On the Hamiltonian Structure of the Linearized Maxwell-Vlasov System

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by

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A detailed analysis of the noncanonical structure of the linearized Maxwell-Vlasov equations is presented. The full Maxwell-Vlasov bracket is linearized about a stable, homogeneous and isotropic equilibrium. Velocity space moments are taken leading to a natural decomposition of the system into longitudinal and transverse parts. This bracket together with the linearized energy is shown to give the usual linearized moment equations. A family of integral transforms whose kernels are closely related to singular eigenfunctions are introduced. It is shown that by means of these transformations, both the bracket and energy can be brought into diagonal form. The diagonalizing transformation is essentially a transformation to linear action-angle variables for this infinite-dimensional Hamiltonian system. The resulting energy expression, which depends on the Fourier transform of the electric field, has physical meaning as the energy of the perturbations and in

general is *not* equal to the usual expression for the wave energy in a dielectric. Equilibria that support discrete modes are also studied. It is shown that the eigenfunctions corresponding to the discrete modes enter as a natural result of regularizing the (now singular) inverse transform. It is seen that in the case of neutral modes, the transformed variables must be interpreted as generalized functions. Lastly, quadrature rules for Cauchy integrals are discussed and an efficient, high accuracy algorithm for computing Hilbert transforms is developed. This algorithm is used to evaluate the exact solution of the longitudinal equations for different initial conditions.

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Introduction

Classical mechanics is a fascinating subject with a long history. Canonical coordinates and the symplectic structure of phase space are central figures in this history. These concepts, which grew from the work of Hamilton, Jacobi and others, opened the door to a geometrical understanding of systems whose dynamics is described by Hamilton's equations. Classical mechanics, in particular the Hamiltonian formulation of classical mechanics, is of great importance to many areas of physics, indeed it can be justifiably claimed as the foundation of all modern physics.

When considering a canonical Hamiltonian system, an alternative formulation may be more convenient: certain symmetries may be averaged over; degrees of freedom may be eliminated to simplify the system; one may wish to include dissipative effects; and so on. This leaves one with three possibilities: the new system may be canonically Hamiltonian; it may still be Hamiltonian but not *manifestly* Hamiltonian or it may no longer be Hamiltonian at all. It is systems in this second category, which are called noncanonical Hamiltonian systems, that are of special interest to us. Noncanonical Hamiltonian dynamics is a generalization of the ideas of canonical mechanics and is closely related to the remarkable work of Lie. These systems possess a number of interesting features not found in canonical systems. The geometry of phase space can be much more complicated than that of canonical systems. There exists a special class of constants of motion, Casimir invariants, that enjoy a status above all other constants of motion — they result from the geometry of phase space as embodied in the Poisson bracket and as such their constancy is *independent* of the Hamiltonian.

Continuous systems (field theories) can be classified in much the same way. There are field theories, most notably those of importance in quantum theory, that possess canonically conjugate variables and are thus of the canonically Hamiltonian type. In addition, there are field theories that possess a noncanonical Hamiltonian structure. Most, if not all, Eulerian theories of matter are in this category. For this reason, noncanonical field theories are the subject of much study.

It is into this framework that the Maxwell-Vlasov model of a plasma finds itself. This system is known to be of the noncanonical type which possess an infinity of Casimirs. The full Maxwell-Vlasov system is a rich (and correspondingly complex) set of coupled, nonlinear integro-differential equations. There is little in the way of mathematical understanding of the structure of this system — there is no proof of existence of solutions and the structure of the phase space (which is an infinite dimensional function space) is stunningly complicated. Fortunately, a great deal can be learned in the study of perturbations about an equilibrium. Moreover, in general, and for the above system in particular, noncanonical structure survives linearization.

In this work we consider the linearized Maxwell-Vlasov system. We begin in Chapter 2 by linearizing about a homogeneous and isotropic equilibrium and further simplifying the systems by taking velocity moments in two directions and only allowing spatial variations in the third. The resulting system is a noncanonical field theory containing both longitudinal and transverse degrees of freedom. These linearized moment equations (both longitudinal and transverse) are singular integral equations. As such they possess a continuous spectrum and singular eigenfunctions. In Chapter 3 we develop the theory of a family of integral transforms that is closely related to these singular eigenfunctions. Following this, in Chapter 4 we show that certain members of this family can be used to solve the linearized equations. In Chapter 5 we apply the same transform as in Chapter 4 as a coordinate change. In these new coordinates, the Hamiltonian and bracket obtain canonical diagonal form. A further coordinate change yields action-angle variables for this system. Thus we succeed in discovering canonical coordinates for the linearized system. We extend the longitudinal formalism in Chapter 6 to include neutral discrete modes and we see that they are a natural by-product of regularizing a singularity. By similar methods, in Chapter 7 we make a further extension to include the effects of unstable modes. We then move on in Chapter 8 to develop and implement the necessary numerical algorithms to allow numerical evaluation of the integral transform solution of the longitudinal equations. These results demonstrate the efficacy of singular eigenfunction expansions. We summarize our results in Chapter 9.

2

Noncanonical Hamiltonian Structure of the Maxwell-Vlasov Equations

The starting point of our analysis is the Vlasov–Maxwell system

$$\dot{f}_\alpha + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_v f_\alpha = 0\,, \tag{1a}$$

$$\dot{\boldsymbol{E}} + c\boldsymbol{\nabla}\times\boldsymbol{B} = -4\pi\sum_\alpha q_\alpha \int d^3\boldsymbol{v}\,\boldsymbol{v} f_\alpha\,, \tag{1b}$$

$$\dot{\boldsymbol{B}} + c\boldsymbol{\nabla}\times\boldsymbol{E} = 0\,, \tag{1c}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} - 4\pi\sum_\alpha q_\alpha \int d^3\boldsymbol{v}\, f_\alpha = 0\,, \tag{1d}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, \tag{1e}$$

were $f_\alpha(\boldsymbol{r},\boldsymbol{v},t)$ is the phase space distribution function for species α having mass m_α and charge q_α. It is well known[1–7] that these equations constitute an infinite dimensional, noncanonical Hamiltonian system (*i.e.* a noncanonical field theory). The Poisson bracket for this system is given by

$$\begin{aligned}\{F,G\} = \sum_\alpha \int d^3\boldsymbol{r}\int d^3\boldsymbol{v}\,\frac{f_\alpha}{m_\alpha}\left[\frac{\delta F}{\delta f_\alpha},\frac{\delta G}{\delta f_\alpha}\right] \\ + 4\pi\sum_\alpha \frac{q_\alpha}{m_\alpha}\int d^3\boldsymbol{r}\int d^3\boldsymbol{v}\,\boldsymbol{\nabla}_v f_\alpha\cdot\left(\frac{\delta F}{\delta \boldsymbol{E}}\frac{\delta G}{\delta f_\alpha} - \frac{\delta G}{\delta \boldsymbol{E}}\frac{\delta F}{\delta f_\alpha}\right) \\ + 4\pi\sum_\alpha \frac{q_\alpha}{m_\alpha^2}\int d^3\boldsymbol{r}\int d^3\boldsymbol{v}\, f_\alpha \boldsymbol{B}\cdot\left(\boldsymbol{\nabla}_v\frac{\delta F}{\delta f_\alpha}\times\boldsymbol{\nabla}_v\frac{\delta G}{\delta f_\alpha}\right)\end{aligned}$$

$$+ 4\pi c \int d^3\boldsymbol{r} \, \left(\frac{\delta F}{\delta \boldsymbol{E}} \cdot \boldsymbol{\nabla} \times \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \boldsymbol{\nabla} \times \frac{\delta F}{\delta \boldsymbol{B}} \right) , \tag{2}$$

where F and G are arbitrary functionals, $[\,,]$ is the ordinary Poisson bracket

$$[a\,,b] = \boldsymbol{\nabla} a \cdot \boldsymbol{\nabla}_v b - \boldsymbol{\nabla}_v a \cdot \boldsymbol{\nabla} b, \tag{3}$$

and $\delta F/\delta f_\alpha$ is the functional derivative. To satisfy the Jacobi identity, we require $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. With this bracket the Vlasov–Maxwell equations (1a)–(1c) are now Hamilton's equations:

$$\dot{f}_\alpha = \{f_\alpha\,, H\}, \tag{4a}$$

$$\dot{\boldsymbol{E}} = \{\boldsymbol{E}\,, H\}, \tag{4b}$$

$$\dot{\boldsymbol{B}} = \{\boldsymbol{B}\,, H\}. \tag{4c}$$

where the Hamiltonian, H, is the total energy

$$H = \frac{1}{2} \sum_\alpha \int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, m_\alpha v^2 f_\alpha + \frac{1}{8\pi} \int d^3\boldsymbol{r} \, \left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2 \right) . \tag{5}$$

Note that equations (1d) and (1e) are not obtained from Hamilton's equations. The reason for this is simple: neither of these equations are dynamical in nature — they are constraints on $\boldsymbol{E}$ and $\boldsymbol{B}$. These equations may be viewed as initial conditions; zero being a particular value. In the context of linear theory it is possible, as we will see, to directly incorporate these constraints into the dynamical equations.

I. Linear Theory

We begin with a two component plasma with dynamic electrons and a fixed neutralizing ionic background and consider first order perturbations to $f^{(1)}$, $\boldsymbol{E}^{(1)}$ and $\boldsymbol{B}^{(1)}$ about a stable, monotonic, homogeneous and isotropic equilibrium electron distribution $f^{(0)}(\boldsymbol{r}, \boldsymbol{v}; t) = f^{(0)}(v)$. For such an equilibrium, $\boldsymbol{E}^{(0)} = 0$, $\boldsymbol{B}^{(0)} = 0$ and

$$\nabla_v f^{(0)} = \widehat{\boldsymbol{v}}\, f^{(0)\prime}. \tag{6}$$

The linearized dynamics will involve only electrons and we take the charge to be e and the mass to be m. Linearizing about this equilibrium the bracket becomes:

$$\begin{aligned}\{F, G\}_L = &\frac{1}{m}\int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, f^{(0)} \left[\frac{\delta F}{\delta f^{(1)}}, \frac{\delta G}{\delta f^{(1)}}\right] \\ &+ \frac{4\pi e}{m}\int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, f^{(0)\prime}\, \widehat{\boldsymbol{v}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}}\frac{\delta G}{\delta f^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}}\frac{\delta F}{\delta f^{(1)}}\right) \\ &+ 4\pi c \int d^3\boldsymbol{r} \left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}} \cdot \nabla \times \frac{\delta G}{\delta \boldsymbol{B}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}} \cdot \nabla \times \frac{\delta F}{\delta \boldsymbol{B}^{(1)}}\right).\end{aligned} \tag{7}$$

An expression for the energy of linear perturbations about monotonic equilibria was obtained by Kruskal and Oberman:[8]

$$H^{(2)} = -\frac{m}{2}\int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, v\, \frac{f^{(1)2}}{f^{(0)\prime}} + \frac{1}{8\pi}\int d^3\boldsymbol{r}\, \left(\left|\boldsymbol{E}^{(1)}\right|^2 + \left|\boldsymbol{B}^{(1)}\right|^2\right). \tag{8}$$

In the above, we can easily see the need for monotonicity of $f^{(0)}$ — clearly $H^{(2)}$ is not well defined, for arbitrary $f^{(1)}$, if we allow $f^{(0)\prime}$ to change sign. We will come back to this when we discuss dynamic accessibility.

We introducing the perturbed vector potential, $\boldsymbol{A}^{(1)}$, in place of the magnetic field which both simplifies the bracket and identically satisfies $\nabla \cdot \boldsymbol{B}^{(1)} = 0$. One can easily show

$$\left|\boldsymbol{B}^{(1)}\right|^2 = \frac{1}{2}\nabla^2\left|\boldsymbol{A}^{(1)}\right|^2 - \boldsymbol{\nabla}\cdot\left(\boldsymbol{A}^{(1)}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{A}^{(1)}\right) - \boldsymbol{A}^{(1)}\cdot\nabla^2\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(1)}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{A}^{(1)} \quad (9)$$

and

$$\frac{\delta F}{\delta \boldsymbol{A}^{(1)}} = \boldsymbol{\nabla}\times\frac{\delta F}{\delta \boldsymbol{B}^{(1)}}\,. \quad (10)$$

Thus

$$\begin{aligned}\{F,G\}_L &= \frac{1}{m}\int d^3\boldsymbol{r}\int d^3\boldsymbol{v}\, f^{(0)}\left[\frac{\delta F}{\delta f^{(1)}},\frac{\delta G}{\delta f^{(1)}}\right] \\ &\quad + \frac{4\pi e}{m}\int d^3\boldsymbol{r}\int d^3\boldsymbol{v}\, f^{(0)\prime}\,\widehat{\boldsymbol{v}}\cdot\left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}}\frac{\delta G}{\delta f^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}}\frac{\delta F}{\delta f^{(1)}}\right) \\ &\quad + 4\pi c\int d^3\boldsymbol{r}\left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}}\cdot\frac{\delta G}{\delta \boldsymbol{A}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}}\cdot\frac{\delta F}{\delta \boldsymbol{A}^{(1)}}\right) \quad (11)\end{aligned}$$

and

$$\begin{aligned}H^{(2)} &= -\frac{m}{2}\int d^3\boldsymbol{r}\int d^3\boldsymbol{v}\, v\,\frac{f^{(1)2}}{f^{(0)\prime}} \\ &\quad + \frac{1}{8\pi}\int d^3\boldsymbol{r}\left(\left|\boldsymbol{E}^{(1)}\right|^2 + \boldsymbol{A}^{(1)}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{A}^{(1)} - \boldsymbol{A}^{(1)}\cdot\nabla^2\boldsymbol{A}^{(1)}\right)\,. \quad (12)\end{aligned}$$

For our spatial domain we assume a periodic box of (arbitrarily large) volume V and allow only spatial variation in the (fixed) direction defined by $\widehat{\boldsymbol{k}}$. Taking the Fourier amplitudes as our new dynamical variables the bracket becomes (see Appendix D-2 for details)

$$\begin{aligned}\{F,G\}_L &= \frac{4}{mV^2}\sum_{k,k'=-\infty}^{\infty}\int d^3\boldsymbol{r}\, e^{i(k+k')\widehat{\boldsymbol{k}}\cdot\boldsymbol{r}}\Bigg\{ \\ &\quad \int d^3\boldsymbol{v}\, f^{(0)}\left(-ik\frac{\delta F}{\delta f_k^{(1)}}\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}_v\frac{\delta G}{\delta f_{k'}^{(1)}} + ik'\frac{\delta G}{\delta f_{k'}^{(1)}}\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}_v\frac{\delta F}{\delta f_k^{(1)}}\right)\end{aligned}$$

$$+ 4\pi e \int d^3\boldsymbol{v}\, f^{(0)\prime} \widehat{\boldsymbol{v}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{E}_k^{(1)}} \frac{\delta G}{\delta f_{k'}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{k'}^{(1)}} \frac{\delta F}{\delta f_k^{(1)}} \right)$$

$$+ 4\pi mc \left(\frac{\delta F}{\delta \boldsymbol{E}_k^{(1)}} \cdot \frac{\delta G}{\delta \boldsymbol{A}_{k'}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{k'}^{(1)}} \cdot \frac{\delta F}{\delta \boldsymbol{A}_k^{(1)}} \right) \Bigg\}. \tag{13}$$

The spatial integral can easily be evaluated having the result of setting $k' = -k$ in the sum, giving

$$\{F, G\}_L = \frac{4}{mV} \sum_{k=-\infty}^{\infty} \Bigg\{ \int d^3\boldsymbol{v}\, f^{(0)} \left(-ik \frac{\delta F}{\delta f_k^{(1)}} \widehat{\boldsymbol{k}} \cdot \boldsymbol{\nabla}_v \frac{\delta G}{\delta f_{k'}^{(1)}} - ik \frac{\delta G}{\delta f_{-k}^{(1)}} \widehat{\boldsymbol{k}} \cdot \boldsymbol{\nabla}_v \frac{\delta F}{\delta f_k^{(1)}} \right)$$

$$+ 4\pi e \int d^3\boldsymbol{v}\, f^{(0)\prime} \widehat{\boldsymbol{v}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{E}_k^{(1)}} \frac{\delta G}{\delta f_{-k}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{-k}^{(1)}} \frac{\delta F}{\delta f_k^{(1)}} \right)$$

$$+ 4\pi mc \left(\frac{\delta F}{\delta \boldsymbol{E}_k^{(1)}} \cdot \frac{\delta G}{\delta \boldsymbol{A}_{-k}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{-k}^{(1)}} \cdot \frac{\delta F}{\delta \boldsymbol{A}_k^{(1)}} \right) \Bigg\}$$

$$= \frac{4}{mV} \sum_{k=-\infty}^{\infty} \Bigg\{ ik \int d^3\boldsymbol{v}\, \frac{\partial f^{(0)}}{\partial v_{\parallel}} \frac{\delta F}{\delta f_k^{(1)}} \frac{\delta G}{\delta f_{-k}^{(1)}}$$

$$+ 4\pi e \int d^3\boldsymbol{v}\, \boldsymbol{\nabla}_v f^{(0)} \cdot \left(\frac{\delta F}{\delta \boldsymbol{E}_k^{(1)}} \frac{\delta G}{\delta f_{-k}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{-k}^{(1)}} \frac{\delta F}{\delta f_k^{(1)}} \right)$$

$$+ 4\pi mc \left(\frac{\delta F}{\delta \boldsymbol{E}_k^{(1)}} \cdot \frac{\delta G}{\delta \boldsymbol{A}_{-k}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{-k}^{(1)}} \cdot \frac{\delta F}{\delta \boldsymbol{A}_k^{(1)}} \right) \Bigg\}. \tag{14}$$

In terms of these variables the linearized energy is given by

$$H^{(2)} = \frac{V}{4} \sum_{k=-\infty}^{\infty} \Bigg\{ -\frac{m}{2} \int d^3\boldsymbol{v}\, v \frac{|f_k^{(1)}|^2}{f^{(0)\prime}}$$

$$+ \frac{1}{8\pi} \left(|\boldsymbol{E}_k^{(1)}|^2 + k^2 |\boldsymbol{A}_k^{(1)}|^2 - |\boldsymbol{k} \cdot \boldsymbol{A}_k^{(1)}|^2 \right) \Bigg\}. \tag{15}$$

In the expression for $H^{(2)}$ we assume $f_k^{(1)}$ to be complex and make use of the "reality condition" $f_k^{(1)*} = f_{-k}^{(1)}$. Likewise for $\boldsymbol{E}_k^{(1)}$ and $\boldsymbol{A}_k^{(1)}$.

One can easily obtain the linearized equations of motion from Hamilton's equations:

$$\dot{f}_k^{(1)} + i\widehat{\boldsymbol{k}} \cdot \boldsymbol{v} f_k^{(1)} + \frac{e}{m} \boldsymbol{E}_k^{(1)} \cdot \widehat{\boldsymbol{v}}\, f^{(0)\prime} = 0, \tag{16a}$$

$$-\dot{\boldsymbol{E}}_k^{(1)} + ck^2 \boldsymbol{A}_k^{(1)} - c\boldsymbol{k}\,\boldsymbol{k} \cdot \boldsymbol{A}_k^{(1)} = 4\pi e \int d^3\boldsymbol{v}\, \boldsymbol{v} f_k^{(1)}, \tag{16b}$$

$$\dot{\boldsymbol{A}}_k^{(1)} + c\boldsymbol{k} \times \boldsymbol{E}_k^{(1)} = 0. \tag{16c}$$

In these variables, Poisson's equation reads

$$i\boldsymbol{k} \cdot \boldsymbol{E}_k^{(1)} = 4\pi e \int d^3\boldsymbol{v}\, f_k^{(1)}. \tag{16d}$$

II. Derivation of the Moment System

We now replace $f_k^{(1)}$ as our dynamical variable with selected velocity moments. The moments that we will take do not involve integration over all velocity variables; thus our moment variables will have a continuum label and we will still be dealing with a field theory. The moments that are of interest to us here involve velocity integrations in plane perpendicular to the direction of $\widehat{\boldsymbol{k}}$. To this end we wish to decompose the velocity, $\boldsymbol{v}$, into its component along $\widehat{\boldsymbol{k}}$:

$$v_{\parallel} = \widehat{\boldsymbol{k}} \cdot \boldsymbol{v} \tag{17}$$

and its projection into the perpendicular plane:

$$\boldsymbol{v}_{\perp} = \boldsymbol{v} - \widehat{\boldsymbol{k}} \cdot \boldsymbol{v}\,. \tag{18}$$

We will use this same notation for other vectors as necessary. For any function, $f(\boldsymbol{v})$, we define two moments

$$f_{\parallel} = \int d\boldsymbol{v}_{\perp}\, f\,, \tag{19a}$$

$$\boldsymbol{f}_{\perp} = \int d\boldsymbol{v}_{\perp}\, \boldsymbol{v}_{\perp} f\,. \tag{19b}$$

It is possible to decompose the bracket and, as we will see, the Hamiltonian (and thus the equations of motion) into longitudinal and transverse pieces by restricting to functionals that depend only on these moments of $f_k^{(1)}$, *i.e.* to functionals of the form

$$F\left[f_k^{(1)}\right] = F\left[f_{\parallel k}^{(1)}, \boldsymbol{f}_{\perp k}^{(1)}\right], \tag{20}$$

in which case

$$\frac{\delta F}{\delta f_k^{(1)}} = \frac{\delta F}{\delta f_{\parallel k}^{(1)}} + \boldsymbol{v}_{\perp} \cdot \frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}}\,. \tag{21}$$

We could consider moments other than these but they prove to be of less interest for several reasons. If one goes above second degree in $\boldsymbol{v}_{\perp}$, then the bracket of two such moments will always involve moments of $f^{(0)}$ of greater degree. Thus, for example, the bracket of a third and fifth degree moment will involve a sixth degree moment of the equilibrium. This situation is unsatisfactory in that it requires us to take higher moments of the equilibrium than of the perturbations. This not withstanding, there are sets of higher degree moments that do yield closed sets equations of motion, however *none* of these higher moments appear as *source terms* in Maxwell's equations. The same is true of the moments of degree two and lower other than the above. In fact *only* the moments defined above will couple self-consistently to the fields. Thus the dynamics associated with these other moments is in some sense trivial; it amounts to passive advection. In the end, our choice of moment variables was dictated by the manner in which Maxwell's equations couple to matter.

We are now in a position to evaluate the linearized bracket, (14) in terms of the moment variables. Although we have introduced the vector potential, we have yet to take advantage of the resulting gauge freedom. It will prove convenient to choose our gauge such that $\boldsymbol{k}\cdot\boldsymbol{A}_k^{(1)} = 0$[9] which ensures that $E_{\parallel}^{(1)}$ is entirely electrostatic and, *via* the chain rule for functional derivatives (see Appendix D-1), that

$$\boldsymbol{k}\cdot\frac{\delta F}{\delta \boldsymbol{A}_{\perp k}^{(1)}} = 0\,. \tag{22}$$

(This also applies to functional derivatives with respect to $\boldsymbol{f}_{\perp k}^{(1)}$ and $\boldsymbol{E}_{\perp k}^{(1)}$, independent of the gauge.)

