \text{(C.25)} \\
+ \quad & \mathcal{P}\int_{-1}^{1} \mathrm{d}\xi \Bigg[[E_{k_a}(u_b - (k_c/k_b)\delta\xi)a_0(u_b + (k_a/k_b)\delta\xi) \\
& -E_{k_a}(u_b + (k_c/k_a)\delta\xi)a_0(u_b - \delta\xi)]\frac{1}{\xi}\log\left(\frac{1-\xi}{1+\xi}\right) \\
& +\delta[E_{k_a}(u_b - (k_c/k_b)\delta\xi)a_1(u_b + (k_a/k_b)\delta\xi) \\
& -E_{k_a}(u_b + (k_c/k_a)\delta\xi)a_1(u_b - \delta\xi)]\left(\frac{2}{\xi} + \log\left(\frac{1-\xi}{1+\xi}\right)\right)\ \Bigg]\ \Bigg\} \qquad \text{(C.26)} \\
+ \quad & \mathcal{O}(\delta^2)\,.
\end{aligned}
$$

In the following, we will break the above into $I_2 \approx I_{2\xi}^{-} + I_{2\xi}^{+} + I_{2\xi}^{i}$, where each term in the sum corresponds to the numbered terms above, repectively.

Truncating terms (C.24) and (C.25) at $\mathcal{O}(\delta)$

We have finally come to the point where we can pull the δ dependence completely out of the integrands. We now consider terms (C.24) and (C.25).

The first thing to notice is that both the log and the function Υ are singular at $\xi = \pm 1$ in (C.24) and (C.25) respectively. (Also, we should here note that the integrability of $\frac{1}{\xi^2}\Upsilon(\xi)$ follows from (C.23), as can be seen by multiplying through by $|\xi|$.) Thus, the ξ integrals will be dominated by the values of the unknowns near $\xi = \pm 1$. So, we expand the factors of $(1/\xi)\log(\frac{1-\xi}{1+\xi})$ and $\Upsilon(\xi)$ around $\xi = \pm 1$, keeping terms to $\mathcal{O}(\delta)$. This expansion is more complicated than usual: since ξ appears in the denominator of the arguments, every order of ξ derivative contributes to $\mathcal{O}(\delta)$. To see this, we start with the first derivative:

$$\begin{aligned}
&\frac{\partial}{\partial \xi}[E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))a_i(u_b + (k_a/k_b)(\delta/\xi)) \\
&\qquad -E_{k_a}(u_b + (k_c/k_a)(\delta/\xi))a_i(u_b - (\delta/\xi)] = \\
&\delta\frac{1}{\xi^2}\left[\frac{k_c}{k_b}\frac{\partial E_{k_a}}{\partial u_b}(u_b - (k_c/k_b)(\delta/\xi))a_i(u_b + (k_a/k_b)(\delta/\xi))\right. \\
&-\frac{k_a}{k_b}E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))\frac{\partial a_i}{\partial u_b}(u_b + (k_a/k_b)(\delta/\xi)) \\
&+\frac{k_c}{k_a}\frac{\partial E_{k_a}}{\partial u_b}(u_b + (k_c/k_a)(\delta/\xi))a_i(u_b - (\delta/\xi)) \\
&\qquad \left. - E_{k_a}(u_b + (k_c/k_a)(\delta/\xi))\frac{\partial a_i}{\partial u_b}(u_b - (\delta/\xi))\right] \\
&+\mathcal{O}(\delta^2)\,,
\end{aligned} \tag{C.27}$$

where $i = 0, 1$.

A fact that is apparent from (C.27) is that the higher order ξ derivatives are also $\mathcal{O}(\delta)$. Fortunately, though, only the factor $1/\xi^2$ contributes at $\mathcal{O}(\delta)$ to these higher derivatives. Thus we can easily calculate the nth ξ derivative to be

$$\begin{aligned}
&\frac{\partial^n}{\partial \xi^n}[E_{k_a}(u_b - (k_c/k_b)(\delta/\xi))a_i(u_b + (k_a/k_b)(\delta/\xi)) \\
&\qquad -E_{k_a}(u_b + (k_c/k_a)(\delta/\xi))a_i(u_b - (\delta/\xi)] =
\end{aligned}$$