Substituting (21) into the expression for the bracket and taking our gauge condition into account gives

$$\begin{aligned}
\{F,G\}_L = \frac{4}{mV}\sum_{k=-\infty}^{\infty}\Bigg\{ & ik\int d^3\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_{\parallel}}\,\frac{\delta F}{\delta f_{\parallel k}^{(1)}}\,\frac{\delta G}{\delta f_{\parallel -k}^{(1)}} \\
& + ik\int d^3\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_{\parallel}}\,\boldsymbol{v}_{\perp}\cdot\left(\frac{\delta F}{\delta f_{\parallel k}^{(1)}}\,\frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}} + \frac{\delta G}{\delta f_{\parallel -k}^{(1)}}\,\frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}}\right) \\
& + ik\int d^3\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_{\parallel}}\,v_{\perp i}v_{\perp j}\left(\frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}}\,\frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}}\right)_{ij} \\
& + 4\pi e\,\frac{\delta F}{\delta E_{\parallel k}^{(1)}}\int d^3\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_{\parallel}}\left(\frac{\delta G}{\delta f_{\parallel -k}^{(1)}} + \boldsymbol{v}_{\perp}\cdot\frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}}\right) \\
& - 4\pi e\,\frac{\delta G}{\delta E_{\parallel -k}^{(1)}}\int d^3\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_{\parallel}}\left(\frac{\delta F}{\delta f_{\parallel k}^{(1)}} + \boldsymbol{v}_{\perp}\cdot\frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}}\right) \\
& + 4\pi e\,\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}}\cdot\int d^3\boldsymbol{v}\,\nabla_{v_{\perp}} f^{(0)}\left(\frac{\delta G}{\delta f_{\parallel -k}^{(1)}} + \boldsymbol{v}_{\perp}\cdot\frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}}\right)
\end{aligned}$$

$$
- 4\pi e \frac{\delta G}{\delta \boldsymbol{E}^{(1)}_{\perp -k}} \cdot \int d^3\boldsymbol{v}\, \boldsymbol{\nabla}_{v_\perp} f^{(0)} \left(\frac{\delta F}{\delta f^{(1)}_{\parallel k}} + \boldsymbol{v}_\perp \cdot \frac{\delta F}{\delta \boldsymbol{f}^{(1)}_{\perp k}} \right)
$$

$$
+ \pi mc \left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}_{\perp k}} \cdot \frac{\delta G}{\delta \boldsymbol{A}^{(1)}_{\perp -k}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}_{\perp -k}} \cdot \frac{\delta F}{\delta \boldsymbol{A}^{(1)}_{\perp k}} \right) \Bigg\}. \tag{23}
$$

The next step is to separate the $\boldsymbol{v}_\perp$ and $v_\parallel$ integrations. Doing so, using

$$
\boldsymbol{\nabla}_{v_\perp} f^{(0)} = \frac{\boldsymbol{v}_\perp}{v_\parallel} \frac{\partial f^{(0)}}{\partial v_\parallel}, \tag{24}
$$

and rearranging terms gives

$$
\{F, G\}_L = \frac{4}{mV} \sum_{k=-\infty}^{\infty} \Bigg\{ ik \int d^3\boldsymbol{v}\, \frac{\partial f^{(0)}}{\partial v_\parallel} \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}}
$$

$$
+ ik \int dv_\parallel \left(\frac{\delta F}{\delta f^{(1)}_{\parallel k}} \frac{\delta G}{\delta \boldsymbol{f}^{(1)}_{\perp -k}} + \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} \frac{\delta F}{\delta \boldsymbol{f}^{(1)}_{\perp k}} \right) \cdot \int d^2\boldsymbol{v}_\perp\, \boldsymbol{v}_\perp \frac{\partial f^{(0)}}{\partial v_\parallel}
$$

$$
+ ik \int dv_\parallel \left(\frac{\delta F}{\delta \boldsymbol{f}^{(1)}_{\perp k}} \frac{\delta G}{\delta \boldsymbol{f}^{(1)}_{\perp -k}} \right)_{ij} \int d^2\boldsymbol{v}_\perp\, \frac{\partial f^{(0)}}{\partial v_\parallel} v_{\perp i} v_{\perp j}
$$

$$
+ 4\pi e \int dv_\parallel \left(\frac{\delta F}{\delta E^{(1)}_{\parallel k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} - \frac{\delta G}{\delta E^{(1)}_{\parallel -k}} \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \right) \int d^2\boldsymbol{v}_\perp\, \frac{\partial f^{(0)}}{\partial v_\parallel}
$$

$$
+ 4\pi e \int dv_\parallel \left(\frac{\delta F}{\delta E^{(1)}_{\parallel k}} \frac{\delta G}{\delta \boldsymbol{f}^{(1)}_{\perp -k}} - \frac{\delta G}{\delta E^{(1)}_{\parallel -k}} \frac{\delta F}{\delta \boldsymbol{f}^{(1)}_{\perp k}} \right) \cdot \int d^2\boldsymbol{v}_\perp\, \boldsymbol{v}_\perp \frac{\partial f^{(0)}}{\partial v_\parallel}
$$

$$
+ 4\pi e \int dv_\parallel \frac{1}{v_\parallel} \left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}_{\perp k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}_{\perp -k}} \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \right) \cdot \int d^2\boldsymbol{v}_\perp\, \boldsymbol{v}_\perp \frac{\partial f^{(0)}}{\partial v_\parallel}
$$

$$
+ 4\pi e \int dv_\parallel \frac{1}{v_\parallel} \left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}_{\perp k}} \frac{\delta G}{\delta \boldsymbol{f}^{(1)}_{\perp -k}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}_{\perp -k}} \frac{\delta F}{\delta \boldsymbol{f}^{(1)}_{\perp k}} \right)_{ij} \int d^2\boldsymbol{v}_\perp\, \frac{\partial f^{(0)}}{\partial v_\parallel} v_{\perp i} v_{\perp j}
$$

$$
+ \pi mc \left(\frac{\delta F}{\delta \boldsymbol{E}^{(1)}_{\perp k}} \cdot \frac{\delta G}{\delta \boldsymbol{A}^{(1)}_{\perp -k}} - \frac{\delta G}{\delta \boldsymbol{E}^{(1)}_{\perp -k}} \cdot \frac{\delta F}{\delta \boldsymbol{A}^{(1)}_{\perp k}} \right) \Bigg\}. \tag{25}
$$

To carry out the integration over $\boldsymbol{v}_{\perp}$, we make use of the following results established in Appendix E:

$$\int d^2\boldsymbol{v}_{\perp}\, \frac{\partial f^{(0)}}{\partial v_{\parallel}} = f_{\parallel}^{(0)\prime}, \tag{570}$$

$$\int d^2\boldsymbol{v}_{\perp}\, \boldsymbol{v}_{\perp} \frac{\partial f^{(0)}}{\partial v_{\parallel}} = 0, \tag{573}$$

$$\int d^2\boldsymbol{v}_{\perp}\, v_{\perp i} v_{\perp j} \frac{\partial f^{(0)}}{\partial v_{\parallel}} = -\left(\delta_{ij} - \widehat{k}_i \widehat{k}_j\right) v_{\parallel} f_{\parallel}^{(0)}. \tag{580}$$

The above allows us to immediately evaluate all of the terms in (25) giving

$$\begin{aligned}
\{F, G\}_L = \frac{4}{mV} \sum_{k=-\infty}^{\infty} \Bigg\{ & ik \int dv_{\parallel}\, f_{\parallel}^{(0)\prime} \frac{\delta F}{\delta f_{\parallel k}^{(1)}} \frac{\delta G}{\delta f_{\parallel -k}^{(1)}} - ik \int dv_{\parallel}\, v_{\parallel} f_{\parallel}^{(0)} \frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}} \cdot \frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}} \\
& + 4\pi e \int dv_{\parallel}\, f_{\parallel}^{(0)\prime} \left(\frac{\delta F}{\delta E_{\parallel k}^{(1)}} \frac{\delta G}{\delta f_{\parallel -k}^{(1)}} - \frac{\delta G}{\delta E_{\parallel -k}^{(1)}} \frac{\delta F}{\delta f_{\parallel k}^{(1)}} \right) \\
& - 4\pi e \int dv_{\parallel}\, f_{\parallel}^{(0)} \left(\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}} \cdot \frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{\perp -k}^{(1)}} \cdot \frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}} \right) \\
& + 4\pi mc \left(\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}} \cdot \frac{\delta G}{\delta \boldsymbol{A}_{\perp -k}^{(1)}} - \frac{\delta G}{\delta \boldsymbol{E}_{\perp -k}^{(1)}} \cdot \frac{\delta F}{\delta \boldsymbol{A}_{\perp k}^{(1)}} \right) \Bigg\}.
\end{aligned} \tag{26}$$

III. Dynamic Accessibility

Having calculated the bracket in terms of the moment variables, we are in the position of not having an explicit expression for the energy in these variables.[10] Since our transformation to the moment variables is not invertible, we cannot necessarily express $f_k^{(1)}$ in terms of $f_{\parallel k}^{(1)}$ and $\boldsymbol{f}_{\perp k}^{(1)}$. This non-invertibility tells us that in the moment projection some of the information contained in $f_k^{(1)}$ is eliminated.

Thus any functional that depends on this eliminated information will not have an exact representation in terms of the moment variables. We need to determine a general form for $f_k^{(1)}$ which contains only information that contributes to $f_{\parallel k}^{(1)}$ and $\boldsymbol{f}_{\perp k}^{(1)}$. It is the determination of this form that brings us to the issue of dynamic accessibility.

An arbitrary perturbation will not, in general, preserve the equilibrium value of the Casimirs.[10,11] The essence of dynamic accessibility is that the Casimirs act as a constraint on the perturbations; only those perturbations that lie in the constraint surface (in function space) defined by the equilibrium value of the Casimirs are guaranteed to conserve those Casimirs. Leaving the equilibrium value of the Casimirs unchanged is necessary for the linearized theory to be consistent with the full nonlinear theory. One can make a convincing argument that if dynamically inaccessible perturbations appear to be essential in describing some phenomena, then there is likely to be some important physics missing from the nonlinear model.[10]

To satisfy dynamic accessibility, we need a method of projecting an arbitrary perturbation into this constraint surface. As we will see, this can be easily accomplished by making use of the nonlinear bracket. The dynamically accessible variation in any functional F, δF_{DA}, is that which arises from dynamically accessible variations in f, $\boldsymbol{E}$ and $\boldsymbol{A}$. To first order, we will denote such perturbations by $f_{\text{DA}}^{(1)}$, $\boldsymbol{E}_{\text{DA}}^{(1)}$ and $\boldsymbol{A}_{\text{DA}}^{(1)}$ respectively.

By definition, the bracket of a Casimir and any other functional is identically zero, hence variations generated using the bracket will be such that the Casimirs are unchanged. That is, for any first order quantity $G^{(1)}$, if we define

$$\delta F_{\text{DA}} = \{F, G^{(1)}\} \tag{27}$$

then the corresponding variations in $f^{(1)}_{\rm DA}$, $\boldsymbol{E}^{(1)}_{\rm DA}$ and $\boldsymbol{A}^{(1)}_{\rm DA}$, will be such that the Casimirs will be left unchanged. We can express $\delta F_{\rm DA}$ in terms of the various functional derivatives of F, namely

$$F^{(1)}_{\rm DA} = \int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, \frac{\delta F}{\delta f} f^{(1)}_{\rm DA} + \int d^3\boldsymbol{r}\, \frac{\delta F}{\delta \boldsymbol{E}} \cdot \boldsymbol{E}^{(1)}_{\rm DA} + \int d^3\boldsymbol{r}\, \frac{\delta F}{\delta \boldsymbol{A}} \cdot \boldsymbol{A}^{(1)}_{\rm DA}. \qquad (28)$$

By comparing (27) and (28) we can use the chain rule to obtain expressions for the dynamically accessible perturbations:

$$f^{(1)}_{\rm DA} = \{f\,, G^{(1)}\}, \qquad (29a)$$

$$\boldsymbol{E}^{(1)}_{\rm DA} = \{\boldsymbol{E}\,, G^{(1)}\}, \qquad (29b)$$

$$\boldsymbol{A}^{(1)}_{\rm DA} = \{\boldsymbol{A}\,, G^{(1)}\}. \qquad (29c)$$

To proceed it is convenient express the nonlinear bracket in terms $\boldsymbol{A}$ and consider only electrons. In this case, the bracket becomes

$$\begin{aligned}\{F,G\} = &\int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, \frac{f}{m} \left[\frac{\delta F}{\delta f}\,, \frac{\delta G}{\delta f}\right] \\ &+ \frac{4\pi e}{m} \int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, \boldsymbol{\nabla}_v f \cdot \left(\frac{\delta F}{\delta \boldsymbol{E}} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta \boldsymbol{E}} \frac{\delta F}{\delta f}\right) \\ &+ \frac{4\pi e}{m^2} \int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, f \left(\boldsymbol{\nabla}_v \frac{\delta F}{\delta f} \times \boldsymbol{\nabla}_v \frac{\delta G}{\delta f}\right) \cdot \boldsymbol{\nabla} \times \boldsymbol{A} \\ &+ 4\pi c \int d^3\boldsymbol{r} \left(\frac{\delta F}{\delta \boldsymbol{E}} \cdot \frac{\delta G}{\delta \boldsymbol{A}} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \frac{\delta F}{\delta \boldsymbol{A}}\right). \qquad (30)\end{aligned}$$

Taking

$$G^{(1)} = \int d^3\boldsymbol{r} \int d^3\boldsymbol{v}\, g\, f + \int d^3\boldsymbol{r}\, \boldsymbol{g}^A \cdot \boldsymbol{E} + \int d^3\boldsymbol{r}\, \boldsymbol{g}^E \cdot \boldsymbol{A}, \qquad (31)$$

where g, $\boldsymbol{g}^A$ and $\boldsymbol{g}^E$ are first order quantities (29) gives

$$f^{(1)}_{\rm DA} = \frac{1}{m}\,[g\,, f] - \frac{4\pi e}{m} \boldsymbol{\nabla}_v f \cdot \boldsymbol{g}^A - \frac{4\pi e}{m^2} \left(\boldsymbol{\nabla}_v f \times \boldsymbol{\nabla}_v g\right) \cdot \boldsymbol{\nabla} \times \boldsymbol{A}, \qquad (32a)$$

$$\boldsymbol{E}^{(1)}_{\rm DA} = 4\pi c\, \boldsymbol{g}^{E} + \frac{4\pi e}{m} \int d^3\boldsymbol{v}\, \boldsymbol{\nabla}_v f^{(0)} g\,, \tag{32b}$$

$$\boldsymbol{A}^{(1)}_{\rm DA} = -4\pi c\, \boldsymbol{g}^{E}. \tag{32c}$$

Evaluating the above on our equilibrium: $f = f^{(0)}(v)$, $\boldsymbol{E}^{(0)} = 0$ and $\boldsymbol{A}^{(0)} = 0$ we find

$$f^{(1)}_{\rm DA} = \frac{1}{m} \boldsymbol{\nabla}_v f^{(0)} \cdot (\boldsymbol{\nabla} g - 4\pi e\, \boldsymbol{g}^{A})\,, \tag{33a}$$

$$\boldsymbol{E}^{(1)}_{\rm DA} = 4\pi c\, \boldsymbol{g}^{E} + \frac{4\pi e}{m} \int d^3\boldsymbol{v}\, \boldsymbol{\nabla}_v f^{(0)} g\,, \tag{33b}$$

$$\boldsymbol{A}^{(1)}_{\rm DA} = -4\pi c\, \boldsymbol{g}^{A}. \tag{33c}$$

Clearly $\boldsymbol{E}^{(1)}_{\rm DA}$ and $\boldsymbol{A}^{(1)}_{\rm DA}$ are arbitrary. This is what we expect since $\boldsymbol{E}^{(1)}$ and $\boldsymbol{A}^{(1)}$ are canonically conjugate variables. Now consider $f^{(1)}_{\rm DA}$. Define $\widehat{g}$ to be the solution of

$$\nabla^2 \widehat{g} = \nabla^2 g - 4\pi e\, \boldsymbol{\nabla} \cdot \boldsymbol{g}^{A}. \tag{34}$$

As this is just Laplace's equation, we can uniquely determine $\widehat{g}$ for any g and

$$\boldsymbol{\nabla} \widehat{g} = \boldsymbol{\nabla} g - 4\pi e\, \boldsymbol{g}^{A}, \tag{35}$$

Thus the dynamically accessible form of $f^{(1)}$ is

$$f^{(1)}_{\rm DA} = \frac{1}{m} \boldsymbol{\nabla}_v f^{(0)} \cdot \boldsymbol{\nabla} g = \frac{1}{m} \left[g\,, f^{(0)} \right], \tag{36}$$

where g is any function.

In terms of the Fourier transformed variables we have

$$f^{(1)}_{k\,\rm DA} = g_k \frac{\partial f^{(0)}}{\partial v_{\parallel}}\,, \tag{37}$$

while $\boldsymbol{E}^{(1)}_{k\,\mathrm{DA}}$ and $\boldsymbol{A}^{(1)}_{k\,\mathrm{DA}}$ are still arbitrary. Note that g_k is the Fourier transform of a gradient and as such, the $k = 0$ contribution to $f^{(1)}$ is *not* dynamically accessible. From the expression for $H^{(2)}$, we see that by restricting to dynamically accessible perturbations, any singularity associated with zeros of $f^{(0)\prime}$ is removable and thus we are no longer constrained to monotonic equilibria.

We now turn to the problem of projecting the constraint of dynamic accessibility onto the moment variables. The simplest approach seems to be to start with $f^{(1)}_{k\,\mathrm{DA}}$ and take moments. As we argued above, we need only consider generating functions g that contain only information that survives the taking of moments. It is straightforward to show that the most general such function is

$$g_k(v_{\parallel}, \boldsymbol{v}_{\perp}) = g_{\parallel k}(v_{\parallel}, v_{\perp}) + \boldsymbol{v}_{\perp} \cdot \boldsymbol{g}_{\perp k}(v_{\parallel}, v_{\perp}) \,. \tag{38}$$

Using the results in Appendix E, we find

$$f^{(1)}_{\parallel k\,\mathrm{DA}} = \int d^2\boldsymbol{v}_{\perp}\, g_{\parallel k}(v_{\parallel}, v_{\perp}) \frac{\partial f^{(0)}}{\partial v_{\parallel}} \,, \tag{39}$$

and

$$\boldsymbol{f}^{(1)}_{\perp k\,\mathrm{DA}} = \int d^2\boldsymbol{v}_{\perp}\, \boldsymbol{v}_{\perp}\boldsymbol{v}_{\perp} \cdot \boldsymbol{g}_{\perp k} \frac{\partial f^{(0)}}{\partial v_{\parallel}} = \frac{1}{2} v_{\parallel} \int d^2\boldsymbol{v}_{\perp}\, \boldsymbol{g}_{\perp k}\, v_{\perp} \frac{\partial f^{(0)}}{\partial v_{\perp}} \,. \tag{40}$$

Expanding $g_{\parallel k}$ and $\boldsymbol{g}_{\perp k}$ in power series in $v_{\perp}$:

$$g_{\parallel k} = \sum_{n=0}^{\infty} g^{(n)}_{\parallel k} v_{\perp}^n \,, \tag{41a}$$

$$\boldsymbol{g}_{\perp k} = \sum_{n=0}^{\infty} \boldsymbol{g}^{(n)}_{\perp k} v_{\perp}^n \,, \tag{41b}$$

where

$$g^{(n)}_{\parallel k} = \frac{1}{n!} \frac{\partial^n}{\partial v_{\perp}^n} g_{\parallel k} \bigg|_{v_{\perp} = 0} \,, \tag{42}$$

and similarly for $\boldsymbol{g}_{\perp k}^{(n)}$, one can readily compute

$$f_{\parallel k\,\mathrm{DA}}^{(1)} = \sum_{n=0}^{\infty} g_{\parallel k}^{(n)} \frac{\partial}{\partial v_{\parallel}} \left\langle v_{\perp}^{n} f^{(0)} \right\rangle, \tag{43}$$

and

$$\boldsymbol{f}_{\perp k\,\mathrm{DA}}^{(1)} = \frac{1}{2} v_{\parallel} \sum_{n=0}^{\infty} \boldsymbol{g}_{\perp k}^{(n)} (n+2) \left\langle v_{\perp}^{n} f^{(0)} \right\rangle, \tag{44}$$

where

$$\left\langle v_{\perp}^{n} f^{(0)} \right\rangle = \int d^{2}\boldsymbol{v}_{\perp}\, v_{\perp}^{n} f^{(0)}. \tag{45}$$

Notice that $f_{\parallel k\mathrm{DA}}^{(1)}$ and $\boldsymbol{f}_{\perp k\mathrm{DA}}^{(1)}$, in principle, contain arbitrarily high degree perpendicular moments of $f^{(0)}$. As we have argued above, it seems inconsistent to retain such moments of the equilibrium distribution but not of the perturbations. We resolve this by considering only generating functions of the form

$$g_{k}(v_{\parallel}, \boldsymbol{v}_{\perp}) = g_{\parallel k}(v_{\parallel}) + \boldsymbol{v}_{\perp} \cdot \boldsymbol{g}_{\perp k}(v_{\parallel}). \tag{46}$$

In which case, keeping only the $n = 0$ term on the above, we have

$$f_{\parallel k\,\mathrm{DA}}^{(1)} = g_{\parallel k}(v_{\parallel})\, f_{\parallel}^{(0)\prime}, \tag{47}$$

and

$$\boldsymbol{f}_{\perp k\,\mathrm{DA}}^{(1)} = v_{\parallel}\, \boldsymbol{g}_{\perp k}(v_{\parallel})\, f_{\parallel}^{(0)}. \tag{48}$$

IV. Energy for the Moment System

Now that we have determined a general, dynamically accessible form for $f_k^{(1)}$, we can use this to compute the particle contribution to the linearized energy, (15). Recall that the requirement of dynamically accessibility has the side effect of ensuring the expression for the Hamiltonian is well defined. Consider the linearized particle energy:

$$H^{(2)}_{\text{particles}} = -\frac{mV}{8} \sum_{k=-\infty}^{\infty} \int d^3\boldsymbol{v}\, v\, \frac{|f_k^{(1)}|^2}{f^{(0)\prime}}, \tag{49}$$

where we assume that $f_k^{(1)}$ is dynamically accessible, *i.e.*

$$f_k^{(1)} = \left[g_{\parallel k}(v_\parallel) + \boldsymbol{v}_\perp \cdot \boldsymbol{g}_{\perp k}(v_\parallel)\right] \frac{\partial f^{(0)}}{\partial v_\parallel}. \tag{50}$$

Now

$$|f_k^{(1)}|^2 = \left(\frac{\partial f^{(0)}}{\partial v_\parallel}\right)^2 \left[g_{\parallel k}(v_\parallel) + \boldsymbol{v}_\perp \cdot \boldsymbol{g}_{\perp k}(v_\parallel)\right] \left[g_{\parallel -k}(v_\parallel) + \boldsymbol{v}_\perp \cdot \boldsymbol{g}_{\perp -k}(v_\parallel)\right] \tag{51}$$

and

$$\frac{1}{v} f^{(0)\prime} = \frac{1}{v_\parallel} \frac{\partial f^{(0)}}{\partial v_\parallel} \tag{569}$$

giving

$$\begin{aligned} H^{(2)}_{\text{particles}} &= -\frac{mV}{8} \sum_{k=-\infty}^{\infty} \int d^3\boldsymbol{v}\, v_\parallel \frac{\partial f^{(0)}}{\partial v_\parallel} \Big\{ |g_{\parallel k}|^2 \\ &\qquad + \boldsymbol{v}_\perp \cdot \left(g_{\parallel k} \boldsymbol{g}_{\perp -k} + g_{\parallel -k} \boldsymbol{g}_{\perp k}\right) + \boldsymbol{g}_{\perp k} \boldsymbol{g}_{\perp -k} : \boldsymbol{v}_\perp \boldsymbol{v}_\perp \Big\} \\ &= -\frac{mV}{8} \sum_{k=-\infty}^{\infty} \int dv_\parallel \left\{ v_\parallel |g_{\parallel k}|^2 f_\parallel^{(0)\prime} - v_\parallel^2 |\boldsymbol{g}_{\perp k}|^2 f_\parallel^{(0)} \right\}. \end{aligned} \tag{52}$$

Since we are assuming dynamically accessible perturbations, we can use (47) and (48). Making this substitution, we find

$$H^{(2)}_{\text{particles}} = \frac{mV}{8} \sum_{k=-\infty}^{\infty} \int d^3\boldsymbol{v} \left\{ \frac{\left|\boldsymbol{f}^{(1)}_{\perp k}\right|^2}{f^{(0)}_{\parallel}} - v_{\parallel} \frac{\left|f^{(1)}_{\parallel k}\right|^2}{f^{(0)\prime}_{\parallel}} \right\}. \tag{53}$$

Thus the complete expression for the linearized energy is

$$H^{(2)} = \frac{V}{8} \sum_{k=-\infty}^{\infty} \left\{ m \int dv_{\parallel} \left(\frac{\left|\boldsymbol{f}^{(1)}_{\perp k}\right|^2}{f^{(0)}_{\parallel}} - v_{\parallel} \frac{\left|f^{(1)}_{\parallel k}\right|^2}{f^{(0)\prime}_{\parallel}} \right) + \frac{1}{4\pi} \left(\left|E^{(1)}_{\parallel k}\right|^2 + \left|\boldsymbol{E}^{(1)}_{\perp k}\right|^2 + k^2 \left|\boldsymbol{A}^{(1)}_{\perp k}\right|^2 \right) \right\}. \tag{54}$$

It is interesting to note that had we allowed a more general $v_{\perp}$ dependence in g_k, then it would not have been possible to exactly express $H^{(2)}$ in terms of $f^{(1)}_{\parallel}$ and $\boldsymbol{f}^{(1)}_{\perp}$. This helps to support the our argument for restricting the form of g_k.

One can deduce the same expression for the energy by first obtaining the equations of motion for the moment variables and then considering the energy balance between the particles and fields. Doing so, one finds

$$0 = \frac{d}{dt} H^{(2)} = \frac{d}{dt} \mathcal{E}_{\text{particles}} + \frac{d}{dt} \mathcal{E}_{\text{fields}}\,. \tag{55}$$

V. Equations of Motion

Now that we have expressions for the bracket, (26), and energy, (54) in terms of the moment variables we can directly obtain the equations of motion (which are now Hamilton's equations):

$$\dot{f}^{(1)}_{\parallel} = \left\{ f^{(1)}_{\parallel}, H^{(2)} \right\}_L, \tag{56a}$$

$$\dot{\boldsymbol{f}}_{\perp}^{(1)} = \left\{ \boldsymbol{f}_{\perp}^{(1)}, H^{(2)} \right\}_L, \tag{56b}$$

$$\dot{E}_{\parallel k}^{(1)} = \left\{ E_{\parallel k}^{(1)}, H^{(2)} \right\}_L, \tag{56c}$$

$$\dot{\boldsymbol{E}}_{\perp k}^{(1)} = \left\{ \boldsymbol{E}_{\perp k}^{(1)}, H^{(2)} \right\}_L, \tag{56d}$$

$$\dot{\boldsymbol{A}}_{\perp k}^{(1)} = \left\{ \boldsymbol{A}_{\perp k}^{(1)}, H^{(2)} \right\}_L. \tag{56e}$$

One can see from the form of the bracket and Hamiltonian that there will be no coupling between the longitudinal and transverse degrees of freedom. Carrying out this calculation yields

$$\dot{f}_{\parallel k}^{(1)} + ikv_{\parallel} f_{\parallel k}^{(1)} + \frac{e}{m} E_{\parallel k}^{(1)} f_{\parallel}^{(0)\prime} = 0, \tag{57a}$$

$$-\dot{E}_{\parallel k}^{(1)} = 4\pi e \int dv_{\parallel}\, v_{\parallel} f_{\parallel k}^{(1)}, \tag{57b}$$

and

$$\dot{\boldsymbol{f}}_{\perp k}^{(1)} + ikv_{\parallel} \boldsymbol{f}_{\perp k}^{(1)} - \frac{e}{m} \boldsymbol{E}_{\perp k}^{(1)} f_{\parallel}^{(0)} = 0, \tag{58a}$$

$$-\dot{\boldsymbol{E}}_{\perp k}^{(1)} + ck^2 \boldsymbol{A}_{\perp k}^{(1)} = 4\pi e \int dv_{\parallel}\, \boldsymbol{f}_{\perp k}^{(1)}, \tag{58b}$$

$$\dot{\boldsymbol{A}}_{\perp k}^{(1)} + c\boldsymbol{E}_{\perp k}^{(1)} = 0. \tag{58c}$$

Note that the longitudinal projection of Poisson's equation,

$$ik\, E_{\parallel k}^{(1)} = 4\pi e \int dv_{\parallel}\, f_{\parallel k}^{(1)}, \tag{59}$$

is again not obtained by this method. It is possible, however, to incorporate this constraint into the longitudinal dynamics. By means of the longitudinal Vlasov equation, one can show that Ampere's law, (57*b*), is equivalent to (in fact it is the time derivative of) Poisson's equation, (59). Thus we can replace a dynamical equation (Ampere's law) with a kinematic constraint (Poisson's equation).

Furthermore, we can use Poisson's equation to eliminate $E^{(1)}_{\parallel k}$ as a dynamical variable.[12] That is, through Poisson's equation, we can view a functional of $f^{(1)}_{\parallel k}$ and $E^{(1)}_{\parallel k}$ as a functional of $f^{(1)}_{\parallel k}$ alone, *i.e.*

$$F\left[f^{(1)}_{\parallel k}, E^{(1)}_{\parallel k}\right] = \widehat{F}\left[f^{(1)}_{\parallel k}\right] \tag{60}$$

and

$$\frac{\delta \widehat{F}}{\delta f^{(1)}_{\parallel k}} = \frac{\delta F}{\delta f^{(1)}_{\parallel k}} + \frac{\delta F}{\delta E^{(1)}_{\parallel k}} \frac{\delta E^{(1)}_{\parallel k}}{\delta f^{(1)}_{\parallel k}}. \tag{61}$$

From Poisson's equation we see that

$$\frac{\delta E^{(1)}_{\parallel k}}{\delta f^{(1)}_{\parallel k}} = \frac{4\pi e}{ik}, \tag{62}$$

which gives us

$$\frac{\delta \widehat{F}}{\delta f^{(1)}_{\parallel k}} = \frac{\delta F}{\delta f^{(1)}_{\parallel k}} + \frac{4\pi e}{ik} \frac{\delta F}{\delta E^{(1)}_{\parallel k}}. \tag{63}$$

For the moment, consider only the longitudinal part of the bracket:

$$\begin{aligned} \{F, G\}_{\parallel} = \frac{4}{mV} \sum_{k=-\infty}^{\infty} \Bigg\{ & ik \int dv_{\parallel}\, f^{(0)\prime}_{\parallel} \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} \\ & + 4\pi e \int dv_{\parallel}\, f^{(0)\prime}_{\parallel} \left(\frac{\delta F}{\delta E^{(1)}_{\parallel k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} - \frac{\delta G}{\delta E^{(1)}_{\parallel -k}} \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \right) \Bigg\}. \end{aligned} \tag{64}$$

Rearranging terms we get

$$\begin{aligned} \{F, G\}_{\parallel} = \frac{4}{mV} \sum_{k=-\infty}^{\infty} \frac{ik}{m} \int dv_{\parallel}\, f^{(0)\prime}_{\parallel} \Bigg\{ & \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} \\ & + \frac{4\pi e}{ki} \left(\frac{\delta F}{\delta E^{(1)}_{\parallel k}} \frac{\delta G}{\delta f^{(1)}_{\parallel -k}} - \frac{\delta G}{\delta E^{(1)}_{\parallel -k}} \frac{\delta F}{\delta f^{(1)}_{\parallel k}} \right) \Bigg\} \end{aligned}$$

$$= \frac{4}{mV}\sum_{k=-\infty}^{\infty} ik\int dv_{\parallel}\, f_{\parallel}^{(0)\prime}\left\{\left(\frac{\delta F}{\delta f_{\parallel k}^{(1)}}+\frac{4\pi e}{ki}\frac{\delta F}{\delta E_{\parallel k}^{(1)}}\right)\right.$$

$$\left.\times\left(\frac{\delta G}{\delta f_{\parallel -k}^{(1)}}-\frac{4\pi e}{ki}\frac{\delta G}{\delta E_{\parallel -k}^{(1)}}\right)-\frac{16\pi^2e^2}{k^2}\frac{\delta F}{\delta E_{\parallel k}^{(1)}}\frac{\delta G}{\delta E_{\parallel -k}^{(1)}}\right\}. \tag{65}$$

The last term in the above depends on $v_{\parallel}$ only through $f^{(0)\prime}$ and is therefore proportional to a surface term which is assumed to vanish. Making use of (63) we obtain

$$\{F,G\}_{\parallel} = \frac{4}{mV}\sum_{k=-\infty}^{\infty} ik\int dv_{\parallel}\, f_{\parallel}^{(0)\prime}\frac{\delta\widehat{F}}{\delta f_{\parallel k}^{(1)}}\frac{\delta\widehat{G}}{\delta f_{\parallel -k}^{(1)}}. \tag{66}$$

Dropping the $\widehat{\ }$'s, the complete bracket becomes

$$\{F,G\}_{L} = \frac{4}{mV}\sum_{k=-\infty}^{\infty}\left\{ ik\int dv_{\parallel}\, f_{\parallel}^{(0)\prime}\frac{\delta F}{\delta f_{\parallel k}^{(1)}}\frac{\delta G}{\delta f_{\parallel -k}^{(1)}} - ik\int dv_{\parallel}\, v_{\parallel} f_{\parallel}^{(0)}\frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}}\cdot\frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}}\right.$$

$$-4\pi e\int dv_{\parallel}\, f_{\parallel}^{(0)}\left(\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}}\cdot\frac{\delta G}{\delta \boldsymbol{f}_{\perp -k}^{(1)}}-\frac{\delta G}{\delta \boldsymbol{E}_{\perp -k}^{(1)}}\cdot\frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}}\right)$$

$$\left.+4\pi mc\left(\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}}\cdot\frac{\delta G}{\delta \boldsymbol{A}_{\perp -k}^{(1)}}-\frac{\delta G}{\delta \boldsymbol{E}_{\perp -k}^{(1)}}\cdot\frac{\delta F}{\delta \boldsymbol{A}_{\perp k}^{(1)}}\right)\right\}. \tag{67}$$

We now have a single longitudinal equation of motion:

$$\dot{f}_{\parallel k}^{(1)} + ikv_{\parallel} f_{\parallel k}^{(1)} + \frac{e}{m}E_{\parallel k}^{(1)} f_{\parallel}^{(0)\prime} = 0\,, \tag{68}$$

with the understanding that $E_{\parallel k}^{(1)}$ is be shorthand for the solution of Poisson's equation.

3

Van Kampen Modes and Integral Transforms

Exact solutions of the linearized Maxwell-Vlasov equations have been known for some time. The longitudinal equation was first solved by van Kampen[13] for the case of stable equilibria by means of a singular eigenfunction expansion. Later Felderhof[14] applied these same methods to the transverse equations again for stable equilibria. Case[15,16] and others[17,18] computed the eigenfunctions associated with neutrally stable and unstable modes for longitudinal motion which rounded out the treatment plasma oscillations in Vlasov theory. The drawback to the Case-van Kampen picture of plasma oscillations is the seemingly *ad hoc* manner in which the discrete modes, especially the embedded neutral modes, must be added to the continuum.

We approach the solution of linearized equations from a different perspective — that of integral transforms. Our ultimate goal is a coordinate transformation that will simultaneously diagonalize the bracket and the Hamiltonian. The language of integral transforms seems naturally suited to this line of investigation. As we shall see in subsequent chapters, this approach has the added benefit of giving rise, in a natural way, to the necessary discrete modes in the unstable and marginally stable cases and to providing a clear prescription for computing the relevant discrete eigenfunctions.

In this chapter we develop the theory of specific members of a large family of linear integral transforms, whose kernels are intimately related to van Kampen modes, that will accomplish the above goal. Although the longitudinal and transverse modes have much different transforms, they belong to the same general family which we can represent as a linear functional, $\mathcal{G}[\phi]$, defined by

$$\mathcal{G}[\phi](x) = \int dy\, \mathcal{G}(x,y)\, \phi(y) = \alpha\overline{\phi} + \beta\,\phi\,, \tag{69}$$

where the overbar indicates the Hilbert transform. (See Appendix A for a review of Hilbert transform theory.) Explicitly the kernel is,

$$\mathcal{G}(x,y) = \alpha(x)\,\frac{1}{\pi}\,\mathrm{P}\frac{1}{y-x} + \beta(x)\,\delta(y-x)\,, \tag{70}$$

where P denotes the Cauchy principal value. Notice that the inverse of $\mathcal{G}$, if it exists, is essentially a solution of the Riemann-Hilbert problem on the real axis, *i.e.* solving $\psi = \mathcal{G}[\phi]$ for ϕ amounts to solving the Riemann-Hilbert problem.[19,20]

Before embarking on a detailed study of the relevant members of this family, we provide some insight into the connection between this transform and the linearized Maxwell-Vlasov equations.

I. Motivation

To motivate the choice of this particular family, consider the following integro-differential equation:

$$\frac{\partial}{\partial t}\, g(x,t) + x\, g(x,t) + \frac{1}{\pi}\,\rho(x) \int_{-\infty}^{\infty} dx'\, g(x',t) = 0\,. \tag{71}$$

Clearly the longitudinal Vlasov equation is of this form.

Faced with a complicated ordinary or partial differential equation, one possible course of action is to seek an integral transform such that the differential operator acting on the transformed variable is more tractable.[21,22] This technique is often referred to as Laplace's method. As an example, let $\mathfrak{L}_x$ be a linear differential operator in x and suppose we wish to solve

$$\mathfrak{L}_x\,[u(x)] = 0\,. \tag{72}$$

Let A be a linear functional and consider the change of variables

$$u(x) = A[v(\xi)](x) \equiv \int_C d\xi\, \mathcal{A}(\xi,x)\, v(\xi)\,. \tag{73}$$

The differential equation now reads

$$0 = \mathfrak{L}_x\,[A\,[v(\xi)]] \equiv A\left[\mathfrak{M}_\xi^\dagger\,[v(\xi)]\right], \tag{74}$$

where $\mathfrak{M}_\xi^\dagger$ is the adjoint of the operator defined by

$$\mathfrak{L}_x\,[\mathcal{A}(\xi,x)] = \mathfrak{M}_\xi\,[\mathcal{A}(\xi,x)]\,. \tag{75}$$

Assuming that A has a trivial null space, the differential equation is equivalent to

$$\mathfrak{M}_\xi^\dagger\,[v(\xi)] = 0\,. \tag{76}$$

The strategy is to use the freedom in the choice of A and the contour of integration, C, to make $\mathfrak{M}_\xi$ as simple as possible.

We approach the solution of (71) with Laplace's method in mind. It is the presence of the integral term in (71) that greatly complicates finding the solution. Having made this observation, it seems natural to wonder if there exists a linear transformation,

$$g(x,t) = A\,[q(\xi,t)] \equiv \int_{-\infty}^{\infty} d\xi\, \mathcal{A}(x,\xi)\, q(\xi,t)\,, \tag{77}$$

such that (71) becomes

$$A\left[\frac{\partial}{\partial t}q(\xi,t)+c(\xi)\,q(\xi,t)\right]=0\,. \tag{78}$$

We have chosen the contour C to be the real axis which we now take to be our domain of integration. Thus we want our transform to have the property

$$x\,A\,[q]+\frac{1}{\pi}\,\rho(x)\int dx'\,A\,[q]\,(x')=A\,[c\,q]\,. \tag{79}$$

Although one might think the existence of an A with this property unlikely, we do have considerable freedom since $c(\xi)$ can be chosen as needed. We are considering a linear transformation, therefore it is reasonable to expect that we can normalize $\mathcal{A}$ such that

$$\int dx'\,A\,[q]=\int d\xi'\,q(\xi',t)\,, \tag{80}$$

or equivalently

$$\int dx'\,\mathcal{A}(x',\xi)=1\,. \tag{81}$$

Using this normalization, we can write the required property of A as

$$\int d\xi\,\left[x\mathcal{A}(x,\xi)+\frac{1}{\pi}\,\rho(x)-c(\xi)\mathcal{A}(x,\xi)\right]\,q(\xi,t)=0\,. \tag{82}$$

Certainly one way to satisfy this equation is

$$\frac{1}{\pi}\,\rho(x)+[x-c(\xi)]\,\mathcal{A}(x,\xi)=0\,. \tag{83}$$

We have not yet had to choose $c(\xi)$; the simplest reasonable choice is $c(\xi)=\xi$. With this choice, the equation for the kernel becomes

$$\frac{1}{\pi}\,\rho(x)+(x-\xi)\mathcal{A}(x,\xi)=0\,, \tag{84}$$

which has as its solution

$$\mathcal{A}(x,\xi) = \frac{1}{\pi}\,\mathrm{P}\frac{\rho(x)}{\xi - x} + \lambda(\xi)\,\delta(x-\xi)\,, \tag{85}$$

where P denotes the Cauchy Principal value and $\lambda(\xi)$ is, in general, an arbitrary function. The normalization that we have chosen for $\mathcal{A}$ fixes λ:

$$1 = \int dx'\,\mathcal{A}(x',\xi) = -\overline{\rho}(\xi) + \lambda(\xi)\,, \tag{86}$$

implying

$$\lambda = 1 + \overline{\rho}. \tag{87}$$

Combining these results we find that the kernel of the our transformation is given by

$$\mathcal{A}(x,\xi) = \frac{1}{\pi}\,\frac{\rho(x)}{x-\xi} + [1+\overline{\rho}(\xi)]\,\delta(x-\xi)\,, \tag{88}$$

and thus

$$A\,[q] = \rho\,\overline{q} + (1+\overline{\rho})\,q\,. \tag{89}$$

The kernel we have derived here is essentially the longitudinal van Kampen mode. (See Appendix D of Morrison and Pfirsh.[23]) Therefore, it seems reasonable to expect that the appropriate transform for the transverse case will have as its kernel the transverse van Kampen mode. Both the longitudinal and transverse transforms are members of a larger family of transforms:

$$\mathcal{G}\,[\phi] = \alpha\,\overline{\phi} + \beta\,\phi, \tag{90}$$

where the relationship between α and β is arbitrary. This family turns out to have a very rich structure, especially in the case where $\beta - \overline{\alpha}$ is not a constant.

This derivation of the transform overlooks many important issues such as invertibility and the nature of the domain and range of $\mathcal{G}$. These issues will be discussed in some detail below, especially regarding their implications concerning existence and completeness of solutions of the linearized Maxwell-Vlasov equations expressed in terms of transforms.

Prior to a discussion of the properties of these we need to introduce certain concepts from the theory of Hilbert transforms. (See Appendix A for a review of the theory of Hilbert transforms.) Of the ways of formulating this theory our purposes are best suited by casting it in the language of Hölder classes. A function $\phi : R \mapsto R$ is said to satisfy the Hölder condition of index μ if

$$|\phi(x) - \phi(y)| \leq A\,|x - y|^{\mu} \qquad \forall x, y \in R, \tag{91}$$

where $A > 0$ and $0 < \mu \leq 1$. We denote the class of functions satisfying the Hölder condition by $\mathcal{H}^{\mu}$. There is a sub-class of Hölder functions that will be of special interest to us. Namely those functions which, in addition to belonging to $\mathcal{H}^{\mu}$, possess a limit ϕ^{∞}, as $|x| \longrightarrow \infty$ and, for sufficiently large $|x|$, satisfy

$$|\phi(x) - \phi^{\infty}| \leq \frac{A'}{|x|^{\alpha}}, \tag{92}$$

where A', $\alpha > 0$. We will denote this restricted class by $\mathcal{H}^{\mu}_{*}$. the most important property of this class of functions is that the Hilbert transform of any element of $\mathcal{H}^{\mu}_{*}$ us guaranteed to exist and will also belong to $\mathcal{H}^{\mu}_{*}$.

II. The Longitudinal Transform

As we have seen the transform of interest for the longitudinal equations belongs to a restricted form of the general transform where

$$\beta = \overline{\alpha} + \beta^{\infty} \tag{93}$$

and β^{∞} is an arbitrary constant that determines the value of β as $|x| \longrightarrow \infty$, since

$$\lim_{|x| \to \infty} \overline{\alpha}(x) = 0. \tag{94}$$

As we will see below, α is closely related to the equilibrium plasma distribution function which, for compelling physical reasons, one expects to be "gaussian-like" in character — *e.g.* some smooth (polynomial) function multiplying a gaussian. In which case, it is reasonable to assume $\alpha \in \mathcal{H}_*^1$, that all integrals involving polynomials multiplying α will exist and further that $\alpha \in L^{\infty}$. Thus $\beta \in \mathcal{H}_*^1 \cap L^{\infty}$ and $\overline{\beta} = -\alpha$.

To consider the action of $\mathcal{G}$ on $\phi \in \mathcal{H}_*^{\mu}$ for some $\mu \leq 1$ let

$$\psi(x) = \mathcal{G}[\phi](x) = \alpha\overline{\phi} + \beta\phi. \tag{95}$$

Since $\overline{\phi} \in \mathcal{H}_*^{\mu}$ and, by assumption, α and β are both in $\mathcal{H}_*^1$, the properties of Hölder functions, (476), tells us that ψ also belongs to $\mathcal{H}_*^{\mu}$. Thus we see that $\mathcal{G}$ maps $\mathcal{H}_*^{\mu}$ into $\mathcal{H}_*^{\mu}$ for all $\mu \leq 1$.

The ultimate application of this transform requires that $\psi \in L^1$. Define

$$\mathcal{D}_L^{\mu} = \left\{\phi : \phi \in \mathcal{H}_*^{\mu} \cap L^1\right\}, \tag{96}$$

Since α and $\beta \in L^{\infty}$, $\mathcal{G}[\phi] \in L^1$ and thus $\mathcal{G}$ maps $\mathcal{D}_L^{\mu}$ into $\mathcal{D}_L^{\mu}$.

For this transform to be of use, we must have an expression for the inverse. By means of the convolution theorem, (487), we can compute

$$\begin{aligned}\overline{\psi} &= \overline{\alpha\overline{\phi}} + \overline{\beta\phi} \\ &= \overline{\alpha\overline{\phi}} + \beta\overline{\phi} + \overline{\beta}\phi + \overline{\overline{\beta}\overline{\phi}} \\ &= \overline{\alpha\overline{\phi}} + \beta\overline{\phi} - \alpha\phi - \overline{\alpha\overline{\phi}} \\ &= \beta\overline{\phi} - \alpha\phi. \end{aligned} \tag{97}$$

From the expressions for ψ and $\overline{\psi}$ we see that

$$\beta\psi - \alpha\overline{\psi} = \left(\beta^2 + \alpha^2\right)\phi. \tag{98}$$

So provided

$$\alpha^2 + \beta^2 \neq 0 \tag{99}$$

we can solve for ϕ:

$$\phi = \frac{\beta\psi - \alpha\overline{\psi}}{\alpha^2 + \beta^2}. \tag{100}$$

Therefore, subject to the restriction (99), the transform $\mathcal{G}$ has an inverse, $\widetilde{\mathcal{G}}$, given by

$$\widetilde{\mathcal{G}}[\psi] = \zeta\overline{\psi} + \chi\psi, \tag{101}$$

where

$$\chi = \frac{\beta}{\alpha^2 + \beta^2}, \tag{102a}$$

$$\zeta = -\frac{\alpha}{\alpha^2 + \beta^2}. \tag{102b}$$

Notice that

$$\chi + i\zeta = \frac{1}{\beta + i\alpha}. \tag{103}$$

From Hilbert's theorem, (491), we know that $\beta + i\alpha$ is the limiting value to the real axis of a function analytic in the upper half-plane. If (99) was extended to apply in the upper half-plane in addition to on the real axis, then $\chi + i\zeta$ would also be the limiting value of an analytic function and, due to Hilbert's theorem

$$\chi = \overline{\zeta} + \chi^{\infty}, \tag{104}$$

where $\chi^{\infty} = 1/\beta^{\infty}$. In this case $\widetilde{\mathcal{G}}$ satisfies the same restriction on its coefficient functions as does $\mathcal{G}$. As we will see below, transforms satisfying these restrictions form an infinite dimensional group.

Clearly (99) holding on the real axis is sufficient for $\widetilde{\mathcal{G}}$ to be well defined, however, one finds that (104) is required to guarantee that $\mathcal{G}$ maps $\mathcal{D}_L^\mu$ *onto* $\mathcal{D}_L^\mu$. More precisely, if (104) does not hold, then there will be elements of the range of $\mathcal{G}$ that are in the null space of $\widetilde{\mathcal{G}}$. Since we are planning to use $\mathcal{G}$ to represent the solutions of the linearized Vlasov equation and there appears to be no physical reasons to exclude as solutions some elements of $\mathcal{D}_L^\mu$ but not others, we will require $\mathcal{G}$ to be "onto" and thus we require (99) to hold in the upper half-plane. It turns out that (99) is related to the stability of the equilibrium plasma distribution. We relax this condition in Chapters 6 and 7 and find that failure of $\mathcal{G}$ to be "onto" is not indicative of the exclusion of certain elements of $\mathcal{D}_L^\mu$ from the set of solutions of the Vlasov equation but rather of the inability of this transform to represent *all* solutions.

Under the change of variables $\psi = \mathcal{G}[\phi]$, the functional derivative with respect to ϕ transforms according to the adjoint transformation, $\mathcal{G}^\dagger$. (See Appendix D for details.) The adjoint is defined by

$$\int dx\, \vartheta\, \mathcal{G}[\varphi] = \int dx\, \varphi\, \mathcal{G}^{\dagger}[\vartheta]. \tag{105}$$

We can compute $\mathcal{G}^{\dagger}$ directly from this definition *viz.*

$$\begin{aligned} \int dx\, \vartheta\, \mathcal{G}[\varphi] &= \int dx\, \left(\vartheta\, \beta\, \varphi + \vartheta\, \alpha\, \overline{\varphi}\right) = \int dx\, \left(\vartheta\, \beta\, \varphi - \overline{\vartheta\, \alpha}\, \varphi\right) \\ &= \int dx\, \varphi\, \left(\beta\, \vartheta - \overline{\alpha\, \vartheta}\right) \equiv \int dx\, \varphi\, \mathcal{G}^{\dagger}[\vartheta]. \end{aligned} \tag{106}$$

Since (106) must hold for all ϑ and φ, we conclude that

$$\mathcal{G}^{\dagger}[\vartheta] = -\overline{\alpha\vartheta} + \beta\,\vartheta \tag{107}$$

and similarly

$$\widetilde{\mathcal{G}}^{\dagger}[\vartheta] = -\overline{\zeta\vartheta} + \chi\,\vartheta. \tag{108}$$

There are many identities involving $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ that can be proved. Below we give two identities that will subsequently be required for the transformation to diagonal form. Let ϕ and $\psi \in \mathcal{D}_{L}^{\mu}$ with the necessary behaviour for large x such that all integrals exist. We consider:

$$\int dx\, \alpha\, \widetilde{\mathcal{G}}^{\dagger}[\phi]\, \widetilde{\mathcal{G}}^{\dagger}[\psi] = -\int dx\, \zeta\phi\psi; \tag{109}$$

$$\int dx\, \frac{x}{\alpha}\, \mathcal{G}[\phi]\, \mathcal{G}[\psi] = -\int dx\, \frac{x}{\zeta}\, \phi\,\psi - \frac{\beta^{\infty}}{\pi}\int dx\, \phi \int dx\, \psi. \tag{110}$$

The general strategy for proving these and other identities is to alternately apply the convolution theorem, (487), and the two forms of Parseval's formula, (488*a*) and (488*b*), all the while avoiding traveling in circles.