$$
\begin{aligned}
&\delta\frac{(-1)^{n+1}n!}{\xi^{n+1}}\left[\frac{k_c}{k_b}\frac{\partial E_{k_a}}{\partial u_b}(u_b-(k_c/k_b)(\delta/\xi))a_i(u_b+(k_a/k_b)(\delta/\xi))\right.\\
&-\frac{k_a}{k_b}E_{k_a}(u_b-(k_c/k_b)(\delta/\xi))\frac{\partial a_i}{\partial u_b}(u_b+(k_a/k_b)(\delta/\xi))\\
&+\frac{k_c}{k_a}\frac{\partial E_{k_a}}{\partial u_b}(u_b+(k_c/k_a)(\delta/\xi))a_i(u_b-(\delta/\xi))\\
&\qquad\left.-\,E_{k_a}(u_b+(k_c/k_a)(\delta/\xi))\frac{\partial a_i}{\partial u_b}(u_b-(\delta/\xi))\right]\\
&+\mathcal{O}(\delta^2)\,,
\end{aligned}
\tag{C.28}
$$

where again, $i=0,1$.

From (C.28), we can write (C.24), denoted $I^-_{2\xi}$, and (C.25), denoted $I^+_{2\xi}$, in Taylor series about $\xi=-1,+1$ respectively. We first need to evaluate (C.28) at $\xi=\pm1$:

$$
\begin{aligned}
&\frac{\partial^n}{\partial\xi^n}[E_{k_a}(u_b-(k_c/k_b)(\delta/\xi))a_i(u_b+(k_a/k_b)(\delta/\xi))\\
&\qquad-E_{k_a}(u_b+(k_c/k_a)(\delta/\xi))a_i(u_b-(\delta/\xi))]\Big|_{\xi=\pm1}\quad=\\
&\delta\frac{(-1)^{n+1}n!}{(\pm1)^{n+1}}\left[\frac{k_c}{k_b}\frac{\partial E_{k_a}}{\partial u_b}(u_b\mp(k_c/k_b)\delta)a_i(u_b\pm(k_a/k_b)\delta)\right.\\
&-\frac{k_a}{k_b}E_{k_a}(u_b\mp(k_c/k_b)\delta)\frac{\partial a_i}{\partial u_b}(u_b\pm(k_a/k_b)\delta)\\
&\left.+\frac{k_c}{k_a}\frac{\partial E_{k_a}}{\partial u_b}(u_b\pm(k_c/k_a)\delta)a_i(u_b\mp\delta)-\,E_{k_a}(u_b\pm(k_c/k_a)\delta)\frac{\partial a_i}{\partial u_b}(u_b\mp\delta)\right]\\
&+\mathcal{O}(\delta^2)\,.
\end{aligned}
\tag{C.29}
$$

Notice, however, that the arguments of the unknowns still contain δ, and so can be further expanded in δ. But since the derivatives are all of order δ, we need only keep the leading term of this further expansion. And so,

$$
\begin{aligned}
&\frac{\partial^n}{\partial\xi^n}[E_{k_a}(u_b-(k_c/k_b)(\delta/\xi))a_i(u_b+(k_a/k_b)(\delta/\xi))\\
&\qquad-E_{k_a}(u_b+(k_c/k_a)(\delta/\xi))a_i(u_b-(\delta/\xi))]\Big|_{\xi=\pm1}\quad=
\end{aligned}
$$

$$\delta\frac{(-1)^{n+1}n!}{(\pm 1)^{n+1}}\left[\left(\frac{k_c}{k_b}+\frac{k_c}{k_a}\right)\frac{\partial E_{k_a}}{\partial u_b}(u_b)a_i(u_b)-\left(\frac{k_a}{k_b}+1\right)E_{k_a}(u_b)\frac{\partial a_i}{\partial u_b}(u_b)\right]$$
$$+\mathcal{O}(\delta^2)\,. \tag{C.30}$$

To complete our expansion, we still need the zeroth-order term in the ξ series evaluated at $\xi=\pm 1$. This is simply

$$E_{k_a}(u_b\mp(k_c/k_b)\delta)a_i(u_b\pm(k_a/k_b)\delta)-E_{k_a}(u_b\pm(k_c/k_a)\delta)a_i(u_b\mp\delta)=$$
$$\mp\delta\left[\left(\frac{k_c}{k_b}+\frac{k_c}{k_a}\right)\frac{\partial E_{k_a}}{\partial u_b}(u_b)a_i(u_b)-\left(\frac{k_a}{k_b}+1\right)E_{k_a}(u_b)\frac{\partial a_i}{\partial u_b}(u_b)\right].$$
$$+\mathcal{O}(\delta^2) \tag{C.31}$$