We prove the first identity by substitution of the explicit expressions for $\widetilde{\mathcal{G}}^\dagger[\phi]$ and $\widetilde{\mathcal{G}}^\dagger[\psi]$, *viz.*

$$\begin{aligned}\int dx\,\alpha\,\widetilde{\mathcal{G}}^\dagger[\phi]\,\widetilde{\mathcal{G}}^\dagger[\psi] &= \int dx\,\alpha\,(\chi\phi - \overline{\zeta\phi})\,(\chi\psi - \overline{\zeta\psi}) \\ &= \int dx\,\alpha\,(\chi^2\phi + \overline{\zeta\phi}\,\overline{\zeta\psi}) - \int dx\,\alpha\chi\,(\phi\overline{\zeta\psi} + \overline{\zeta\phi}\psi)\,. \end{aligned} \tag{111}$$

First consider the term that contains two Hilbert transforms:

$$\begin{aligned}\int dx\,\alpha\overline{\zeta\phi}\,\overline{\zeta\psi} &= -\int dx\,\overline{\alpha\overline{\zeta\psi}}\,\zeta\phi \\ &= \int dx\,\zeta\phi\,(\alpha\zeta\psi - \overline{\alpha}\,\overline{\zeta\psi} + \overline{\overline{\alpha}\zeta\psi}) \\ &= \int dx\,\alpha\zeta^2\phi\psi - \int dx\,\overline{\alpha}\,(\zeta\phi\overline{\zeta\psi} + \zeta\psi\overline{\zeta\phi})\,, \end{aligned} \tag{112}$$

where we used Parseval's formula on the last term. Using (112), the right-hand side of (111) becomes

$$\int dx\,\alpha\,(\zeta^2 + \chi^2)\,\phi\psi - \int dx\;(\phi\overline{\zeta\psi} + \psi\overline{\zeta\phi})\,(\overline{\alpha}\zeta + \alpha\chi)\,. \tag{113}$$

From the definitions of χ and ζ we see that

$$\alpha\chi = -\beta\,\zeta \tag{114}$$

whence, the second integral in (113) becomes

$$\int dx\;(\phi\overline{\zeta\psi} + \psi\overline{\zeta\phi})\,\zeta\,(\overline{\alpha} - \beta) = -\beta^\infty \int dx\;(\zeta\phi\overline{\zeta\psi} + \zeta\psi\overline{\zeta\phi}) = 0\,. \tag{115}$$

Therefore

$$\int dx\,\alpha\,\widetilde{\mathcal{G}}^\dagger[\phi]\,\widetilde{\mathcal{G}}^\dagger[\psi] = \int dx\,\alpha\,(\chi^2 + \zeta^2)\,\phi\psi = -\int dx\,\zeta\phi\psi\,, \tag{116}$$

where the last step follows since

$$\chi^2 + \zeta^2 = \frac{1}{\alpha^2 + \beta^2} = \frac{\zeta}{\alpha}. \tag{117}$$

To prove the second identity, we first consider

$$\begin{aligned}
\int dx\, \frac{1}{\alpha}\, \mathcal{G}[\phi]\, \mathcal{G}[\psi] &= \int dx\, \frac{1}{\alpha}\left(\alpha\overline{\phi} + \beta\,\phi\right)\left(\alpha\overline{\psi} + \beta\,\psi\right) \\
&= \int dx\, \frac{\beta^2}{\alpha}\,\phi\psi + \int dx\, \alpha\overline{\phi}\,\overline{\psi} + \int dx\, \beta\left(\overline{\phi}\psi + \phi\overline{\psi}\right) \\
&= \int dx\, \frac{\beta^2}{\alpha}\,\phi\psi + \int dx\, \alpha\overline{\phi}\,\overline{\psi} \\
&\qquad + \int dx\, \overline{\alpha}\left(\overline{\overline{\phi}\,\overline{\psi}} - \overline{\overline{\phi}}\,\overline{\overline{\psi}}\right) + \int dx\, \beta^{\infty}\left(\overline{\phi}\,\psi + \phi\overline{\psi}\right) \\
&= \int dx\, \frac{\beta^2}{\alpha}\,\phi\psi + \int dx\, \overline{\alpha}\,\overline{\phi\psi} \\
&= \int dx\, \frac{\beta^2}{\alpha}\,\phi\psi + \int dx\, \alpha\,\phi\,\psi.
\end{aligned} \tag{118}$$

Using the definition of ζ, (102b), we find

$$\int dx\, \frac{1}{\alpha}\, \mathcal{G}[\phi]\, \mathcal{G}[\psi] = -\int dx\, \zeta\phi\psi. \tag{119}$$

To obtain (110) we make the substitution $\phi \longrightarrow x\,\phi$ in (118). Using (490a), we see that

$$\mathcal{G}[x\,\phi] = x\,\mathcal{G}[\phi] + \frac{\alpha}{\pi}\int dx\,\phi, \tag{120}$$

giving

$$\int dx\, \frac{x}{\alpha}\, \mathcal{G}[\phi]\, \mathcal{G}[\psi] + \frac{1}{\pi}\int dx\,\phi \int dx\, \mathcal{G}[\psi] = -\int dx\, x\,\zeta\phi\,\psi. \tag{121}$$

Now

$$\mathcal{G}^{\dagger}[1] = \beta - \overline{\alpha} = \beta^{\infty}. \tag{122}$$

Thus

$$\int dx\, \frac{x}{\alpha}\, \mathcal{G}[\phi]\, \mathcal{G}[\psi] = -\frac{\beta^{\infty}}{\pi} \int dx\, \phi \int dx\, \psi - \int dx\, x\, \zeta\, \phi\, \psi\,. \tag{123}$$

As we alluded to above, the restricted family of transforms that we have been studying form an infinite dimensional group. We now describe its group composition law, adopting a slightly different notation for convenience. We denote $\mathcal{G}$ by

$$\mathcal{G}[\phi;\beta,\alpha] = \alpha\,\overline{\phi} + \beta\,\phi\,. \tag{124}$$

The composition of two such transforms can be written out explicitly as follows:

$$\begin{aligned}
\mathcal{G}[\mathcal{G}[\phi;\beta_1,\alpha_2];\beta_2,\alpha_2] &= \alpha_2\,\overline{\alpha_1\,\overline{\phi} + \beta_1\,\phi} + \beta_2\left(\alpha_1\,\overline{\phi} + \beta_1\,\phi\right) \\
&= \alpha_2\left(\beta_1\,\overline{\phi} - \alpha_1\,\phi\right) + \beta_2\left(\alpha_1\,\overline{\phi} + \beta_1\,\phi\right) \\
&= (\alpha_1\,\beta_2 + \beta_1\,\alpha_2)\,\overline{\phi} + (\beta_1\,\beta_2 - \alpha_1\,\alpha_2)\,\phi \\
&= \mathcal{G}[\phi;\beta_3,\alpha_3]\,,
\end{aligned} \tag{125}$$

where

$$\beta_3 + i\,\alpha_3 = (\beta_1 + i\,\alpha_1)(\beta_2 + i\,\alpha_2)\,. \tag{126}$$

The identity transformation can be expressed as $\mathcal{G}[\phi;1,0]$. Suppose that (β',α') are the parameters for the inverse of the transformation with parameters (β,α). From the composition rule, we see immediately that

$$\beta' + i\,\alpha' = \frac{1}{(\beta + i\,\alpha)}\,, \tag{127}$$

which is exactly what we found above by directly solving for the inverse. The condition (99) guarantees that α' and β' are well defined.

III. Transverse Transform

The transform of interest for the transverse equations has a much richer structure than the longitudinal transform. The transverse transform is defined by

$$\beta = \overline{\alpha} + x - \frac{\gamma^2}{x}, \tag{128}$$

where γ is a constant. Thus

$$\mathcal{G}[\phi] = \alpha\overline{\phi} + \overline{\alpha}\phi + \left(x - \frac{\gamma^2}{x}\right)\phi. \tag{129}$$

One might justifiably be concerned about the behaviour of β at the origin and at infinity. As it turns out, we will only be interested in the action of $\mathcal{G}$ on functions ϕ such that $\beta\phi$ is well behaved at both of these points. For the same reasons as in the longitudinal case, we will assume that $\alpha \in \mathcal{H}_*^1$. In addition, we also assume that $x\alpha \in \mathcal{H}_*^1$. Clearly $\beta \notin \mathcal{H}_*^\mu$ for any μ and moreover $\overline{\beta}$ does not exist. It is this property of β that gives this transform the richer structure mentioned above.

We begin the study of this transform by defining a subspace of $\mathcal{H}_*^\mu$. Define

$$\mathcal{D}_T^\mu = \left\{\phi : \phi, \frac{\phi}{x} \in \mathcal{H}_*^\mu \cap L^1\right\}. \tag{130}$$

Let $\psi = \mathcal{G}[\phi]$ where $\phi \in \mathcal{D}_T^\mu$ for some $\mu \leq 1$. From the definition of $\mathcal{D}_T^\mu$, we see that $\varphi = \phi/x$ belongs to $\mathcal{H}_*^\mu \cap L^1$. We can write ψ as

$$\begin{aligned}
\psi = \mathcal{G}[x\varphi] &= \alpha\overline{x\varphi} + \beta x\varphi \\
&= \alpha x\overline{\varphi} + \frac{1}{\pi}\alpha\int dx'\,\varphi(x') + \overline{\alpha}x\varphi + x^2\varphi - \gamma\varphi \\
&= x\left(\alpha\overline{\varphi} + \overline{\alpha}\varphi + x\varphi\right) - \gamma\varphi + \frac{1}{\pi}\alpha\int dx'\,\varphi(x').
\end{aligned} \tag{131}$$

By virtue of the properties of ϕ and our assumption about α, all of the terms in the above are either in $\mathcal{H}_*^\mu$ or $\mathcal{H}_*^1$. Thus we see that $\psi \in \mathcal{H}_*^\mu$ and therefore $\mathcal{G}$ maps $\mathcal{D}_T^\mu$ into $\mathcal{H}_*^\mu$. At this point it is unclear whether this mapping is "onto". We will explore this question by computing the inverse of $\mathcal{G}$.

We can use the convolution theorem, (487), to rewrite ψ as

$$\psi = \overline{\alpha\phi} - \overline{\alpha}\overline{\phi} + \left(x - \frac{\gamma^2}{x}\right)\phi. \tag{132}$$

To compute $\overline{\psi}$, we make use of (490a) and (490b) and the definition of β, giving

$$\overline{\psi} = \beta\overline{\phi} - \alpha\phi + \frac{\gamma^2}{x}A + B, \tag{133}$$

where

$$A = \frac{1}{\pi}\int dx\, \frac{\phi}{x} \tag{134}$$

and

$$B = \frac{1}{\pi}\int dx\, \phi. \tag{135}$$

By taking an appropriate combination of (132) and (133) we can eliminate $\overline{\phi}$ and obtain an equation for ϕ:

$$\beta\psi - \alpha\overline{\psi} = (\beta^2 + \alpha^2)\phi - \frac{\gamma^2}{x}\alpha A - \alpha B. \tag{136}$$

Thus we have an equation that can be solved for ϕ provided that

$$\beta^2 + \alpha^2 \neq 0 \tag{137}$$

for all x. In this case we find

$$\phi = x\chi\psi + x\zeta\overline{\psi} - \gamma^2\zeta A - x\zeta B, \tag{138}$$

where

$$\chi = \frac{\beta}{x\,(\beta^2 + \alpha^2)} = \frac{x\,\beta}{(x\,\beta)^2 + (x\,\alpha)^2}, \tag{139a}$$

$$\zeta = -\frac{\alpha}{x\,(\beta^2 + \alpha^2)} = -\frac{x\,\alpha}{(x\,\beta)^2 + (x\,\alpha)^2}. \tag{139b}$$

Note that χ, ζ and ζ/x are bounded as $x \longrightarrow 0$ and χ, ζ, $x\,\chi$ and $x\,\zeta$ are bounded as $x \longrightarrow \infty$. Furthermore, since α, $x\,\alpha \in \mathcal{H}^1_*$ and $[(x\,\beta)^2 + (x\,\alpha)^2]^{-1}$ is essentially bounded, we see that ζ, $x\,\zeta$ and ζ/x belong to $\mathcal{H}^1_* \cap L^1$ and thus ζ, $x\,\zeta$ are in $\mathcal{D}^{\mu}_{T}$. One can readily see that

$$\chi + i\,\zeta = \frac{1}{x(\beta + i\,\alpha)}. \tag{140}$$

Extending the condition (137) to the upper half-plane, $\chi + i\,\zeta$ is then boundary value of an analytic function and Hilbert's theorem, (491), tells us that

$$\chi = \overline{\zeta} + \chi^{\infty}. \tag{141}$$

From the explicit expression for χ, we see that $\chi^{\infty} = 0$ and thus

$$\chi = \overline{\zeta} \tag{142}$$

and $\chi \in \mathcal{H}^1_*$. As in the longitudinal case, (137) is related to stability of the equilibrium distribution and thus the requirement that (137) holds in the upper-half plane is physically justified.

Note in our expression relating ϕ to ψ and $\overline{\psi}$, there are two (seemingly) arbitrary constants, A and B. Before we say more about the significance of these constants, we need to establish some properties of ζ.

We start with the observation that

$$\overline{\alpha}\zeta + \alpha\overline{\zeta} = \frac{\alpha(x^2 - \gamma^2)}{x^2\,(\beta^2 + \alpha^2)} = -\left(x - \frac{\gamma^2}{x}\right)\zeta. \tag{143}$$

An immediate consequence of this, in view of Parseval's formula, (488b), is that

$$\int dx\, x\,\zeta = \gamma^2 \int dx\, \frac{\zeta}{x}. \tag{144}$$

Now

$$\int dx\, \frac{\zeta}{x} = \pi\,\overline{\zeta}(0) = \pi\,\chi(0) = -\frac{\pi}{\gamma^2}, \tag{145}$$

from which, through (144), we conclude

$$\int dx\, x\,\zeta = -\pi\,. \tag{146}$$

As we saw above, $\chi + i\,\zeta$ is the limiting value, as $\mathrm{Im}(z) \longrightarrow 0^+$, of an analytic function which implies, in view of the behaviour of $\chi + i\,\zeta$ for large x, that $x(\chi + i\,\zeta)$ is also the boundary value of an analytic function. Thus

$$x\,\chi = \overline{x\,\zeta} + (x\,\chi)^\infty. \tag{147}$$

From (139a), we see that $(x\,\chi)^\infty = 0$ and so

$$x\,\chi = \overline{x\,\zeta} \tag{148}$$

and $x\,\chi \in \mathcal{H}^1_*$. Furthermore,

$$\int dx\, \zeta = \pi\,\overline{x\,\zeta}(0) = x\,\chi\Big|_{x=0} = 0\,. \tag{149}$$

Now

$$x\,\chi = x\,\frac{x\,\beta}{(x\beta)^2 + (x\,\alpha)^2} \tag{150}$$

and $x\,\beta \longrightarrow -\gamma^2$ as $x \longrightarrow 0$, giving

$$\int dx\, \zeta = 0\,. \tag{151}$$

Using this result, we see that

$$\overline{x\zeta} = x\overline{\zeta} \tag{152}$$

which, when combined with (143) and (488*b*), gives

$$\int dx\,(x^2-\gamma^2)\zeta = \int dx\ \left(\overline{\alpha}\,x\,\zeta + \alpha\overline{x\,\zeta}\right) = 0 \tag{153}$$

implying

$$\int dx\,x^2\zeta = 0\,. \tag{154}$$

For completeness, we state one further relation that will be of use later:

$$x\,(\overline{\alpha}\,\overline{\zeta} - \alpha\,\zeta) = 1 - (x^2-\gamma^2)\,\overline{\zeta}\,. \tag{155}$$

We now return to the question of the meaning of the constants A and B in

$$\phi = x\chi\psi + x\zeta\overline{\psi} - \gamma^2\zeta A - x\zeta B\,. \tag{138}$$

To understand their rôle, consider

$$\begin{aligned}\frac{1}{\pi}\int dx\,\frac{\phi}{x} &= \frac{1}{\pi}\int dx\ \left(\overline{\zeta}\,\psi + \zeta\overline{\psi} - \gamma^2\frac{\zeta}{x}A - \zeta B\right)\\ &= -A\,\frac{\gamma^2}{\pi}\int dx\,\frac{\zeta}{x}\\ &= A\,,\end{aligned} \tag{156}$$

where we have used (145) and (151). Further, consider

$$\begin{aligned}\frac{1}{\pi}\int dx\,\phi &= \frac{1}{\pi}\int dx\ \left(x\,\overline{\zeta}\,\psi + x\,\zeta\overline{\psi} - \gamma^2\zeta A - x\,\zeta B\right)\\ &= -B\,\frac{1}{\pi}\int dx\,x\,\zeta\\ &= B\,.\end{aligned} \tag{157}$$

This is in complete agreement with the original definitions of A and B and so our solution is seen to be self-consistent. Thus, when we solve $\psi = \mathcal{G}[\phi]$ for ϕ, we find that ϕ is only partially determined; the integrals given by (134) and (135) are arbitrary. This tells us that the functions multiplying A and B in the equation for ϕ, (138), are null vectors of the transform $\mathcal{G}$. Specifically, defining

$$\eta_1 = \zeta\,, \tag{158a}$$

$$\eta_2 = x\zeta\,, \tag{158b}$$

it is easy to verify that $\mathcal{G}[\eta_1] = 0$ and $\mathcal{G}[\eta_2] = 0$. For a function ϕ such that $A = 0$ and $B = 0$, $\psi = \mathcal{G}[\phi]$ can be uniquely solved for ϕ and

$$\phi = x\chi\psi + x\zeta\overline{\psi} \equiv \widetilde{\mathcal{G}}[\psi]\,. \tag{159}$$

We are thus lead to seek a method of splitting the domain of $\mathcal{G}$, $\mathcal{D}^{\mu}_{T}$, into two parts: those functions for which $A = 0$ and $B = 0$ — *i.e.* those functions that can be represented by $\widetilde{\mathcal{G}}$ — and those that cannot be so represented. This brings us to the topic of projection operators.

A. Projector Operators

As we have seen $\mathcal{G}$ has a non-trivial null space and thus the inverse transformation is only well defined on the space of functions that have no projection into this null space. Here we define two complimentary projection operators $\mathcal{P}$ and $\mathcal{Q}$ that divide the domain of $\mathcal{G}$ into two spaces; the space of all functions which can be represented by $\widetilde{\mathcal{G}}$ and the null space of $\mathcal{G}$. Let $\phi \in \mathcal{D}^{\mu}_{T}$ and define $\mathcal{P}$ by

$$\mathcal{P}[\phi] = \phi + \frac{1}{\pi}x\zeta\int dx\,\phi + \frac{\gamma^2}{\pi}\zeta\int dx\,\frac{\phi}{x}\,. \tag{160}$$

Since $\mathcal{Q}$ is complimentary to $\mathcal{P}$,

$$\mathcal{Q}[\phi] = -\frac{1}{\pi} x \zeta \int dx\, \phi - \frac{\gamma^2}{\pi} \zeta \int dx\, \frac{\phi}{x}. \tag{161}$$

Using the various properties of ζ derived above, it is easy to show that

$$\int dx\, \mathcal{P}[\phi] = 0 \tag{162}$$

and

$$\int dx\, \frac{\mathcal{P}[\phi]}{x} = 0\,, \tag{163}$$

from which we can see that, as claimed, $\mathcal{P}$ is a projection operator:

$$\mathcal{P}[\mathcal{P}[\phi]] = \mathcal{P}[\phi] + \frac{1}{\pi} x \zeta \int dx'\, \mathcal{P}[\phi] + \frac{\gamma^2}{\pi} \zeta \int dx'\, \frac{\mathcal{P}[\phi]}{x'} = \mathcal{P}[\phi]\,, \tag{164}$$

and that $\mathcal{P}[\phi] \in \mathcal{D}_T^\mu$. Since $\mathcal{Q}[\phi]$ is a linear combination of the null vectors of $\mathcal{G}$, we see that $\mathcal{P}$, being complimentary to $\mathcal{Q}$, projects into the non-null space of $\mathcal{G}$. Thus if $\phi = \mathcal{P}[\phi]$, then $\psi = \mathcal{G}[\phi]$ can be uniquely solved for ϕ. Hence defining

$$\phi_{\mathcal{P}} = \mathcal{P}[\phi]\,, \tag{165a}$$

$$\phi_{\mathcal{Q}} = \mathcal{Q}[\phi]\,, \tag{165b}$$

we can write any element $\phi \in \mathcal{D}_T^\mu$ as

$$\phi = \phi_{\mathcal{P}} + \phi_{\mathcal{Q}} = \widetilde{\mathcal{G}}\,[\psi] + \phi_{\mathcal{Q}} \tag{166}$$

for some ψ.

A functional of ϕ may also be thought of as a functional of $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$. Thus

$$\delta F = \int dx\, \frac{\delta f}{\delta \phi} \delta\phi = \int dx\, \left\{ \frac{\delta F}{\delta \phi_{\mathcal{P}}} \delta\phi_{\mathcal{P}} + \frac{\delta F}{\delta \phi_{\mathcal{Q}}} \delta\phi_{\mathcal{Q}} \right\}. \tag{167}$$

There is a one-to-one relationship between $\delta\phi$ and $\delta\phi_{\mathcal{P}}$ and $\delta\phi_{\mathcal{Q}}$:

$$\delta\phi = \delta\phi_{\mathcal{P}} + \delta\phi_{\mathcal{Q}}, \tag{168a}$$

$$\delta\phi_{\mathcal{P}} = \mathcal{P}[\delta\phi], \tag{168b}$$

$$\delta\phi_{\mathcal{Q}} = \mathcal{Q}[\delta\phi]. \tag{168c}$$

Since $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{Q}^2 = \mathcal{Q}$,

$$\begin{aligned} \delta F &= \int dx \left\{ \frac{\delta F}{\delta\phi} \mathcal{P}[\delta\phi_{\mathcal{P}}] + \frac{\delta F}{\delta\phi} \mathcal{Q}[\delta\phi_{\mathcal{Q}}] \right\} \\ &= \int dx \left\{ \mathcal{P}^\dagger \left[\frac{\delta F}{\delta\phi} \right] \delta\phi_{\mathcal{P}} + \mathcal{Q}^\dagger \left[\frac{\delta F}{\delta\phi} \right] \delta\phi_{\mathcal{Q}} \right\}. \end{aligned} \tag{169}$$

Comparing expressions for δF, we see

$$\frac{\delta F}{\delta\phi_{\mathcal{P}}} = \mathcal{P}^\dagger \left[\frac{\delta F}{\delta\phi} \right], \tag{170a}$$

$$\frac{\delta F}{\delta\phi_{\mathcal{Q}}} = \mathcal{Q}^\dagger \left[\frac{\delta F}{\delta\phi} \right]. \tag{170b}$$

To find $\mathcal{P}^\dagger$ we use the definition of the adjoint:

$$\begin{aligned} \int dx\, \psi\, \mathcal{P}[\phi] &\equiv \int dx\, \phi\, \mathcal{P}^\dagger[\psi] \\ &= \int dx \left\{ \psi\,\phi + \frac{1}{\pi} x\,\zeta\,\psi \int dx'\, \phi(x') + \frac{\gamma^2}{\pi} \zeta\,\psi \int dx'\, \frac{\phi(x')}{x'} \right\} \\ &= \int dx'\, \phi(x') \Big\{ \psi(x') + \frac{1}{\pi} \int dx''\, x''\, \zeta(x'')\, \psi(x'') \\ &\qquad + \frac{1}{x'} \frac{\gamma^2}{\pi} \int dx''\, \zeta(x'')\, \psi(x'') \Big\}. \end{aligned} \tag{171}$$

Therefore,

$$\mathcal{P}^\dagger[\psi] = \psi + \frac{1}{\pi} \int dx'\, x'\, \zeta\,\psi + \frac{1}{x} \frac{\gamma^2}{\pi} \int dx'\, \zeta\,\psi. \tag{172}$$

Since $\mathcal{Q}^\dagger = 1 - \mathcal{P}^\dagger$,

$$\mathcal{Q}^\dagger[\psi] = -\frac{1}{\pi}\int dx'\, x'\,\zeta\psi - \frac{1}{x}\frac{\gamma^2}{\pi}\int dx'\,\zeta\psi\,. \tag{173}$$

There are two useful identities that are proved by direct calculation[24] using the definitions of $\mathcal{P}$ and $\mathcal{P}^\dagger$:

$$\int dx\,\frac{1}{x\,\zeta}\,\mathcal{P}[\phi]\,\mathcal{P}[\psi] = \int dx\,\frac{1}{x\,\zeta}\,\phi\psi + \frac{1}{\pi}\int dx\,\phi\int dx\,\psi + \frac{\gamma^2}{\pi}\int dx\,\frac{\phi}{x}\int dx\,\frac{\psi}{x}, \tag{174}$$

and

$$\int dx\,x^2\zeta\mathcal{P}^\dagger[\phi]\,\mathcal{P}^\dagger[\psi] = \int dx\,x^2\zeta\phi\psi$$
$$+\frac{1}{\pi}\int dx\,x\zeta\phi\int dx\,x^2\zeta\psi + \frac{\gamma^2}{\pi}\int dx\,x^2\zeta\phi\int dx\,x\zeta\psi\,. \tag{175}$$

From the definition of $\widetilde{\mathcal{G}}$, we see that $\widetilde{\mathcal{G}}[\psi]$ is well defined for any $\psi \in \mathcal{H}^\mu_*$, in particular it is well defined for $\psi = \mathcal{G}[\phi]$ for any $\phi \in \mathcal{D}^\mu_T$ and

$$\begin{aligned}\widetilde{\mathcal{G}}\,[\mathcal{G}\,[\phi]] &= x\,\chi\left(\alpha\overline{\phi} + \beta\,\phi\right) + x\,\zeta\left(\beta\,\overline{\phi} - \alpha\,\phi + \frac{1}{\pi}\int dx\,\frac{\phi}{x} + \frac{\gamma^2}{x}\int dx\,\phi\right)\\ &= x\,\overline{\phi}\,(\alpha\chi + \beta\zeta) + x\,\phi\,(\beta\chi - \alpha\zeta) + \frac{1}{\pi}x\,\zeta\int dx\,\frac{\phi}{x} + \frac{\gamma^2}{x}\int dx\,\phi\\ &= \phi + \frac{1}{\pi}x\,\zeta\int dx\,\frac{\phi}{x} + \frac{\gamma^2}{x}\int dx\,\phi\\ &= \mathcal{P}[\phi]\,.\end{aligned} \tag{176}$$

This is another way of stating that $\widetilde{\mathcal{G}}$ can only represent functions without projections into the null space of $\mathcal{G}$. Thus we see that $\mathcal{G}$ maps $\mathcal{P}[\mathcal{D}^\mu_T]$ onto $\mathcal{H}^\mu_*$ and $\widetilde{\mathcal{G}}$ maps $\mathcal{H}^\mu_*$ onto $\mathcal{P}[\mathcal{D}^\mu_T]$.