One thing we immediately notice is that the ξ Taylor series are $\mathcal{O}(\delta)$. Since the coefficients of Υ in (C.24) and (C.25), are already $\mathcal{O}(\delta)$, these terms only contribute to $\mathcal{O}(\delta^2)$. Hence, the sum of (C.24) and (C.25) is just

$$\begin{aligned}
I_{2\xi}^{-}+I_{2\xi}^{+} &= \frac{\delta}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}u_b\,E_{k_b}(u_b)\\
&\times\left[\left(\frac{k_c}{k_b}+\frac{k_c}{k_a}\right)\frac{\partial E_{k_a}}{\partial u_b}(u_b)a_0(u_b)-\left(\frac{k_a}{k_b}+1\right)E_{k_a}(u_b)\frac{\partial a_0}{\partial u_b}(u_b)\right]\\
&\times\sum_{n=0}^{\infty}\left\{\int_{-1}^{0}\mathrm{d}\xi\frac{(\xi+1)^n}{\xi}\log\left(\frac{1-\xi}{1+\xi}\right)\right. &\text{(C.32)}\\
&\qquad\left.+\int_{0}^{1}\mathrm{d}\xi\,(-1)^{n+1}\frac{(\xi-1)^n}{\xi}\log\left(\frac{1-\xi}{1+\xi}\right)\right\} &\text{(C.33)}\\
&+\ \mathcal{O}(\delta^2)\,.
\end{aligned}$$

But making the change of variable $\xi\to-\xi$ in either (C.32) or (C.33) shows that the integrals cancel. Therefore, $I_{2\xi}^{-}+I_{2\xi}^{+}=\mathcal{O}(\delta^2)$.

Truncating term (C.26) at $\mathcal{O}(\delta)$

Since the arguments of the unknowns in (C.26), denoted $I_{2\xi}^{i}$, do not contain ξ in denominators, we may expand the unknowns in a Taylor series about $\xi=0$.

Clearly, the zeroth-order terms in this expansion cancel, leaving

$$
\begin{aligned}
I^i_{2\xi} \quad = \quad & \frac{\delta}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}u_b \, E_{k_b}(u_b) \\
& \times(-1)\left[\left(\frac{k_c}{k_b}+\frac{k_c}{k_a}\right)\frac{\partial E_{k_a}}{\partial u_b}(u_b)a_0(u_b) - \left(\frac{k_a}{k_b}+1\right)E_{k_a}(u_b)\frac{\partial a_0}{\partial u_b}(u_b)\right] \\
& \times\int_{-1}^{1}\mathrm{d}\xi\,\log\left(\frac{1-\xi}{1+\xi}\right) \\
+ \quad & \mathcal{O}(\delta^2)\,.) \qquad \text{(C.34)}
\end{aligned}
$$

But the integrand of the ξ integral is odd, and therefore the integral vanishes. Thus, $I^i_{2\xi}$ is also $\mathcal{O}(\delta^2)$. And hence $I_2 \approx I^-_{2\xi} + I^+_{2\xi} + I^i_{2\xi} = \mathcal{O}(\delta^2)$.

In summary, we have shown that after averaging, term (4.57) contributes only at $\mathcal{O}(\delta^2)$.

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Vita

Thomas William Yudichak was born on December 6, 1971 in Niagara Falls, New York, to William Thomas Yudichak and Janet Elaine Yudichak. He completed his secondary education at Belmont Central School in Belmont, New York in June 1990, and entered Clarkson University in Potsdam, New York in September of the same year. There, he majored in physics and mathematics, receiving a Bachelor of Science with Great Distinction in May 1994. In August of 1994, he entered the graduate program in physics at the University of Texas at Austin where he was employed as a teaching assistant by the Department of Physics until August 1996. Since June of 1996, he has been employed as a graduate research assistant by the Institute for Fusion Studies. Thomas was married in July 1994 to Julie Lynn Normand. In December 1996, their son Justin David Normand Yudichak was born. The three currently live together near Round Rock, Texas.

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