The adjoint transformation, $\mathcal{G}^{\dagger}$, and its inverse, $\widetilde{\mathcal{G}}^{\dagger}$, are of interest because of the properties of functional derivatives. It is a straightforward calculation to show

$$\mathcal{G}^{\dagger}[\phi] = \beta\phi - \overline{\alpha\phi} \tag{177}$$

and

$$\widetilde{\mathcal{G}}^{\dagger}[\phi] = x\chi\phi - \overline{x\zeta\phi}. \tag{178}$$

There are three identities that will be of great use for diagonalizing the bracket and Hamiltonian. Let ϕ and $\psi \in \mathcal{D}^{\mu}_{T}$ with the necessary behaviour for large x such that all integrals exist. Consider

$$\int dx\, x\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi] = -\int dx\, x^2\,\zeta\phi\psi \\ -\frac{1}{\pi}\int dx\, x\zeta\phi\int dx\, x^2\,\zeta\psi - \frac{1}{\pi}\int dx\, x^2\,\zeta\phi\int dx\, x\zeta\psi, \tag{179}$$

$$\int dx\,\frac{1}{\alpha}\,\mathcal{G}[\phi]\,\mathcal{G}[\psi] = -\int dx\,\frac{1}{x\zeta}\,\phi\psi - \frac{1}{\pi}\int dx\,\phi\int dx\,\psi - \frac{\gamma^2}{\pi}\int dx\,\frac{\phi}{x}\int dx\,\frac{\psi}{x}, \tag{180}$$

and

$$\widetilde{\mathcal{G}}[\alpha] = -(x^2-\gamma^2)\zeta. \tag{181}$$

We begin the proof (179) by first considering the related expression

$$\int dx\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi], \tag{182}$$

which is the transverse counterpart of (109). Comparing the definitions χ and ζ in the longitudinal and transverse cases, we see that by making the identification[25]

$$X = x\chi, \tag{183a}$$

$$Y = x\zeta\,, \tag{183b}$$

then

$$X + iY = \frac{1}{\beta + i\alpha}\,. \tag{184}$$

By an identical calculation that led to (113), we find

$$\int dx\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi] = \int dx\,\alpha\left(Y^2 + X^2\right)\phi\psi - \int dx\,\left(\phi\overline{Y\psi} + \psi\overline{Y\phi}\right)\left(\overline{\alpha}Y + \alpha X\right). \tag{185}$$

Here $\overline{\alpha} - \beta$ is not a constant so we must proceed in a different way than in the longitudinal case. Writing (185) in terms of ζ and $\overline{\zeta}$, we obtain

$$\int dx\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi] = -\int dx\,x\zeta\phi\psi - \int dx\,\left(\phi\overline{x\zeta\psi} + \psi\overline{x\zeta\phi}\right)\left(\overline{\alpha}x\zeta + \alpha x\overline{\zeta}\right). \tag{186}$$

Now we make the substitution $\phi \longrightarrow x\phi$ in the above. Using

$$\widetilde{\mathcal{G}}^{\dagger}[x\phi] = x\,\widetilde{\mathcal{G}}^{\dagger}[\phi] - \frac{1}{\pi}\int dx\,x\zeta\phi \tag{187}$$

gives

$$\int dx\,x\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi] = \int dx\,x^2\zeta\phi\psi + \frac{1}{\pi}\int dx\,x\zeta\phi\int dx\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\psi]$$
$$- \int dx\,\left(x\,\phi\overline{x\zeta\psi} + \psi\overline{x^2\zeta\phi}\right)\left(\overline{\alpha}x\zeta + \alpha x\overline{\zeta}\right). \tag{188}$$

Expanding $\widetilde{\mathcal{G}}^{\dagger}$ and using (490a) and (488b), we can write the above as

$$\int dx\,x\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi] = \int dx\,x^2\zeta\phi\psi - \int dx\,\left(x\,\phi\overline{x\zeta\psi} + x\,\psi\overline{x\zeta\phi}\right)\left(\overline{\alpha}x\zeta + \alpha x\overline{\zeta}\right)$$
$$+ \frac{1}{\pi}\int dx\,x\zeta\phi\int dx\,\left(\alpha x\overline{\zeta}\psi + \overline{\alpha}x\zeta\psi\right)$$

$$
\begin{aligned}
&-\frac{1}{\pi}\int dx\ \left(\overline{\alpha}x\zeta+\alpha x\overline{\zeta}\right)\psi\int dx\ x\zeta\phi\\
&=\int dx\ x^2\zeta\phi\psi-\int dx\ \left(x\,\phi\,\overline{x\zeta\psi}+x\,\psi\,\overline{x\zeta\phi}\right). \qquad (189)
\end{aligned}
$$

Now, we use (143) giving,

$$
\int dx x\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi]=\int dx x^2\zeta\phi\psi-\int dx\ \left(x\,\phi\,\overline{x\zeta\psi}+x\,\psi\,\overline{x\zeta\phi}\right)\left(x^2-\gamma^2\right)\zeta. \quad (190)
$$

The terms multiplying γ^2 cancel by virtue of Parseval's theorem, leaving

$$
\begin{aligned}
\int dx\ x\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}[\phi]\,\widetilde{\mathcal{G}}^{\dagger}[\psi]&=\int dx\ x^2\zeta\phi\psi-\int dx\ \left(x^2\,\phi\,x\,\overline{x\zeta\psi}+x^2\,\psi\,x\,\overline{x\zeta\phi}\right)\\
&=\int dx\ x^2\zeta\phi\psi-\int dx\ \left(x^2\,\phi\,\overline{x^2\zeta\psi}+x^2\,\psi\,\overline{x^2\zeta\phi}\right)\\
&\qquad-\frac{1}{\pi}\int dx\ x^2\zeta\phi\int dx\ x\zeta\psi-\frac{1}{\pi}\int dx\ x^2\zeta\psi\int dx\ x\zeta\phi\\
&=\int dx\ x^2\zeta\phi\psi-\frac{1}{\pi}\int dx\ x^2\zeta\phi\int dx\ x\zeta\psi\\
&\qquad-\frac{1}{\pi}\int dx\ x^2\zeta\psi\int dx\ x\zeta\phi. \qquad (191)
\end{aligned}
$$

The proof of (180) is a straightforward matter. We begin by substituting the explicit expressions for $\mathcal{G}[\phi]$ and $\mathcal{G}[\psi]$:

$$
\begin{aligned}
\int dx\ \frac{1}{\alpha}\,\mathcal{G}[\phi]\,\mathcal{G}[\psi]&=\int dx\ \frac{1}{\alpha}\left(\alpha\overline{\phi}+\beta\,\phi\right)\left(\alpha\overline{\psi}+\beta\,\psi\right)\\
&=\int dx\ \frac{\beta^2}{\alpha}\,\phi\psi+\int dx\ \alpha\overline{\phi}\,\overline{\psi}+\int dx\ \beta\left(\overline{\phi}\psi+\phi\overline{\psi}\right)\\
&=\int dx\ \frac{\beta^2}{\alpha}\,\phi\psi+\int dx\ \alpha\overline{\phi}\,\overline{\psi}+\int dx\ \overline{\alpha}\left(\overline{\phi\psi}-\overline{\overline{\phi}\,\overline{\psi}}\right)\\
&\qquad+\int dx\left(\beta-\overline{\alpha}\right)\left(\overline{\phi}\psi+\phi\overline{\psi}\right), \qquad (192)
\end{aligned}
$$

where we used the convolution theorem in the last step. Using Parseval's formula,

$$\int dx \, \frac{1}{\alpha} \, \mathcal{G}[\phi] \, \mathcal{G}[\psi] = \int dx \, \frac{\alpha^2 + \beta^2}{\alpha} \, \phi \, \psi + \int dx \, (\beta - \overline{\alpha}) \left(\overline{\phi} \, \psi + \phi \, \overline{\psi} \right)$$
$$= - \int dx \, \frac{1}{x \zeta} \, \phi \, \psi + \int dx \, \left(x - \frac{\gamma^2}{x} \right) \left(\overline{\phi} \, \psi + \phi \, \overline{\psi} \right) . \quad (193)$$

Using (490a) and (490b) the left-hand side of (193) becomes

$$- \int dx \, \frac{1}{x \zeta} \, \phi \, \psi + \int dx \, \left(\overline{x \phi} \, \psi + x \, \phi \, \overline{\psi} \right) - \frac{1}{\pi} \int dx \, \phi \int dx \, \psi$$
$$- \gamma^2 \int dx \, \left[\overline{\left(\frac{\phi}{x} \right)} \, \psi + \frac{\phi}{x} \, \overline{\psi} \right] - \frac{\gamma^2}{\pi} \int dx \, \frac{\phi}{x} \int dx \, \frac{\psi}{x}$$
$$= - \int dx \, \frac{1}{x \zeta} \, \phi \, \psi - \frac{1}{\pi} \int dx \, \phi \int dx \, \psi - \frac{\gamma^2}{\pi} \int dx \, \frac{\phi}{x} \int dx \, \frac{\psi}{x} , \quad (194)$$

thus establishing (180).

The last identity, (181), follows directly from the definition of $\widetilde{\mathcal{G}}$ and (143):

$$\widetilde{\mathcal{G}}[\alpha] = x \, \overline{\zeta} \alpha + x \, \zeta \overline{\alpha} = -x \left(x - \frac{\gamma^2}{x} \right) \zeta = - \left(x^2 - \gamma^2 \right) \zeta . \quad (195)$$

IV. Connection with the Riemann-Hilbert Problem

Above we mentioned that there was a connection between the inverse transform and solutions of the Riemann-Hilbert problem on the real axis. By the Riemann-Hilbert problem, we mean the following: Given functions $a(x)$, $b(x)$ and $c(x)$ on R, find the function $f(z) = u(x, y) + i \, v(x, y)$ analytic in the upper half-plane such that

$$a(x) \, u(x) + b(x) \, v(x) = c(x) , \quad (196)$$

where $u(x)$ and $v(x)$ are the limits as $y \longrightarrow 0^+$ of $u(x,y)$ and $v(x,y)$ respectively. Due to Hilbert's theorem, (491), we know that $u = \overline{v}$ and we can write above as

$$a(x)\,\overline{v}(x) + b(x)\,v(x) = c(x)\,. \tag{197}$$

In this form, the connection to $\widetilde{\mathcal{G}}$ is evident. This problem has received thorough treatment by several Soviet mathematicians.[19,20] In their formulation, the functions a, b and c are all assumed to belong to $\mathcal{H}_*^{\mu}$. The Riemann-Hilbert problem associated with the longitudinal transform fits into this formulation though one could argue that the expression (101) is a substantially more compact form that that of Gakhov *et al.*

In the case of the transverse transform, β is clearly not Hölder and thus the methods of Gakhov cannot be used to obtain a solution. There is a large class of Riemann-Hilbert problems — those where $b - \overline{a}$ is a rational function that can be solved using the methods presented above but are excluded in the standard treatment of this problem.

Furthermore, our expressions for the solution of the Riemann-Hilbert problem are well suited for direct numerical evaluation. Algorithms for numerical evaluation of the $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ along with examples are presented in Chapter 8.

4

Solution of the Maxwell-Vlasov Equations by Integral Transform

Here we use the integral transforms introduced in Chapter 3 to directly solve the linearized Maxwell-Vlasov equations. Recall that we linearized about a homogeneous equilibrium and thus there are no equilibrium electric or magnetic fields. This is a crucial ingredient to our treatment; the presence of equilibrium fields represents a significant complication that we will not address. These transforms were designed to eliminate the integral term in the Vlasov equation. In the longitudinal case, changing variables through the transform is a straightforward procedure that has the expected result. In the transverse case, changing variables is significantly more complicated because of the null space of the transform and because the transverse Vlasov equation does not commute with the projection operators $\mathcal{P}$ and $\mathcal{Q}$. Further complications are due to the transverse field equations, though their presence is intimately connected to the dimensionally and existence of the null space of the transform.

I. Solution of the Longitudinal Equation

Recall the longitudinal Vlasov equation:

$$\dot{f}^{(1)}_{\parallel k} + ikv_{\parallel} f^{(1)}_{\parallel k} + \frac{e}{m} E^{(1)}_{\parallel k} f^{(0)\prime}_{\parallel} = 0\,, \tag{68}$$

where $E^{(1)}_{\parallel k}$ is the solution of Poisson's equation

$$E^{(1)}_{\parallel k} = \frac{4\pi e}{ik} \int dv_{\parallel}\, f^{(1)}_{\parallel k}. \tag{59}$$

Let $\xi_k \in \mathcal{D}^{\mu}_{L}$ be defined by

$$f^{(1)}_{\parallel k} = \frac{ik}{4\pi e}\, \mathcal{G}[\xi_k], \tag{198}$$

where $\mathcal{G}$ is defined by (95) and where α and β are as yet unknown but assumed to satisfy

$$\alpha^2 + \beta^2 \neq 0. \tag{99}$$

Hence the inverse, $\widetilde{\mathcal{G}}$, is well defined and

$$\xi_k = \frac{4\pi e}{ik}\, \widetilde{\mathcal{G}}\left[f^{(1)}_{\parallel k}\right]. \tag{199}$$

Given that we are representing solutions of (68) by means of $\mathcal{G}$, we are implicitly restricting to solutions belonging to $\mathcal{D}^{\mu}_{L}$. Since we expect (198) to hold for all time, we are also restricting our initial conditions to the class $\mathcal{D}^{\mu}_{L}$. Physically this is a reasonable restriction, however, one can make a case for considering initial conditions, such as step functions, that are clearly outside of this class but still physically reasonable. We will comment on this and other extensions in Chapter 9.

Since $\mathcal{G}$ is linear,

$$\dot{f}^{(1)}_{\parallel k} = \frac{ik}{4\pi e}\, \mathcal{G}[\dot{\xi}_k], \tag{200}$$

and we can write the longitudinal Vlasov equation as

$$\mathcal{G}\left[\dot{\xi}_k\right] + iku\, \mathcal{G}\left[\xi_k\right] + \frac{4\pi e^2}{mk} f^{(0)\prime}_{\parallel} \int du\, \mathcal{G}\left[\xi_k\right] = 0. \tag{201}$$

Now

$$\widetilde{\mathcal{G}}[1] = \beta - \overline{\alpha} = \beta^{\infty} \tag{202}$$

and

$$\begin{aligned} \mathcal{G}[u\,\xi_k] &= \alpha\overline{u\,\xi_k} + \beta\, u\,\xi_k \\ &= u\left(\alpha\overline{\xi_k} + \beta\,\xi_k\right) + \frac{1}{\pi}\,\alpha \int du\,\xi_k \\ &= u\,\mathcal{G}\left[\xi_k\right] + \frac{1}{\pi}\,\alpha \int du\,\xi_k, \end{aligned} \tag{203}$$

which enables us to write (201) as

$$\mathcal{G}\left[\dot{\xi}_k + iku\,\xi_k\right] - \frac{ik}{\pi}\left(\alpha + \beta^{\infty}\frac{4\pi^2 e^2}{mk^2}\,f_{\parallel}^{(0)\prime}\right)\int du\,\xi_k = 0\,. \tag{204}$$

The purpose of the transform was to remove the integral term from the Vlasov equation; thus we take

$$\alpha = -\beta^{\infty}\frac{4\pi^2 e^2}{mk^2}\,f_{\parallel}^{(0)\prime} = \beta^{\infty}\epsilon_{L}^{I} \tag{205}$$

and hence

$$\beta = \beta^{\infty} + \overline{\alpha} = \beta^{\infty}\left(1 - \frac{4\pi^2 e^2}{mk^2}\mathrm{P}\!\int du'\,\frac{f_{\parallel}^{(0)\prime}}{u' - u}\right) = \beta^{\infty}\epsilon_{L}^{R}, \tag{206}$$

where ϵ_L^R and ϵ_L^I are the real and imaginary parts of the longitudinal dielectric function (see Appendix B for details). Since β^{∞} enters as an overall scale factor we have complete freedom choosing its value; for simplicity, we take $\beta^{\infty} = 1$. The condition for the existence of the inverse transformation becomes

$$0 \neq \epsilon_L^R + i\,\epsilon_L^I = \epsilon_L \tag{207}$$

in the upper half-plane including the axis. This is nothing more that our original requirement that the equilibrium be stable and not support neutral modes.

With these definitions of α and β, (204) becomes

$$\mathcal{G}\left[\dot{\xi}_k + iku\,\xi_k\right] = 0\,. \tag{208}$$

Since $\widetilde{\mathcal{G}}$ is well defined for all $\phi \in \mathcal{D}^{\mu}_{L}$, this is equivalent to

$$\dot{\xi}_k + iku\,\xi_k = 0\,. \tag{209}$$

Based on the arguments that led us to define $\mathcal{G}$, this is just the result we expected: We have reduced an integro-differential equation into a simple ordinary differential equation by means of a carefully constructed integral transform.

This equation is easily solved, giving

$$\xi_k(u,t) = \xi_k(u)\,e^{-ikut}, \tag{210}$$

where $\xi_k(u) = \xi_k(u,0)$. We determine $\xi_k(u)$ from the initial value of the perturbation by the inverse transform:

$$\xi_k(u) = \frac{4\pi e}{ik}\,\widetilde{\mathcal{G}}\left[F_{\parallel k}\right], \tag{211}$$

where $F_{\parallel k}(u) = f^{(1)}_{\parallel k}(t=0)$. Using the expression for $\xi_k(u,t)$ we can write $f^{(1)}_{\parallel k}(v,t)$ in terms of its initial value:

$$\begin{aligned} f^{(1)}_{\parallel k}(v,t) &= \frac{ik}{4\pi e}\,\mathcal{G}\left[\xi_k(u)\,e^{-ikut}\right] \\ &= \frac{ik}{4\pi e}\,\mathcal{G}\left[\frac{4\pi e}{ik}\,\widetilde{\mathcal{G}}\left[F_{\parallel k}\right]e^{-ikut}\right] \\ &= \mathcal{G}\left[\widetilde{\mathcal{G}}\left[F_{\parallel k}\right]e^{-ikut}\right]. \end{aligned} \tag{212}$$

In Chapter 3 we saw that $\widetilde{\mathcal{G}}$ maps $\mathcal{D}^{\mu}_{L}$ onto $\mathcal{D}^{\mu}_{L}$. By assumption $F_{\parallel k} \in \mathcal{D}^{\mu}_{L}$, thus $\widetilde{\mathcal{G}}[F_{\parallel k}] \in \mathcal{D}^{\mu}_{L}$. It is a simple matter to show that $e^{-ikut} \in \mathcal{H}^1$ for all t and therefore $\widetilde{\mathcal{G}}[F_{\parallel k}]e^{-ikut} \in \mathcal{D}^{\mu}_{L}$ for all t. Since $\mathcal{G}$ maps $\mathcal{D}^{\mu}_{L}$ onto $\mathcal{D}^{\mu}_{L}$, we see that $f^{(1)}_{\parallel k}(v,t) \in \mathcal{D}^{\mu}_{L}$ for all t, *i.e.* the Vlasov equation maps $\mathcal{D}^{\mu}_{L}$ onto $\mathcal{D}^{\mu}_{L}$.

We can now compute $E^{(1)}_{\parallel k}(t)$ from our solution of the transformed Vlasov equation:

$$E^{(1)}_{\parallel k}(t) = \int du\, \xi(u,t) = \int du\, \xi(u)\, e^{-ikut}. \tag{213}$$

This immediately gives us a physical interpretation of $\xi_k(u)$ — it is the temporal Fourier transform of the perturbed electric field corresponding to frequency $\omega = ku$. Interestingly, we see that there is a one-to-one connection between the initial perturbed particle distribution and the frequency spectrum of the perturbed field. This is a direct consequence of Poisson's equation — it is a sufficiently rigid constraint on the dynamics that it ties the electric field unambiguously to the particle motion. This is the fundamental difference between the longitudinal and transverse motions. In the transverse system, we will see that the particle distribution alone is not sufficient to determine the fields for all time.

Suppose that we have a dynamically accessible initial condition. Since the dynamics are generated by the bracket, one certainly expects that $f^{(1)}_{\parallel k}(t)$ will remain dynamically accessible for all time. That this is so can we readily confirmed. Recall that the condition for the perturbation to be dynamically accessible is

$$f^{(1)}_{\parallel k\,\mathrm{DA}} = g_{\parallel k}(v_{\parallel})\, f^{(0)\prime}_{\parallel}. \tag{47}$$

For our initial condition, this is equivalent to

$$F_{\parallel k} = g(v_{\parallel})\, \alpha\,. \tag{214}$$

To see the implications of this form on $f_{\parallel k}^{(1)}$, consider

$$\begin{aligned} \mathcal{G}\left[\lambda\widetilde{\mathcal{G}}[\alpha g]\right] &= \alpha\overline{\widetilde{\mathcal{G}}[\alpha g]} + \beta\lambda(\chi\alpha g + \zeta\overline{\alpha g}) \\ &= \alpha\left\{\overline{\widetilde{\mathcal{G}}[\alpha g]} + \beta\lambda\chi g - \chi\lambda\overline{\alpha g}\right\}, \end{aligned} \tag{215}$$

where the last step follows from $\beta\zeta = -\alpha\chi$. Thus we see that given a dynamically accessible initial condition, the solution of the Vlasov equation remains dynamically accessible for all time.

II. Solution of the Transverse Equations

We now consider the transverse Maxwell-Vlasov equations:

$$\dot{\boldsymbol{f}}_{\perp k}^{(1)} + ikv_{\parallel}\boldsymbol{f}_{\perp k}^{(1)} - \frac{e}{m}\boldsymbol{E}_{\perp k}^{(1)} f_{\parallel}^{(0)} = 0, \tag{58a}$$

$$-\dot{\boldsymbol{E}}_{\perp k}^{(1)} + ck^2\boldsymbol{A}_{\perp k}^{(1)} = 4\pi e\int dv_{\parallel}\, \boldsymbol{f}_{\perp k}^{(1)}, \tag{58b}$$

$$\dot{\boldsymbol{A}}_{\perp k}^{(1)} + c\boldsymbol{E}_{\perp k}^{(1)} = 0. \tag{58c}$$

Following in the spirit of the longitudinal case, we propose the change of variables

$$\boldsymbol{f}_{\perp k}^{(1)} = \frac{ik}{4\pi e}\mathcal{G}\left[\boldsymbol{\xi}_k\right] = \frac{ik}{4\pi e}\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}k}\right], \tag{216}$$

where $\mathcal{G}$ is defined by (129) with α and β unspecified but satisfying

$$\alpha^2 + \beta^2 \neq 0 \tag{137}$$

in the upper half-plane and $\boldsymbol{\xi}_k \in \mathcal{D}_T^{\mu}$. Given that this transformation only relates $\boldsymbol{f}_{\perp k}^{(1)}$ and $\boldsymbol{\xi}_{\mathcal{P}k}$, one might be tempted to pick $\boldsymbol{\xi}_k$ such that $\boldsymbol{\xi}_{\mathcal{Q}k} \equiv 0$. Unfortunately such a choice is does not result in any simplification of the Vlasov equation. Thus we must find another way to determine $\boldsymbol{\xi}_{\mathcal{Q}k}$.

Consider the first two terms in the Vlasov equation. Since $\mathcal{G}$ is linear

$$\dot{f}^{(1)}_{\parallel k} = \frac{ik}{4\pi e}\,\mathcal{G}\left[\dot{\boldsymbol{\xi}}_{\mathcal{P}k}\right]. \tag{217}$$

Furthermore

$$\int du\,\boldsymbol{\xi}_{\mathcal{P}k} = 0\,, \tag{218}$$

giving

$$\mathcal{G}\left[u\,\boldsymbol{\xi}_{\mathcal{P}k}\right] = u\,\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}k}\right] \tag{219}$$

and we can write the Vlasov equation as

$$\mathcal{G}\left[\dot{\boldsymbol{\xi}}_{\mathcal{P}k} + iku\,\boldsymbol{\xi}_{\mathcal{P}k}\right] + \frac{4\pi e^2}{mk}\,i\,f^{(0)}_{\parallel}\,\boldsymbol{E}^{(1)}_{\perp k} = 0\,. \tag{220}$$

We also need to express the right-hand side of Ampere's law, (58c), in terms of $\boldsymbol{\xi}_k$. From the definition of the transform, (129), we find

$$\begin{aligned}\int dv_{\parallel}\,\boldsymbol{f}^{(1)}_{\perp k} &= \frac{ik}{4\pi e}\int du\,\left[\alpha\,\overline{\boldsymbol{\xi}_{\mathcal{P}k}} + \overline{\alpha}\,\boldsymbol{\xi}_{\mathcal{P}k} + \left(u - \frac{c^2}{u}\right)\boldsymbol{\xi}_{\mathcal{P}k}\right] \\ &= \frac{ik}{4\pi e}\int du\,\left(u - \frac{c^2}{u}\right)\boldsymbol{\xi}_{\mathcal{P}k} \\ &= \frac{ik}{4\pi e}\int du\,u\,\boldsymbol{\xi}_{\mathcal{P}k}\end{aligned} \tag{221}$$

and Ampere's law becomes

$$-\dot{\boldsymbol{E}}^{(1)}_{\perp k} + ck^2\boldsymbol{A}^{(1)}_{\perp k} = ik\int du\,u\,\boldsymbol{\xi}_{\mathcal{P}k}. \tag{222}$$

Obviously we would prefer (220) to be in the form

$$\mathcal{G}\left[\mathcal{L}\left[\boldsymbol{\xi}_k\right]\right] = 0\,, \tag{223}$$

where $\mathfrak{L}$ is some operator in t and u. In Chapter 3 we saw that

$$\widetilde{\mathcal{G}}[\alpha] = -(u^2 - \gamma^2)\,\zeta\,, \tag{181}$$

which tells us that

$$\mathcal{G}\left[(u^2 - \gamma^2)\,\zeta\right] = -\alpha\,. \tag{224}$$

Thus we take

$$\alpha = \frac{4\pi^2 e^2}{mk^2}\, f_{\parallel}^{(0)} = u\,\epsilon_{T}^{I}, \tag{225}$$

where ϵ_{T}^{I} is the imaginary part of the transverse dielectric function (see Appendix B for details). The Vlasov equation now reads

$$\mathcal{G}\left[\dot{\boldsymbol{\xi}}_{\mathcal{P}k} + iku\,\boldsymbol{\xi}_{\mathcal{P}k} - \frac{ik}{\pi}\,(u^2 - \gamma^2)\,\zeta\,\boldsymbol{E}_{\perp k}^{(1)}\right] = 0\,. \tag{226}$$

We can compute β:

$$\begin{aligned}
\beta &= \overline{\alpha} + u - \frac{c^2}{u} \\
&= \frac{4\pi^2 e^2}{mk^2}\,\mathrm{P}\!\int dv_{\parallel}\, \frac{f_{\parallel}^{(0)}}{v_{\parallel} - u} + u - \frac{\gamma^2}{u} \\
&= u\left(1 + \frac{4\pi^2 e^2}{mk^2}\,\frac{1}{u}\,\mathrm{P}\!\int dv_{\parallel}\, \frac{f_{\parallel}^{(0)}}{v_{\parallel} - u}\right) - \frac{\gamma^2}{u} \\
&= u\,\epsilon_{T}^{R}(u) - \frac{\gamma^2}{u}\,,
\end{aligned} \tag{227}$$

where ϵ_{T}^{R} is the real part of the transverse dielectric function. The condition on α and β becomes

$$\beta + i\,\alpha = u\left[\epsilon_{T}^{R} + i\,\epsilon_{T}^{I} - \frac{\gamma^2}{u^2}\right] \neq 0\,, \tag{228}$$

which is equivalent to

$$\epsilon_{T} \neq \frac{\gamma^2}{u^2} \tag{229}$$

in the upper half-plane. On the other hand, the condition that the equilibrium be stable is

$$\epsilon_T \neq \frac{c^2}{u^2}, \tag{230}$$

in the upper half-plane. In the longitudinal case we had a direct connection between the condition for the existence of the inverse and the equilibrium stability. Having a physical interpretation for the invertibility condition is clearly very attractive. With this in mind, the natural choice is $\gamma = c$.

Since $\mathcal{G}$ has a non-trivial null space, $\mathcal{G}[\phi] = 0$ does not imply $\phi = 0$ but rather that $\phi = \mathcal{Q}[\phi]$ or, equivalently, $\mathcal{P}[\phi] = 0$. Hence the transformed Vlasov equation reads

$$\mathcal{P}\left[\dot{\boldsymbol{\xi}}_{\mathcal{P}k} + iku\,\boldsymbol{\xi}_{\mathcal{P}k} - \frac{ik}{\pi}(u^2 - c^2)\zeta \boldsymbol{E}^{(1)}_{\perp k}\right] = 0. \tag{231}$$

We can readily evaluate each term in the above:

$$\mathcal{P}\left[\dot{\boldsymbol{\xi}}_{\mathcal{P}k}\right] = \dot{\boldsymbol{\xi}}_{\mathcal{P}k}; \tag{232}$$

$$\begin{aligned} \mathcal{P}\left[u\,\boldsymbol{\xi}_{\mathcal{P}k}\right] &= u\,\boldsymbol{\xi}_{\mathcal{P}k} + \frac{1}{\pi}u\zeta\int du\, u\,\boldsymbol{\xi}_{\mathcal{P}k} + \frac{c^2}{\pi}\zeta\int du\,\boldsymbol{\xi}_{\mathcal{P}k} \\ &= u\,\boldsymbol{\xi}_{\mathcal{P}k} + \frac{1}{\pi}u\zeta\int du\, u\,\boldsymbol{\xi}_{\mathcal{P}k}; \end{aligned} \tag{233}$$

and

$$\begin{aligned} \mathcal{P}\left[(u^2 - c^2)\zeta\right] &= (u^2 - c^2)\zeta + \frac{c^2}{\pi}\zeta\int du\,\left(u - \frac{c}{u}\right)\zeta \\ &= (u^2 - c^2)\zeta; \end{aligned} \tag{234}$$

obtaining

$$\dot{\boldsymbol{\xi}}_{\mathcal{P}k} + iku\,\boldsymbol{\xi}_{\mathcal{P}k} + \frac{ik}{\pi}\,u\zeta\int du\, u\,\boldsymbol{\xi}_{\mathcal{P}k} - \frac{ik}{\pi}(u^2 - c^2)\zeta \boldsymbol{E}^{(1)}_{\perp k} = 0. \tag{235}$$

On the surface (235), appears to be more unpleasant than our original equation. Fortunately, the integral term in (235) is precisely that appearing on the right-hand side of Ampere's law. Using Ampere's law to eliminate this term leaves

$$\dot{\boldsymbol{\xi}}_{\mathcal{P}k} + iku\,\boldsymbol{\xi}_{\mathcal{P}k} + \frac{1}{\pi}\,u\zeta\left(ck^2\boldsymbol{A}^{(1)}_{\perp k} - \dot{\boldsymbol{E}}^{(1)}_{\perp k}\right) - \frac{ik}{\pi}(u^2 - c^2)\,\zeta\boldsymbol{E}^{(1)}_{\perp k} = 0\,. \tag{236}$$

We can express $\boldsymbol{\xi}_{\mathcal{P}k}$ in terms of $\boldsymbol{\xi}_k$ using the definition of $\mathcal{P}$:

$$\begin{aligned}\boldsymbol{\xi}_{\mathcal{P}k} &= \boldsymbol{\xi}_k + \frac{1}{\pi}\,u\zeta\int du\,\boldsymbol{\xi}_k + \frac{c^2}{\pi}\,\zeta\int du\,\frac{\boldsymbol{\xi}_k}{u} \\ &= \boldsymbol{\xi}_k + \frac{1}{\pi}\,u\zeta\,\boldsymbol{\xi}_{1k} + \frac{c^2}{\pi}\,\zeta\,\boldsymbol{\xi}_{2k},\end{aligned} \tag{237}$$

where we have introduced

$$\boldsymbol{\xi}_{1k} = \int du\,\boldsymbol{\xi}_k, \tag{238a}$$

$$\boldsymbol{\xi}_{2k} = \int du\,\frac{\boldsymbol{\xi}_k}{u}. \tag{238b}$$

We can then write

$$\dot{\boldsymbol{\xi}}_{\mathcal{P}k} = \dot{\boldsymbol{\xi}}_k + \frac{1}{\pi}\,u\zeta\,\dot{\boldsymbol{\xi}}_{1k} + \frac{c^2}{\pi}\,\zeta\,\dot{\boldsymbol{\xi}}_{2k}. \tag{239}$$

Using Faraday's law, (58*c*), and rearranging terms, the transformed Vlasov equation takes the suggestive form:

$$\begin{aligned}&\dot{\boldsymbol{\xi}}_k + iku\boldsymbol{\xi}_k + \frac{1}{\pi}\,u\zeta\left[\dot{\boldsymbol{\xi}}_{1k} - \dot{\boldsymbol{E}}^{(1)}_{\perp k}\right] + \frac{ik}{\pi}\,u^2\zeta\left[\boldsymbol{\xi}_{1k} - \boldsymbol{E}^{(1)}_{\perp k}\right] \\ &\qquad + \frac{c^2}{\pi}\,\zeta\left[\dot{\boldsymbol{\xi}}_{2k} - \frac{ik}{c}\dot{\boldsymbol{A}}^{(1)}_{\perp k}\right] - \frac{ik}{\pi}\gamma^2 u\zeta\left[\boldsymbol{\xi}_{2k} - \frac{ik}{c}\boldsymbol{A}^{(1)}_{\perp k}\right] = 0\,.\end{aligned} \tag{240}$$

We are free to choose $\boldsymbol{\xi}_{1k}$ and $\boldsymbol{\xi}_{2k}$ (and thereby determining $\boldsymbol{\xi}_{\mathcal{Q}k}$) as we wish. Taking

$$\boldsymbol{\xi}_{1k} = \boldsymbol{E}^{(1)}_{\perp k}, \tag{241a}$$

$$\boldsymbol{\xi}_{2k} = \frac{ik}{c}\boldsymbol{A}^{(1)}_{\perp k}, \tag{241b}$$

the Vlasov equation reduces to the remarkably compact expression:[26]

$$\dot{\boldsymbol{\xi}}_k + iku\boldsymbol{\xi}_k = 0. \tag{242}$$

We can solve this equation readily,

$$\boldsymbol{\xi}_k(u,t) = \boldsymbol{\xi}_k(u)e^{-ikut}, \tag{243}$$

where $\boldsymbol{\xi}_k(u) = \boldsymbol{\xi}_k(u, t=0)$ is determined by the initial perturbation through the inverse transform:

$$\boldsymbol{\xi}_k(u) = \frac{4\pi e}{ik}\widetilde{\mathcal{G}}\left[\boldsymbol{F}_{\perp k}\right] + \boldsymbol{\xi}_{\Omega k}(u, t=0), \tag{244}$$

where $\boldsymbol{F}_{\perp k}(v) = \boldsymbol{f}^{(1)}_{\perp k}(v, t=0)$. Now

$$\boldsymbol{\xi}_{\Omega k}(u, t=0) = -\frac{1}{\pi}u\zeta\boldsymbol{E}^{(1)}_{\perp k}(0) - \frac{ikc}{\pi}\zeta\boldsymbol{A}^{(1)}_{\perp k}(0), \tag{245}$$

giving

$$\boldsymbol{\xi}_k(u) = \frac{4\pi e}{ik}\widetilde{\mathcal{G}}\left[\boldsymbol{F}_{\perp k}\right] - \frac{1}{\pi}u\zeta\boldsymbol{E}^{(1)}_{\perp k}(0) - \frac{ikc}{\pi}\zeta\boldsymbol{A}^{(1)}_{\perp k}(0). \tag{246}$$

We can use the transform to determine $\boldsymbol{f}^{(1)}_{\perp k}(v,t)$ in terms of the initial conditions:

$$\begin{aligned}\boldsymbol{f}^{(1)}_{\perp k}(v,t) &= \frac{4\pi e}{ik}\mathcal{G}\left[\boldsymbol{\xi}_k(u)\,e^{-ikut}\right] \\ &= \mathcal{G}\left[\left\{\widetilde{\mathcal{G}}\left[\boldsymbol{F}_{\perp k}\right] - \frac{1}{\pi}u\zeta\boldsymbol{E}^{(1)}_{\perp k}(0) - \frac{ikc}{\pi}\zeta\boldsymbol{A}^{(1)}_{\perp k}(0)\right\}e^{-ikut}\right].\end{aligned} \tag{247a}$$

Additionally

$$\boldsymbol{E}^{(1)}_{\perp k}(t) = \boldsymbol{\xi}_{1k}(t) = \int du\, \boldsymbol{\xi}_k(u,t)$$
$$= \frac{4\pi e}{ik} \int du \left\{ \widetilde{\mathcal{G}}\left[\boldsymbol{F}_{\perp k}\right] - \frac{1}{\pi} u \zeta\, \boldsymbol{E}^{(1)}_{\perp k}(0) - \frac{ikc}{\pi} \zeta \boldsymbol{A}^{(1)}_{\perp k}(0) \right\} e^{-ikut}, \quad (247b)$$

$$\boldsymbol{A}^{(1)}_{\perp k}(t) = \boldsymbol{\xi}_{2k}(t) = -\frac{ic}{k} \int du\, \frac{\boldsymbol{\xi}_k(u,t)}{u}$$
$$= -\frac{4\pi e}{k^2} \int du \left\{ \widetilde{\mathcal{G}}\left[\boldsymbol{F}_{\perp k}\right] - \frac{1}{\pi} u \zeta \boldsymbol{E}^{(1)}_{\perp k}(0) - \frac{ikc}{\pi} \zeta \boldsymbol{A}^{(1)}_{\perp k}(0) \right\} \frac{e^{-ikut}}{u}. \quad (247c)$$

There are several interesting aspects to this solution. Initially, specifying the particle distribution alone is not sufficient, we also need the initial values of the fields. Compare this with the longitudinal case where the initial particle distribution alone completely determine the future behaviour of the field. Further, suppose the fields where initially zero ($\boldsymbol{\xi}_{\mathcal{Q}k}(u) \equiv 0$), they would *not* necessarily remain so since $\mathcal{Q}[\boldsymbol{\xi}(t)]$ is a function of both $\boldsymbol{\xi}_{\mathcal{P}}(u)$ and $\boldsymbol{\xi}_{\mathcal{Q}}(u)$. As we see from (247), there appears to be no simple way to separate out the contributions to $\boldsymbol{E}^{(1)}_{\perp k}(t)$ and $\boldsymbol{A}^{(1)}_{\perp k}(t)$ that are due to the initial fields from those due to the initial particle distributions.

In Chapter 3 we saw that $\widetilde{\mathcal{G}}$ maps $\mathcal{H}^{\mu}_{*}$ onto $\mathcal{P}[\mathcal{D}^{\mu}_{T}]$. Thus from the initial condition $\boldsymbol{F}_{\perp k} \in \mathcal{H}^{\mu}_{*}$ and the initial value of the fields, $\boldsymbol{E}^{(1)}_{\perp k}(0)$ and $\boldsymbol{A}^{(1)}_{\perp k}(0)$, one can construct $\boldsymbol{\xi}_k(u) \in \mathcal{D}^{\mu}_{T}$. Since $e^{-ikut} \in \mathcal{H}^1$ for all t, we have that $\boldsymbol{\xi}_k(u,t) \in \mathcal{D}^{\mu}_{T}$ for all t. Since $\mathcal{G}$ maps $\mathcal{D}^{\mu}_{T}$ onto $\mathcal{H}^{\mu}_{*}$, we see that $\boldsymbol{f}^{(1)}_{\perp k}(v,t) \in \mathcal{H}^{\mu}_{*}$ for all t. Thus the transverse Vlasov equation maps $\mathcal{H}^{\mu}_{*}$ into $\mathcal{H}^{\mu}_{*}$.

As in the longitudinal case, we can obtain a physical interpretation for $\boldsymbol{\xi}_k(u)$.

From the expression for $\boldsymbol{E}^{(1)}_{\perp k}(t)$:

$$\boldsymbol{E}^{(1)}_{\perp k}(t) = \int du\, \boldsymbol{\xi}_k(u,t) = \int du\, \boldsymbol{\xi}_k(u)\, e^{-ikut}, \tag{248}$$

we see that $\boldsymbol{\xi}_k(u)$ is the temporal Fourier transform of $\boldsymbol{E}^{(1)}_{\perp k}$ corresponding to frequency $\omega = ku$. This is exactly what we found in the longitudinal case.

As in the longitudinal case, it is of interest to consider the fate of a dynamically accessible initial condition. Recall that for a perturbation to be dynamically accessible, it must be of the form

$$\boldsymbol{f}^{(1)}_{\perp k\,\mathrm{DA}} = v_{\parallel}\, \boldsymbol{g}_{\perp k}(v_{\parallel})\, f^{(0)}_{\parallel}. \tag{48}$$

For out initial condition, this reads

$$\boldsymbol{F}_{\perp k} = v_{\parallel}\, \boldsymbol{g}(v_{\parallel})\, \alpha . \tag{249}$$

To see the effect of this form on $\boldsymbol{f}^{(1)}_{\perp k}(t)$, consider

$$\begin{aligned} \mathcal{G}\left[\lambda\, \widetilde{\mathcal{G}}[\alpha g]\right] &= \alpha \overline{\lambda \widetilde{\mathcal{G}}[\alpha g]} + \beta\{\chi \alpha \lambda g + \zeta \lambda \overline{\alpha g}\} \\ &= \alpha\left\{\overline{\lambda \widetilde{\mathcal{G}}[\alpha g]} + \beta \chi \lambda g - \chi \lambda \overline{\alpha g}\right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}[\lambda \zeta] &= \alpha \overline{\lambda \zeta} + \beta \lambda \zeta \\ &= \alpha\{\overline{\lambda \zeta} - \lambda \chi\}. \end{aligned}$$

Together, these results tell us that $\boldsymbol{f}^{(1)}_{\perp k}$ will be proportional to α for all time; *i.e.* that $\boldsymbol{f}^{(1)}_{\perp k}(t)$ will be dynamically accessible for all time if it is so initially.

For both the longitudinal and transverse motions, through the use of the appropriate integral transform, we were able to reduce the velocity dependence in the equations of motion to the level of a parametric dependence yielding first order ordinary differential equations in time that were readily solved. The resulting solutions have the same time dependence that was was *assumed* by van Kampen[13] and Felderhof.[14]

5

Canonization of the Hamiltonian and Bracket

We now make use of the integral transforms studied in the previous chapter to transform the Hamiltonian and bracket to diagonal form by means of a linear coordinate change. As we will see, these new coordinates are essentially action-angle variables. Since neither the Hamiltonian nor the bracket couple longitudinal and transverse degrees of freedom, we can consider the longitudinal and transverse motions separately.

I. Longitudinal Motion

Consider the longitudinal contribution to the energy

$$H_{\parallel}^{(2)} = \frac{V}{4} \sum_{k=-\infty}^{\infty} \left\{ -\frac{m}{2} \int dv_{\parallel}\, v_{\parallel} \frac{\left|f_{\parallel k}^{(1)}\right|^2}{f_{\parallel}^{(0)\prime}} + \frac{1}{8\pi} E_{\parallel k}^{(1)} E_{\parallel -k}^{(1)} \right\}, \tag{250}$$

and the longitudinal part of the bracket

$$\{F, G\}_{\parallel} = \frac{4}{V} \sum_{k=-\infty}^{\infty} \frac{ik}{m} \int dv_{\parallel}\, f_{\parallel}^{(0)\prime} \frac{\delta F}{\delta f_{\parallel k}^{(1)}} \frac{\delta G}{\delta f_{\parallel -k}^{(1)}} . \tag{251}$$

We apply the same coordinate change introduced in Section 4-I:

$$f_{\parallel k}^{(1)} = \frac{ik}{4\pi e} \mathcal{G}\left[\xi_k\right], \tag{252a}$$

with

$$\alpha = -\frac{4\pi^2 e^2}{mk^2} f_{\parallel}^{(0)\prime} \equiv \epsilon_L^I, \tag{252b}$$

$$\beta = 1 + \overline{\alpha} \equiv \epsilon_L^R. \tag{252c}$$

In the expression for $H_{\parallel}^{(2)}$, $E_{\parallel k}^{(1)}$ represents the solution of Poisson's equation:

$$E_{\parallel k}^{(1)} = \frac{4\pi e}{ik} \int dv_{\parallel}\, f_{\parallel k}^{(1)}. \tag{253}$$

Substituting this explicit expression for $E_{\parallel k}^{(1)}$ into the longitudinal energy gives

$$H_{\parallel}^{(2)} = \frac{V}{4} \sum_{k=-\infty}^{\infty} \left\{ -\frac{m}{2} \int dv_{\parallel}\, v_{\parallel} \frac{\left|f_{\parallel k}^{(1)}\right|^2}{f_{\parallel}^{(0)\prime}} + \frac{(4\pi e)^2}{8\pi k^2} \int dv_{\parallel}\, f_{\parallel k}^{(1)} \int dv_{\parallel}\, f_{\parallel -k}^{(1)} \right\}. \tag{254}$$

In terms of the ξ_k coordinates this becomes

$$H_{\parallel}^{(2)} = \frac{V}{32} \sum_{k=-\infty}^{\infty} \left\{ \int du\, \frac{u}{\alpha}\, \mathcal{G}[\xi_k]\, \mathcal{G}[\xi_{-k}] + \frac{1}{\pi} \int du\, \xi_k \int du\, \xi_{-k} \right\}. \tag{255}$$

Recall from Section 3-II the identity

$$\int dx\, \frac{x}{\alpha}\, \mathcal{G}[\phi]\, \mathcal{G}[\psi] = -\int dx\, \frac{x}{\zeta}\, \phi\, \psi - \frac{\beta^{\infty}}{\pi} \int dx\, \phi \int dx\, \psi\,. \tag{110}$$

Using the above, we can write the Hamiltonian in the diagonal form:

$$H_{\parallel}^{(2)} = -\frac{V}{32} \sum_{k=-\infty}^{\infty} \int du\, \frac{u}{\zeta} \left|\xi_k\right|^2. \tag{256}$$

Recalling the definition of ζ, (102b), we see

$$\zeta = -\frac{\alpha}{\alpha^2 + \beta^2} = -\frac{\epsilon_L^I}{\left|\epsilon_L\right|^2} \tag{257}$$

and thus the Hamiltonian is given by[23,27]

$$H_{\parallel}^{(2)} = \frac{V}{32} \sum_{k=-\infty}^{\infty} \int du\, u\, \frac{\left|\epsilon_L\right|^2}{\epsilon_L^I} \left|\xi_k\right|^2 . \tag{258}$$

Notice that $H_{\parallel}^{(2)}$ is quite different from the expression for the energy stored in a dielectric:

$$\mathcal{E}_D = \frac{V}{16\pi} \frac{\partial(\omega \epsilon_L^R)}{\partial \omega} \left|E(k,\omega)\right|^2 , \tag{259}$$

where $E(k,\omega)$ is the Fourier transform in space and time of the electric field and k and ω are related by the dispersion function through $\epsilon_L^R(k,\omega/k) = 0$. The discrepancy between these two expressions stems from the fact that, unlike true dielectrics, plasmas possesses resonant particles. That resonant particles are responsible for this difference will be made clear in Chapter 6 when we expand our assumptions to allow the equilibrium to support neutral modes.

Under the coordinate change (252),

$$\frac{\delta F}{\delta f_{\parallel k}^{(1)}} = \frac{4\pi e}{ik} \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta F}{\delta \xi_k} \right] \tag{260}$$

and the bracket can be written as

$$\begin{aligned} \{F,G\}_{\parallel} &= \frac{4}{V} \sum_{k=-\infty}^{\infty} ik \int dv_{\parallel} \frac{4\pi^2 e^2}{m k^2} f_{\parallel}^{(0)\prime} \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta F}{\delta \xi_k} \right] \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta G}{\delta \xi_{-k}} \right] \\ &= -\frac{16}{V} \sum_{k=-\infty}^{\infty} ik \int du\, \alpha \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta F}{\delta \xi_k} \right] \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta G}{\delta \xi_{-k}} \right] . \end{aligned} \tag{261}$$

Recall from Section 3-II a further identity

$$\int dx\, \alpha \widetilde{\mathcal{G}}^{\dagger}[\phi]\, \widetilde{\mathcal{G}}^{\dagger}[\psi] = -\int dx\, \zeta \phi \psi . \tag{109}$$

Using (109) we obtain

$$\{F,G\}_{\parallel} = \frac{16}{V}\sum_{k=-\infty}^{\infty} ik\int du\,\zeta\,\frac{\delta F}{\delta\xi_k}\frac{\delta G}{\delta\xi_{-k}}. \tag{262}$$

Expressing ζ in terms of ϵ_L gives

$$\{F,G\}_{\parallel} = -\frac{16}{V}\sum_{k=-\infty}^{\infty} ik\int du\,\frac{\epsilon_L^I}{|\epsilon_L|^2}\frac{\delta F}{\delta\xi_k}\frac{\delta G}{\delta\xi_{-k}}. \tag{263}$$

The equation of motion for ξ_k is now Hamilton's equation:

$$\dot{\xi}_k = \{\xi_k, H_{\parallel}^{(2)}\}_{\parallel} = -iku\xi_k, \tag{264}$$

which is the same result obtained by directly transforming the Vlasov equation.

II. Transverse Motion

We now move on to the transverse contribution to the energy,

$$H_{\perp}^{(2)} = \frac{V}{4}\sum_{k=-\infty}^{\infty}\left\{\frac{m}{2}\int dv_{\parallel}\,\frac{|\boldsymbol{f}_{\perp k}^{(1)}|^2}{f_{\parallel}^{(0)}} + \frac{1}{8\pi}\left(|\boldsymbol{E}_{\perp k}^{(1)}|^2 + k^2|\boldsymbol{A}_{\perp k}^{(1)}|^2\right)\right\}, \tag{265}$$

and the transverse part of the bracket,

$$\begin{aligned}\{F,G\}_{\perp} = \frac{4}{V}\sum_{k=-\infty}^{\infty}\Bigg\{&\frac{ik}{m}\int dv_{\parallel}\,v_{\parallel}f_{\parallel}^{(0)}\frac{\delta F}{\delta\boldsymbol{f}_{\perp k}^{(1)}}\cdot\frac{\delta G}{\delta\boldsymbol{f}_{\perp -k}^{(1)}}\\ &-\frac{4\pi e}{m}\int dv_{\parallel}\,f_{\parallel}^{(0)}\left(\frac{\delta F}{\delta\boldsymbol{E}_{\perp k}^{(1)}}\cdot\frac{\delta G}{\delta\boldsymbol{f}_{\perp -k}^{(1)}} - \frac{\delta G}{\delta\boldsymbol{E}_{\perp -k}^{(1)}}\cdot\frac{\delta F}{\delta\boldsymbol{f}_{\perp k}^{(1)}}\right)\\ &+4\pi c\left(\frac{\delta F}{\delta\boldsymbol{E}_{\perp k}^{(1)}}\cdot\frac{\delta G}{\delta\boldsymbol{A}_{\perp -k}^{(1)}} - \frac{\delta G}{\delta\boldsymbol{E}_{\perp -k}^{(1)}}\cdot\frac{\delta F}{\delta\boldsymbol{A}_{\perp k}^{(1)}}\right)\Bigg\}.\end{aligned} \tag{266}$$

We apply the coordinate change introduced in Section 4-II:

$$\boldsymbol{f}_{\perp k}^{(1)} = \frac{ik}{4\pi e}\,\mathcal{G}\left[\boldsymbol{\xi}_k\right] = \frac{ik}{4\pi e}\,\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}k}\right], \tag{267a}$$

$$\boldsymbol{E}_{\perp k}^{(1)} = \int du\,\boldsymbol{\xi}_k, \tag{267b}$$

$$\boldsymbol{A}_{\perp k}^{(1)} = -\frac{ic}{k}\int du\,\frac{\boldsymbol{\xi}_k}{u}, \tag{267c}$$

with

$$\alpha = \frac{4\pi^2 e^2}{mk^2}\,f_{\parallel}^{(0)} = u\,\epsilon_T^I, \tag{267d}$$

$$\beta = u\,\epsilon_T^R(u) - \frac{c^2}{u}. \tag{267e}$$

Substituting the expression for $\boldsymbol{f}_{\perp k}^{(1)}$ into the transverse energy gives

$$H_{\perp}^{(2)} = \frac{V}{32}\sum_{k=-\infty}^{\infty}\left\{\int dv_{\parallel}\,\frac{1}{\alpha}\,\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}k}\right]\,\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}-k}\right] + \frac{1}{\pi}\left(\left|\boldsymbol{E}_{\perp k}^{(1)}\right|^2 + k^2\left|\boldsymbol{A}_{\perp k}^{(1)}\right|^2\right)\right\}. \tag{268}$$

From Section 3-III we have the identity

$$\int dx\,\frac{1}{\alpha}\,\mathcal{G}[\phi]\,\mathcal{G}[\psi] = -\int dx\,\frac{1}{x\zeta}\,\phi\psi - \frac{1}{\pi}\int dx\,\phi\int dx\,\psi - \frac{\gamma^2}{\pi}\int dx\,\frac{\phi}{x}\int dx\,\frac{\psi}{x}. \tag{179}$$

Taking into account that

$$\int du\,\boldsymbol{\xi}_{\mathcal{P}k} = 0, \tag{269a}$$

$$\int du\,\frac{\boldsymbol{\xi}_{\mathcal{P}k}}{u} = 0, \tag{269b}$$

the above identity becomes

$$\int dv_{\parallel}\,\frac{1}{\alpha}\,\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}k}\right]\,\mathcal{G}\left[\boldsymbol{\xi}_{\mathcal{P}-k}\right] = -\int du\,\frac{1}{u\,\zeta}\,\boldsymbol{\xi}_{\mathcal{P}k}\cdot\boldsymbol{\xi}_{\mathcal{P}-k}. \tag{270}$$

Recall from Section 3-III the identity

$$\int du\, \frac{1}{u\zeta}\,\mathcal{P}[\phi]\,\mathcal{P}[\psi] = \int du\, \frac{1}{u\zeta}\,\phi\psi + \frac{1}{\pi}\int du\,\phi \int du\,\psi + \frac{\gamma^2}{\pi}\int du\,\frac{\phi}{u}\int du\,\frac{\psi}{u}, \quad (174)$$

which allows us to write the energy as

$$H_{\perp}^{(2)} = \frac{V}{32}\sum_{k=-\infty}^{\infty}\Bigg\{ -\int du\,\frac{1}{u\zeta}\,|\boldsymbol{\xi}_k|^2 + \frac{1}{\pi}\left(\left|\boldsymbol{E}_{\perp k}^{(1)}\right|^2 + k^2\left|\boldsymbol{A}_{\perp k}^{(1)}\right|^2\right)$$
$$-\frac{1}{\pi}\left|\int du\,\boldsymbol{\xi}_k\right|^2 - \frac{c^2}{\pi}\left|\int du\,\frac{\boldsymbol{\xi}_k}{u}\right|^2\Bigg\}. \quad (271)$$

Using (267*b*) and (267*c*) the last four terms in $H_{\perp}^{(2)}$ cancel leaving the compact diagonal expression

$$H_{\perp}^{(2)} = -\frac{V}{32}\sum_{k=-\infty}^{\infty}\int du\,\frac{1}{u\zeta}\,|\boldsymbol{\xi}_k|^2. \quad (272)$$

From the definition of ζ, (139*b*), we see

$$u\zeta = -\frac{\alpha}{\alpha^2+\beta^2} = -\frac{\epsilon_T^I}{u\left|\epsilon_T - c^2/u^2\right|^2}. \quad (273)$$

Writing the energy in terms of ϵ_T yields

$$H_{\perp}^{(2)} = \frac{V}{32}\sum_{k=-\infty}^{\infty}\int du\,u\,\frac{\left|\epsilon_T - c^2/u^2\right|^2}{\epsilon_T^I}\,\boldsymbol{\xi}_k\cdot\boldsymbol{\xi}_{-k}. \quad (274)$$

Again this is quite different from the expression for electromagnetic wave energy in a dielectric media:

$$\mathcal{E}_D = \frac{V}{32\pi}\,\frac{1}{\omega}\,\frac{\partial\omega^2\epsilon_T^R}{\partial\omega}\,|\boldsymbol{E}(k,\omega)|^2, \quad (275)$$

where $\boldsymbol{E}(k,\omega)$ is the Fourier transform in space and time of the electric field and k and ω are related by the dispersion function through $\epsilon_T(k,k/\omega) = k^2c^2/\omega^2$. Again this difference can be ascribed to the presence of resonant particles. When one computes the energy of transverse neutral modes with super-luminal phase velocities, for which $f_\parallel^{(0)} = 0$, one obtains the dielectric energy.[28]

From the transformation (267) we have

$$\frac{\delta F}{\delta \boldsymbol{f}_{\perp k}^{(1)}} = \frac{4\pi e}{ik}\tilde{\mathcal{G}}^\dagger \left[\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right], \tag{276}$$

and

$$\boldsymbol{\xi}_{\Omega k} = -\frac{1}{\pi}u\zeta\boldsymbol{E}_{\perp k}^{(1)} - \frac{ikc}{\pi}\zeta\boldsymbol{A}_{\perp k}^{(1)}, \tag{277}$$

which implies

$$\delta\boldsymbol{\xi}_{\Omega k} = -\frac{1}{\pi}u\zeta\delta\boldsymbol{E}_{\perp k}^{(1)} - \frac{ikc}{\pi}\zeta\delta\boldsymbol{A}_{\perp k}^{(1)}. \tag{278}$$

Consider the variation in a functional of $\boldsymbol{\xi}_k$ due to variations of $\boldsymbol{\xi}_{\Omega k}$ only. Since there is a one-to-one correspondence between variations of $\boldsymbol{\xi}_{\Omega k}$ and those of $\boldsymbol{E}_{\perp k}^{(1)}$ and $\boldsymbol{A}_{\perp k}^{(1)}$, we have

$$\begin{aligned}\delta F &= \sum_k \left[\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}}\delta\boldsymbol{E}_{\perp k}^{(1)} + \frac{\delta F}{\delta \boldsymbol{A}_{\perp k}^{(1)}}\delta\boldsymbol{A}_{\perp k}^{(1)}\right] \\ &= \sum_k \int du\, \frac{\delta F}{\delta\boldsymbol{\xi}_{\Omega k}}\delta\boldsymbol{\xi}_{\Omega k} \\ &= -\sum_k \left[\frac{1}{\pi}\int du\, u\zeta\frac{\delta F}{\delta\boldsymbol{\xi}_{\Omega k}}\delta\boldsymbol{E}_{\perp k}^{(1)} + \frac{ikc}{\pi}\int du\,\zeta\frac{\delta F}{\delta\boldsymbol{\xi}_{\Omega k}}\delta\boldsymbol{A}_{\perp k}^{(1)}\right].\end{aligned} \tag{279}$$

Comparing these expressions we find

$$\frac{\delta F}{\delta \boldsymbol{E}_{\perp k}^{(1)}} = -\frac{1}{\pi}\int du\, u\zeta\frac{\delta F}{\delta\boldsymbol{\xi}_{\Omega k}}, \tag{280a}$$

$$\frac{\delta F}{\delta \boldsymbol{A}^{(1)}_{\perp k}} = -\frac{ikc}{\pi}\int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{Q}k}}. \tag{280b}$$

Recall that

$$\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{Q}k}} = \mathcal{Q}^{\dagger}\left[\frac{\delta F}{\delta \boldsymbol{\xi}_k}\right] \tag{281}$$

and that

$$\mathcal{Q}[\zeta] = \zeta, \tag{282a}$$

$$\mathcal{Q}[u\zeta] = u\zeta. \tag{282b}$$

These can be combined to yield

$$\frac{\delta F}{\delta \boldsymbol{E}^{(1)}_{\perp k}} = -\frac{1}{\pi}\int du\, u\zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k}, \tag{283a}$$

$$\frac{\delta F}{\delta \boldsymbol{A}^{(1)}_{\perp k}} = -\frac{ikc}{\pi}\int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k}. \tag{283b}$$

Using this and the definition of α in the expression for the transverse bracket we find

$$\begin{aligned}\{F, G\}_{\perp} = -\frac{16}{V}\sum_{k=-\infty}^{\infty} ik\Bigg\{ &\int du\, u\,\alpha\, \widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right]\cdot \widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \boldsymbol{\xi}_{\mathcal{P}-k}}\right] \\ &-\frac{1}{\pi}\int du\, \alpha\, \widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right]\cdot \int du\, \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_{-k}} - \frac{1}{\pi}\int du\, \alpha\, \widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right]\cdot \int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_{-k}} \\ &+\frac{c^2}{\pi}\int du\, u\zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k}\cdot \int du\, \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_{-k}} + \frac{c^2}{\pi}\int du\, u\zeta \frac{\delta G}{\delta \boldsymbol{\xi}_k}\cdot \int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_{-k}}\Bigg\}. \end{aligned} \tag{284}$$

Now

$$\int du\, \alpha\, \widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right] = \int du\, \mathcal{P}\left[\widetilde{\mathcal{G}}[\alpha]\right]\frac{\delta F}{\delta \boldsymbol{\xi}_k}$$

$$= -\int du\, \mathcal{P}\left[(u^2 - c^2)\zeta\right] \frac{\delta F}{\delta \boldsymbol{\xi}_k}$$
$$= -\int du\, (u^2 - c^2)\zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k} \tag{285}$$

and the bracket becomes

$$\{F, G\}_{\perp} = -\frac{16}{V} \sum_{k=-\infty}^{\infty} ik \Bigg\{ \int du\, u\, \alpha\, \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right] \cdot \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta G}{\delta \boldsymbol{\xi}_{\mathcal{P}-k}}\right]$$
$$+ \frac{1}{\pi} \int du\, (u^2 - c^2)\zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k} \cdot \int du\, \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_{-k}} + \frac{1}{\pi} \int du\, (u^2 - c^2)\zeta \frac{\delta G}{\delta \boldsymbol{\xi}_k} \cdot \int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_{-k}}$$
$$+ \frac{c^2}{\pi} \int du\, u\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k} \cdot \int du\, \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_{-k}} + \frac{c^2}{\pi} \int du\, u\, \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_k} \cdot \int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_{-k}} \Bigg\}. \tag{286}$$

The terms involving c^2 cancel, leaving

$$\{F, G\}_{\perp} = -\frac{16}{V} \sum_{k=-\infty}^{\infty} ik \Bigg\{ \int du\, u\, \alpha\, \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta F}{\delta \boldsymbol{\xi}_{\mathcal{P}k}}\right] \cdot \widetilde{\mathcal{G}}^{\dagger} \left[\frac{\delta G}{\delta \boldsymbol{\xi}_{\mathcal{P}-k}}\right]$$
$$+ \frac{1}{\pi} \int du\, u^2 \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_k} \cdot \int du\, \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_{-k}} + \frac{1}{\pi} \int du\, u^2 \zeta \frac{\delta G}{\delta \boldsymbol{\xi}_k} \cdot \int du\, \zeta \frac{\delta F}{\delta \boldsymbol{\xi}_{-k}} \Bigg\}. \tag{287}$$

In Section 3-III we saw

$$\int dx\, x\, \alpha\, \widetilde{\mathcal{G}}^{\dagger}[\phi]\, \widetilde{\mathcal{G}}^{\dagger}[\psi] = -\int dx\, x^2\, \zeta \phi \psi$$
$$-\frac{1}{\pi} \int dx\, x \zeta \phi \int dx\, x^2\, \zeta \psi - \frac{1}{\pi} \int dx\, x^2\, \zeta \phi \int dx\, x \zeta \psi. \tag{179}$$

When $\phi = \mathcal{P}^{\dagger}[\Phi]$ and $\psi = \mathcal{P}^{\dagger}[\Psi]$ this identity can be simplified:

$$\int du\, u\, \alpha\, \widetilde{\mathcal{G}}^{\dagger} \left[\mathcal{P}^{\dagger}[\Phi]\right] \widetilde{\mathcal{G}}^{\dagger} \left[\mathcal{P}^{\dagger}[\Psi]\right] = -\int du\, u^2\, \zeta \mathcal{P}^{\dagger}[\Phi]\, \mathcal{P}^{\dagger}[\Psi]$$

$$-\frac{1}{\pi}\int du\,\mathcal{P}[u\zeta]\Phi\int du\,\mathcal{P}[u^2\zeta]\Psi-\frac{1}{\pi}\int du\,\mathcal{P}[u^2\zeta]\Phi\int du\,\mathcal{P}[u\zeta]\Psi$$
$$=-\int du\,u^2\zeta\mathcal{P}^\dagger[\Phi]\,\mathcal{P}^\dagger[\Psi],\tag{288}$$

since $\mathcal{P}[u\zeta]=0$. We can use this to simplify the first term of the bracket, obtaining:

$$\{F,G\}_\perp=\frac{16}{V}\sum_{k=-\infty}^{\infty}ik\Bigg\{\int du\,u^2\zeta\frac{\delta F}{\delta\boldsymbol{\xi}_{\mathcal{P}k}}\cdot\frac{\delta G}{\delta\boldsymbol{\xi}_{\mathcal{P}-k}}$$
$$-\frac{1}{\pi}\int du\,u^2\zeta\frac{\delta F}{\delta\boldsymbol{\xi}_k}\cdot\int du\,\zeta\frac{\delta G}{\delta\boldsymbol{\xi}_{-k}}-\frac{1}{\pi}\int du\,u^2\zeta\frac{\delta G}{\delta\boldsymbol{\xi}_k}\cdot\int du\,\zeta\frac{\delta F}{\delta\boldsymbol{\xi}_{-k}}\Bigg\}.\tag{289}$$

We need to use one final identity from Section 3-III:

$$\int dx\,x^2\zeta\mathcal{P}^\dagger[\phi]\,\mathcal{P}^\dagger[\psi]=\int dx\,x^2\zeta\phi\psi$$
$$+\frac{1}{\pi}\int dx\,x\zeta\phi\int dx\,x^2\zeta\psi+\frac{\gamma^2}{\pi}\int dx\,x^2\zeta\phi\int dx\,x\zeta\psi\tag{175}$$

to obtain a remarkably simple expression for the transverse part of the bracket:

$$\{F,G\}_\perp=\frac{16}{V}\sum_{k=-\infty}^{\infty}ik\int du\,u^2\zeta\,\frac{\delta F}{\delta\boldsymbol{\xi}_k}\cdot\frac{\delta G}{\delta\boldsymbol{\xi}_{-k}}.\tag{290}$$

Writing $u\zeta$ in terms of the dielectric function, we find

$$\{F,G\}_\perp=-\frac{16}{V}\sum_{k=-\infty}^{\infty}ik\int du\,\frac{\epsilon_T^I}{\left|\epsilon_T-c^2/u^2\right|^2}\,\frac{\delta F}{\delta\boldsymbol{\xi}_k}\cdot\frac{\delta G}{\delta\boldsymbol{\xi}_{-k}}.\tag{291}$$

We now have the necessary ingredients to compute Hamilton's equation giving the equation of motion for $\boldsymbol{\xi}_k$:

$$\dot{\boldsymbol{\xi}}_k=\{\boldsymbol{\xi}_k\,,H_\perp^{(2)}\}_\perp=-iku\,\boldsymbol{\xi}_k,\tag{292}$$

which is the same equation as we obtained from directly transforming the transverse Maxwell-Vlasov equations.

III. Action-Angle Variables for the Linearized Maxwell-Vlasov System

Above we found diagonal forms for the Hamiltonian,

$$H^{(2)} = \frac{V}{32} \sum_{k=-\infty}^{\infty} \int du\; u \left\{ \frac{|\epsilon_L|^2}{\epsilon_L^I} \xi_k \xi_k^* + \frac{|\epsilon_T - c^2/u^2|^2}{\epsilon_T^I} \boldsymbol{\xi}_k \cdot \boldsymbol{\xi}_k^* \right\}, \tag{293}$$

and the bracket,

$$\{F, G\} = -\frac{16}{V} \sum_{k=-\infty}^{\infty} ik \int du \left\{ \frac{\epsilon_L^I}{|\epsilon_L|^2} \frac{\delta F}{\delta \xi_k} \frac{\delta G}{\delta \xi_k^*} + \frac{\epsilon_T^I}{|\epsilon_T - c^2/u^2|^2} \frac{\delta F}{\delta \boldsymbol{\xi}_k} \cdot \frac{\delta G}{\delta \boldsymbol{\xi}_k^*} \right\}. \tag{294}$$

Since ϵ_L and ϵ_T are even functions of k and the $k = 0$ mode is not dynamically accessible, we can rearrange these expressions so that we only need to sum over positive values of k:

$$H^{(2)} = \frac{V}{16} \sum_{k=1}^{\infty} \int du\; u \left\{ \frac{|\epsilon_L|^2}{\epsilon_L^I} \xi_k \xi_k^* + \frac{|\epsilon_T - c^2/u^2|^2}{\epsilon_T^I} \boldsymbol{\xi}_k \cdot \boldsymbol{\xi}_k^* \right\}, \tag{295}$$

$$\begin{aligned} \{F, G\} = -\frac{16}{V} \sum_{k=1}^{\infty} ik \int du \Bigg\{ & \frac{\epsilon_L^I}{|\epsilon_L|^2} \left(\frac{\delta F}{\delta \xi_k} \frac{\delta G}{\delta \xi_k^*} - \frac{\delta G}{\delta \xi_k} \frac{\delta F}{\delta \xi_k^*} \right) \\ & + \frac{\epsilon_T^I}{|\epsilon_T - c^2/u^2|^2} \left(\frac{\delta F}{\delta \boldsymbol{\xi}_k} \cdot \frac{\delta G}{\delta \boldsymbol{\xi}_k^*} - \frac{\delta G}{\delta \boldsymbol{\xi}_k} \cdot \frac{\delta F}{\delta \boldsymbol{\xi}_k^*} \right) \Bigg\}. \end{aligned} \tag{296}$$

Having this diagonal form, it is of interest to compare with the corresponding expressions for a finite degree of freedom system written in action-angle form.[23]

Defining

$$\omega_{\parallel k} = |ku| \operatorname{sgn}\left(u\,\epsilon_L^I\right), \tag{297a}$$

$$\omega_{\perp k} = |ku| \operatorname{sgn}\left(\epsilon_T^I\right), \tag{297b}$$

$$J_{\parallel k} = \frac{1}{\Lambda_\parallel^2(k,u)}\,\xi_k\,\xi_k^*, \tag{297c}$$

$$J_{\perp n,k} = \frac{1}{\Lambda_\perp^2(k,u)}\,\left(\boldsymbol{\xi}_k\right)_n \left(\boldsymbol{\xi}_k^*\right)_n \quad n = 1,2, \tag{297d}$$

where

$$\Lambda_\parallel(k,u) = \sqrt{\frac{16k}{V}\,\frac{\left|\epsilon_L^I\right|}{\left|\epsilon_L\right|^2}}, \tag{298a}$$

$$\Lambda_\perp(k,u) = \sqrt{\frac{16k}{V}\,\frac{\left|\epsilon_L^I\right|}{\left|\epsilon_T - c^2/u^2\right|^2}}, \tag{298b}$$

we can then write

$$H^{(2)} = \sum_{k=1}^{\infty} \int du\, \left\{\omega_{\parallel k}\,J_{\parallel k} + \omega_{\perp k}\,J_{\perp 1,k} + \omega_{\perp k}\,J_{\perp 2,k}\right\}. \tag{299}$$

Combining all of our discrete labels into a single subscript ν this becomes

$$H^{(2)} = \sum_{\nu} \int du\,\omega_\nu\,J_\nu. \tag{300}$$

Introducing the phase angles $\theta_{\parallel k}$ and $\theta_{\perp n,k}$ we can write, for $k > 0$,

$$\xi_k = \Lambda_\parallel \sqrt{J_{\parallel k}}\,e^{i\epsilon_\parallel \theta_{\parallel k}}, \tag{301a}$$

$$\left(\boldsymbol{\xi}_k\right)_n = \Lambda_\perp \sqrt{J_{\perp n,k}}\,e^{i\epsilon_\perp \theta_{\perp n,k}}, \tag{301b}$$

where

$$\epsilon_{\parallel} = \text{sgn}(\epsilon_L^I)\,, \tag{302a}$$

$$\epsilon_{\perp} = \text{sgn}(\epsilon_T^I)\,. \tag{302b}$$

Our restriction to stable equilibria and dynamically accessible perturbations ensure that these relationships are well defined. Using the chain rule to we find

$$\delta\xi_k = \Lambda_{\parallel}\sqrt{J_{\parallel k}}\, e^{i\epsilon_{\parallel}\theta_{\parallel k}} \left(\frac{1}{2J_{\parallel k}}\,\delta J_{\parallel k} + i\epsilon_L\,\delta\theta_{\parallel k} \right), \tag{303a}$$

$$(\delta\boldsymbol{\xi}_k)_n = \Lambda_{\perp}\sqrt{J_{\perp n,k}}\, e^{i\epsilon_{\perp}\theta_{\perp n,k}} \left(\frac{1}{2J_{\parallel k}}\,\delta J_{\perp n,k} + i\epsilon_{\perp}\,\delta\theta_{\perp n,k} \right), \tag{303b}$$

and hence

$$\frac{\delta F}{\delta\xi_k} = \Lambda_{\parallel}\sqrt{J_{\parallel k}}\, e^{i\epsilon_{\parallel}\theta_{\parallel k}} \left(\frac{1}{2J_{\parallel k}}\frac{\delta F}{\delta J_{\parallel k}} + i\epsilon_{\parallel}\frac{\delta F}{\delta\theta_{\parallel k}} \right), \tag{304a}$$

$$\left(\frac{\delta F}{\delta\boldsymbol{\xi}_k}\right)_n = \Lambda_{\perp}\sqrt{J_{\perp n,k}}\, e^{i\epsilon_{\perp}\theta_{\perp n,k}} \left(\frac{1}{2J_{\parallel k}}\frac{\delta F}{\delta J_{\perp n,k}} + i\epsilon_{\perp}\frac{\delta F}{\delta\theta_{\perp n,k}} \right). \tag{304b}$$

Substituting these expressions into the bracket, a straightforward calculation shows

$$\{F, G\} = \sum_{\nu}\int du \left\{ \frac{\delta F}{\delta\theta_{\nu}}\frac{\delta G}{\delta J_{\nu}} - \frac{\delta G}{\delta\theta_{\nu}}\frac{\delta F}{\delta J_{\nu}} \right\}. \tag{305}$$

In these variables, Hamilton's equations yield the expected result:

$$\dot{J}_{\nu} = \{J_{\nu}\,, H\} = 0\,, \tag{306a}$$

$$\dot{\theta}_{\nu} = \{\theta_{\nu}\,, H\} = \omega_{\nu}. \tag{306b}$$

Thus, through a series of coordinate transformations, we have arrived at canonically conjugate coordinates for the linearized Maxwell-Vlasov system.

6

Neutral Modes

We now extend our analysis to include equilibria that support neutral modes *i.e.* equilibria such that the dispersion equation has a discrete, purely real root. As we will see, the modifications that are necessary to treat this case arise as a natural extensions to our transform formalism. In the next chapter, we will consider unstable equilibria. It might seem more logical to consider unstable equilibria first, hoping to recover the neutral mode case as some limit. It is true that the formalism needed for neutral modes is considerably more complicated than that associated with unstable modes, however the neutral mode problem provides a clear path to the handling of the unstable case; the converse is not the case. We proceed as above, beginning with an abstract analysis of the properties of the appropriate transform and then applying this transform to the Vlasov equation, and finally introducing a canonizing coordinate change. In the remaining chapters we will consider only the longitudinal motion purely for reasons of simplicity — the results here are equally applicable to the transverse case.

I. Integral Transforms and Neutral Modes

We now wish to examine the effect of a root of $\beta + i\alpha$. To this end, let α and β have a simple root at $x = x_0$. The key point here is that the longitudinal transform

$$\mathcal{G}[\phi](x) = \alpha\,\overline{\phi} + \beta\,\phi \tag{95}$$

is still well defined. Moreover, it still performs the required task of converting the

Vlasov equation into a simple differential equation. The problem associated with the neutral mode, however, become clear when we examine the inverse transform:

$$\tilde{\mathcal{G}}[\psi] = \zeta\overline{\psi} + \chi\psi. \tag{101}$$

Recall that

$$\chi = \frac{\beta}{\alpha^2 + \beta^2}, \tag{102a}$$

$$\zeta = -\frac{\alpha}{\alpha^2 + \beta^2}. \tag{102b}$$

Evidently, problem is that a root of $\beta + i\alpha$ translates into a pole in $\chi + i\zeta$. As a result we may no longer appeal to Hilbert's theorem to provided a connection between χ and $\overline{\zeta}$. Thus even if we restricted the domain of $\tilde{\mathcal{G}}$ to functions that vanished at x_0, we still would be unable to compute $\mathcal{G}\left[\tilde{\mathcal{G}}\right]$. Furthermore, as we have seen, if $\chi \neq \chi^\infty + \overline{\zeta}$ we expect $\tilde{\mathcal{G}}$ to have a non-trivial null space. This suspicion is borne out by the following calculation. Since $\alpha(x_0) = 0$, $\alpha/(x - x_0)$ is a well behaved function which has a well defined Hilbert transform *viz.*

$$\begin{aligned} H\left[\frac{\alpha}{x - x_0}\right] &= \frac{\overline{\alpha}}{x - x_0} - \frac{1}{\pi}\frac{1}{x - x_0}\int dx\, \frac{\alpha}{x - x_0} \\ &= \frac{\overline{\alpha} - \overline{\alpha}(x_0)}{x - x_0} = \frac{\beta - \beta(x_0)}{x - x_0} \\ &= \frac{\beta}{x - x_0} \end{aligned} \tag{307}$$

and

$$\tilde{\mathcal{G}}\left[\frac{\alpha}{x - x_0}\right] = \frac{\chi\alpha + \beta\zeta}{x - x_0} = 0, \tag{308}$$

since $\chi\alpha + \beta\zeta = 0$.

To proceed further, we will need a method to compute Hilbert transforms of certain singular functions. The most straightforward approach appears to be by means of generalized functions. It turns out that there are only two generalized functions that arise. We begin by computing

$$H\left[\delta(x-x_0)\right] = \frac{1}{\pi}\int dx' \, \frac{\delta(x'-x_0)}{x'-x} = -\frac{1}{\pi}\frac{1}{x-x_0}. \tag{309}$$

This also implies that

$$H\left[\frac{1}{(x-x_0)}\right] = \pi\,\delta(x-x_0)\,, \tag{310}$$

under the assumption that $H[H[\phi]] = -\phi$ holds in the sense of generalized functions. We can, however, verify (310) directly:

$$\begin{aligned}\int dx\,\psi\,H\left[\frac{1}{(x-x_0)}\right] &= -\int dx\,\frac{\overline{\psi}(x)}{x-x_0}\\ &= -\pi\,H\left[\overline{\psi}\right](x_0)\\ &= \pi\,\psi(x_0)\\ &= \pi\int dx\,\psi(x)\,\delta(x-x_0)\,.\end{aligned} \tag{311}$$

Thus, in the sense of generalized functions, we have

$$H\left[\frac{1}{x-x_0}\right] = \pi\,\delta(x-x_0)\,. \tag{312}$$

The obstacle to computing $\overline{\zeta}$ is the pole in ζ at x_0. The only real tool we have at our disposal is Hilbert's theorem, (491), which only applies to functions analytic in the upper half-plane. It is, nonetheless a extremely powerful tool. Consider the function

$$\Xi(z) = \beta(z) + i\,\alpha(z)\,. \tag{313}$$

This is analytic in the upper half-plane having $\beta + i\,\alpha$ for its boundary value on the real axis. Notice that

$$\widehat{\Xi}(z) = \Xi(z) - \frac{1}{z - x_0}\,\Xi'(x_0)\,, \tag{314}$$

is non-zero in the upper half-plane as well as on the real axis. Hence $\widehat{\Xi}^{-1}$ is also an analytic function. Define

$$\widehat{\chi} + i\widehat{\zeta} = \frac{1}{\beta + i\,\alpha} - \frac{1}{x - x_0}\,\frac{1}{\beta'(x_0) + i\,\alpha'(x_0)}\,. \tag{315}$$

Notice that $\widehat{\chi} + i\,\widehat{\zeta}$ is the boundary value of $\widehat{\Xi}^{-1}$. Thus

$$\widehat{\chi} = \widehat{\chi}^{\infty} + \overline{\overline{\zeta}} \tag{316}$$

and $\overline{\overline{\zeta}}$ is unambiguously defined. Splitting (315) into real and imaginary parts we obtain

$$\widehat{\chi} = \chi - \frac{1}{x - x_0}\,\chi^{-1} \tag{317a}$$

and

$$\widehat{\zeta} = \zeta - \frac{1}{x - x_0}\,\zeta^{-1}, \tag{317b}$$

where

$$\chi^{-1} + i\,\zeta^{-1} = \lim_{x \to x_0}\left[(x - x_0)(\chi + i\,\zeta)\right]. \tag{318}$$

That is, $\chi^{-1} + i\,\zeta^{-1}$ is the coefficient of pole in the Laurent series expansion of $\chi + i\,\zeta$ about x_0. We may view (315) as a means of of determining $\overline{\zeta}$ through the formula $\widehat{\chi} = \widehat{\chi}^{\infty} + \overline{\overline{\zeta}}$ namely

$$\widehat{\chi} = \widehat{\chi}^{\infty} + \overline{\zeta} - \pi\,\zeta^{-1}\,\delta(x - x_0)$$

$$= \chi - \frac{1}{x - x_0}\chi^{-1}, \tag{319}$$

implying

$$\chi = \chi^{\infty} + \overline{\zeta} + \frac{1}{x - x_0}\chi^{-1} - \pi\zeta^{-1}\delta(x - x_0), \tag{320}$$

where we have used $\widehat{\chi}^{\infty} = \chi^{\infty}$. We can interpret (320) as the analogue of Hilbert's theorem for the case where the function is analytic in the upper half-plane but has a pole on the axis. The remainder of our analysis will center around determining what effect this different relationship between ζ and χ has on the $\widetilde{\mathcal{G}}$ transform. When studying the transverse motion, we saw that when the functions defining the transform (in that case α and β) are related in a more general way than are boundary values of an analytic function, the associated transform has a non-trivial null space.

The existence of a non-trivial null vector of $\widetilde{\mathcal{G}}$ leads us to expect that $\mathcal{G}\left[\widetilde{\mathcal{G}}[\psi]\right]$ will be a projection operator into the space of functions which can be represented by $\mathcal{G}[\phi]$. An arbitrary function ψ will be representable as

$$\psi = \mathcal{G}[\phi] + A\frac{\alpha}{x - x_0}, \tag{321}$$

where A is a constant. To find A we need to evaluate $\mathcal{G}\left[\widetilde{\mathcal{G}}[\psi]\right]$. To this end, let

$$\phi = \widetilde{\mathcal{G}}[\psi] = \chi\psi + \zeta\overline{\psi}. \tag{322}$$

Using the above result, we can now compute $\overline{\phi}$. Starting with the definition of ϕ we can write

$$\begin{aligned} \phi &= \chi\psi + \zeta\overline{\psi} \\ &= \chi^{\infty}\psi + \overline{\zeta}\psi + \zeta\overline{\psi} + \chi^{-1}\frac{\psi}{x - x_0} - \pi\delta(x - x_0)\zeta^{-1}\psi(x_0). \end{aligned} \tag{323}$$

Taking the Hilbert transform gives

$$\begin{aligned}
\overline{\phi} &= \chi^{\infty}\overline{\psi} - \zeta\psi + \overline{\zeta}\overline{\psi} + \frac{1}{x-x_0}\zeta^{-1}\psi(x_0) \\
&\qquad + \chi^{-1}\left[\pi\,\delta(x-x_0)\psi(x_0) + \frac{1}{x-x_0}\overline{\psi} - \frac{1}{x-x_0}\overline{\psi}(x_0)\right] \\
&= \overline{\psi}\left[\chi^{\infty} + \overline{\psi} + \frac{\chi^{-1}}{x-x_0} - \pi\zeta^{-1}\delta(x-x_0)\right] - \zeta\psi + \pi\,\delta(x-x_0)\chi^{-1}\psi(x_0) \\
&\qquad - \frac{1}{x-x_0}\chi^{-1}\overline{\psi}(x_0) + \frac{\zeta^{-1}}{x-x_0}\psi(x_0) \\
&= \chi\overline{\psi} - \zeta\psi + \frac{1}{x-x_0}\left[\zeta^{-1}\psi(x_0) - \chi^{-1}\overline{\psi}(x_0)\right] \\
&\qquad + \pi\,\delta(x-x_0)\left[\zeta^{-1}\overline{\psi}(x_0) + \chi^{-1}\psi(x_0)\right].
\end{aligned} \tag{324}$$

Using this we can compute $\mathcal{G}[\phi]$:

$$\begin{aligned}
\mathcal{G}[\phi] &= \alpha\overline{\phi} + \beta\phi \\
&= \psi(\beta\chi - \alpha\zeta) + \overline{\psi}(\alpha\chi + \beta\zeta) + \frac{\alpha}{x-x_0}\left[\zeta^{-1}\psi(x_0) - \chi^{-1}\overline{\psi}(x_0)\right] \\
&\qquad + \pi\,\alpha(x_0)\,\delta(x-x_0)\left[\chi^{-1}\psi(x_0) + \zeta^{-1}\overline{\psi}(x_0)\right].
\end{aligned} \tag{325}$$

Using $\alpha(x_0) = 0$, $\alpha\chi + \beta\zeta = 0$ and $\beta\chi - \alpha\zeta = 1$ we can simplify the above to obtain

$$\begin{aligned}
\mathcal{G}\left[\widetilde{\mathcal{G}}[\psi]\right] &= \psi + \frac{\alpha}{x-x_0}\left[\zeta^{-1}\psi(x_0) - \chi^{-1}\overline{\psi}(x_0)\right] \\
&\equiv \mathcal{P}[\psi],
\end{aligned} \tag{326}$$

where $\mathcal{P}$ is the projection operator into the non-null space of $\widetilde{\mathcal{G}}$. That is, $\widetilde{\mathcal{G}}[\mathcal{P}[\psi]] = \widetilde{\mathcal{G}}[\psi]$. We see that the constant A is given by

$$A = \chi^{-1}\overline{\psi}(x_0) - \zeta^{-1}\psi(x_0). \tag{327}$$

Since we claim that $\mathcal{P}$ is a projection operator, then $\mathcal{P}^2 = \mathcal{P}$ should hold. This is straightforward computation:

$$\mathcal{P}[\mathcal{P}[\psi]] = \mathcal{P}[\psi] + \left[\zeta^{-1}\psi(x_0) - \chi^{-1}\overline{\psi}(x_0)\right]\mathcal{P}\left[\frac{\alpha}{x - x_0}\right] \tag{328}$$

and

$$\mathcal{P}\left[\frac{\alpha}{x - x_0}\right] = \frac{\alpha}{x - x_0}\left[1 + \zeta^{-1}\alpha'(x_0) - \chi^{-1}\beta'(x_0)\right] = 0 \tag{329}$$

leaving

$$\mathcal{P}[\mathcal{P}[\psi]] = \mathcal{P}[\psi], \tag{330}$$

verifying our claim that $\mathcal{P}$ is a projection operator. Thus we we see that any function ψ can be written as

$$\psi = \mathcal{G}[\phi] + \phi_0\eta_0, \tag{331}$$

where

$$\eta_0(x) = -\frac{1}{\pi}\frac{\alpha}{x - x_0} \tag{332}$$

and

$$\phi_0 = \pi\left[\zeta^{-1}\psi(x_0) - \chi^{-1}\overline{\psi}(x_0)\right]. \tag{333}$$

The result of the real root of $\beta + i\alpha$ is a single *discrete* contribution to the original transform.

II. Solution of the Vlasov Equation by Integral Transform

We now assume that our equilibrium supports a single neutral mode with phase velocity u_0. Since there are no unstable modes and we are assuming that V is arbitrarily large, this must be an inflection point mode.[29]

Let k_0 be determined by $\epsilon_L^R(k_0, u_0) = 0$. Since ϵ_L is an even function of k, there are two solutions of this equation, giving two neutral modes. Using what we found above, we introduce the change of variables

$$f^{(1)}_{\parallel k} = \frac{ik}{4\pi e}\left\{\mathcal{G}\left[\xi_k\right] + \eta_0(v)\left[\delta_{k\,k_0}\xi_{NM}(k_0) + \delta_{k\,-k_0}\xi_{NM}(-k_0)\right]\right\}. \tag{334}$$

Since we require $f^{(1)}$ to be real, we demand that $f^{(1)*}_{\parallel k} = f^{(1)}_{\parallel -k}$ and thus the two neutral mode amplitudes are related by

$$\xi_{NM}(-k_0) = \xi_{NM}(k_0)^*. \tag{335}$$

For clarity we now drop the argument k_0. The inverse transformation is given by

$$\xi_k(u,t) = \frac{4\pi e}{ik}\widetilde{\mathcal{G}}\left[f^{(1)}_{\parallel k}(v,t)\right], \tag{336a}$$

$$\xi_{NM}(t) = -\frac{4\pi e}{ik}\frac{\pi}{\left|\epsilon_L{}'(u_0)\right|^2}\left[\epsilon_L^{I\prime}(u_0)f^{(1)}_{\parallel k}(u_0,0) + \epsilon_L^{R\prime}(u_0)\overline{f^{(1)}_{\parallel k}}(u_0,t)\right]. \tag{336b}$$

As in the case without neutral modes we take

$$\alpha = -\frac{4\pi^2 e^2}{mk^2}f^{(0)\prime}_{\parallel} = \epsilon_L^I, \tag{337a}$$

$$\beta = 1 + \overline{\alpha} = \epsilon_L^R. \tag{337b}$$

Now

$$\begin{aligned} E^{(1)}_{\parallel k} &= \frac{4\pi e}{ik}\int dv_\parallel\, f^{(1)}_{\parallel k} \\ &= \int du\,\xi_k + \delta_{k\,k_0}\xi_{NM} + \delta_{k\,-k_0}\xi^*_{NM}, \end{aligned} \tag{338}$$

where the last step follows since

$$\int du\,\eta_0(u) = -\overline{\alpha}(u_0) = \beta^\infty = 1\,. \tag{339}$$

The longitudinal Vlasov equation can now be written as

$$\mathcal{G}[\dot{\xi}_k] + iku\,\mathcal{G}[\xi_k] + \frac{ik}{\pi}\alpha\left[\int du\,\xi_k + \delta_{k\,k_0}\xi_{NM} + \delta_{k\,-k_0}\xi^*_{NM}\right]$$
$$+\left[\delta_{k\,k_0}\left(\dot{\xi}_{NM} + iku\xi_{NM}\right) + \delta_{k\,-k_0}\left(\dot{\xi}^*_{NM} + iku\xi^*_{NM}\right)\right]\eta_0(u) = 0\,. \quad (340)$$

As we saw before

$$u\,\mathcal{G}[\psi] = \mathcal{G}[u\,\psi] - \frac{\alpha}{\pi}\int du\,\psi\,, \quad (341)$$

which we can use along with the definition of η_0 to simplify (340), leaving

$$\mathcal{G}[\dot{\xi}_k + iku\,\xi_k] + \eta_0\left[\delta_{k\,k_0}\left(\dot{\xi}_{NM} + iku_0\,\xi_{NM}\right) + \delta_{k\,-k_0}\left(\dot{\xi}^*_{NM} + iku_0\,\xi^*_{NM}\right)\right]. \quad (342)$$

Since η_0 is not in the range of $\mathcal{G}$, (342) is equivalent to

$$\dot{\xi}_k + iku\xi_k = 0\,, \quad (343a)$$

$$\dot{\xi}_{NM} + iku_0\,\xi_{NM} = 0\,, \quad (343b)$$

$$\dot{\xi}^*_{NM} - iku_0\,\xi^*_{NM} = 0\,. \quad (343c)$$

Thus

$$\xi_k(u,t) = \xi_k(u)\,e^{-ikut}, \quad (344a)$$

$$\xi_{NM}(t) = \xi_{NM}(0)\,e^{-iku_0t}, \quad (344b)$$

where $\xi_k(u)$ and $\xi_{NM}(0)$ are determined from the initial perturbation through

$$\xi_k(u) = \frac{4\pi e}{ik}\tilde{\mathcal{G}}\left[f^{(1)}_{\parallel k}(t=0)\right], \quad (345a)$$

$$\xi_{NM}(0) = -\frac{4\pi e}{ik}\frac{\pi}{\left|\epsilon_L{}'(u_0)\right|^2}\left[\epsilon^{I}_L{}'(u_0)f^{(1)}_{\parallel k}(u_0,0) + \epsilon^{R}_L{}'(u_0)\overline{f^{(1)}_{\parallel k}}(u_0,0)\right]. \quad (345b)$$

Notice that for a simple real root of the dispersion relation, we obtain only *one* discrete mode and yet have a *complete* solution. This mode is the same as originally proposed by Case.[15] The difficulty several authors had with Case's original solution is that the prescription given for computing the amplitudes is not entirely correct. In Case's treatment (which followed a considerably different path than ours) his (implicit) regularization of the expression for the inverse transform was incompatible with the definition of the discrete eigenfunction. The additional "discrete" eigenmodes proposed by later authors[16–18] were needed to correct this but these modes are not truly discrete as they have a continuum label. Note that our discrete mode oscillates with a frequency, $k\,u_0$ determined by the phase velocity of the neutral mode while Case's second mode (the so-called Siewert solution), due to the nature of its continuum label, does not oscillate with a single, well defined frequency.

III. Canonizing the Hamiltonian and Bracket

Making the same substitution as in the previous section, we can write the longitudinal contribution to the Hamiltonian as

$$\begin{aligned} H_{\parallel}^{(2)} = \frac{V}{32} \sum_{k=-\infty}^{\infty} \Bigg\{ \int du\, \frac{u}{\alpha} \Big[\mathcal{G}\left[\xi_k\right] + \eta_0(u) \left(\delta_{k\,k_0} \xi_{NM} + \delta_{k\,-k_0} \xi^*_{NM} \right) \Big] \\ \times \Big[\mathcal{G}\left[\xi_{-k}\right] + \eta_0(u) \left(\delta_{-k\,k_0} \xi_{NM} + \delta_{-k\,-k_0} \xi^*_{NM} \right) \Big] \\ + \frac{1}{\pi} \left[\int du\, \xi_k + \delta_{k\,k_0} \xi_{NM} + \delta_{k\,-k_0} \xi^*_{NM} \right] \\ \times \left[\int du\, \xi_{-k} + \delta_{-k\,k_0} \xi_{NM} + \delta_{-k\,-k_0} \xi^*_{NM} \right] \Bigg\}. \qquad (346) \end{aligned}$$

In simplifying this expression, there are two new integrals that we need to evaluate. The first is

$$\begin{aligned}\int du\,\frac{u}{\alpha}\,\eta_0^2(u) &= -\frac{1}{\pi}\int du\,\frac{u}{u-u_0}\,\eta_0(u)\\ &= -\frac{1}{\pi}\int du\,\eta_0 - \frac{u_0}{\pi}\int du\,\frac{\eta_0(u)}{u-u_0}\\ &= -\frac{1}{\pi} + \frac{u_0}{\pi}\overline{\alpha}'(u_0)\\ &= -\frac{1}{\pi} + \frac{u_0}{\pi}\,\beta'(u_0)\,.\end{aligned} \tag{347}$$

Recall that the neutral mode phase velocity corresponds to an inflection point of the equilibrium distribution and thus $\alpha'(u_0) = 0$. Consequently, the above integrals exist in the usual sense and are *not* interpreted as Cauchy principal values. The second integral of interest is

$$\begin{aligned}\int du\,\frac{u}{\alpha}\,\eta_0(u) &= -\frac{1}{\pi}\int du\,\frac{u}{u-u_0}\,\mathcal{G}[\psi]\\ &= -\frac{1}{\pi}\int du\,\mathcal{G}[\psi] + u_0\int du\,\frac{1}{u-u_0}\,\mathcal{G}[\psi]\\ &= -\frac{1}{\pi}\int du\,\psi + u_0\int du\,\psi\,\mathcal{G}^{\dagger}\left[\frac{1}{u-u_0}\right]\\ &= -\frac{1}{\pi}\int du\,\psi + u_0\int du\,\psi\left\{\frac{\beta}{u-u_0} - H\left[\frac{\alpha}{u-u_0}\right]\right\}\\ &= -\frac{1}{\pi}\int du\,\psi\,.\end{aligned} \tag{348}$$

We can now simplify the energy expression to obtain

$$\begin{aligned}H_{\parallel}^{(2)} = \frac{V}{32}\sum_{k=-\infty}^{\infty}&\left\{\int du\,\frac{u}{\alpha}\,\mathcal{G}\left[\xi_k\right]\mathcal{G}\left[\xi_{-k}\right] + \frac{1}{\pi}\int du\,\xi_k\int du\,\xi_{-k}\right\}\\ &- \frac{V}{16\pi}\left\{\xi_{NM}\int du\,\xi_{-k_0} + \xi_{NM}^{*}\int du\,\xi_{k_0}\right\} + \frac{V}{16\pi}\left(u_0\beta'(u_0) - 1\right)\left|\xi_{NM}\right|^2\end{aligned}$$

$$+\frac{V}{16\pi}\left\{\xi_{NM}\int du\,\xi_{-k_0}+\xi^*_{NM}\int du\,\xi_{k_0}\right\}+\frac{V}{16\pi}\left|\xi_{NM}\right|^2$$

$$=\;-\frac{V}{32\pi}\sum_{k=-\infty}^{\infty}\int du\,\frac{u}{\zeta}\left|\xi_k\right|^2+\frac{V}{16\pi}u_0\,\beta'(u_0)\left|\xi_{NM}\right|^2, \qquad (349)$$

where we have used (110) to combine the first two terms. Defining $\omega_0 = ku_0$ and expressing α and β in terms of the dielectric function gives

$$H^{(2)}_{\parallel}=\frac{V}{32}\sum_{k=-\infty}^{\infty}\int du\,u\,\frac{\left|\epsilon_L\right|^2}{\epsilon^I_L}\left|\xi_k\right|^2+\frac{V}{16\pi}\omega_0\,\frac{\partial\epsilon^R_L(\omega_0)}{\partial\omega_0}\left|\xi_{NM}\right|^2. \qquad (350)$$

We see that the neutral mode contribution to the energy is equal to the dielectric energy. It is only for inflection point neutral modes that the energy of the perturbation is given by the dielectric energy.[29] Heuristically, the difference between a neutral mode and the continuum modes is that the discrete eigenfunctions lack the delta function term — *i.e.* there is no interaction between the electric field and resonant particles.[30] *Thus it is the presence of resonant particles that makes the continuum contribution to the energy different from the dielectric energy.*

This first step in computing the bracket in terms of the new variables ξ_k and ξ_{NM} is to use the chain rule to relate functional derivatives with respect to $f^{(1)}_{\parallel k}$ to those with respect to ξ_k and ξ_{NM}. Now

$$\delta F=\sum_{k=-\infty}^{\infty}\int dv_{\parallel}\,\frac{\delta F}{\delta f^{(1)}_{\parallel k}}\,\delta f^{(1)}_{\parallel k}$$

$$=\sum_{k=-\infty}^{\infty}\int du\,\frac{\delta F}{\delta\xi_k}\,\delta\xi_k+\frac{\partial F}{\partial\xi_{NM}}\,\delta\xi_{NM}+\frac{\partial F}{\partial\xi^*_{NM}}\,\delta\xi^*_{NM}$$

$$=\sum_{k=-\infty}^{\infty}\int du\,\left(\frac{\delta F}{\delta\xi_k}\,\frac{\delta\xi_k}{\delta f^{(1)}_{\parallel k}}+\frac{\partial F}{\partial\xi_{NM}}\,\frac{\delta\xi_{NM}}{\delta f^{(1)}_{\parallel k}}+\frac{\partial F}{\partial\xi^*_{NM}}\,\frac{\delta\xi^*_{NM}}{\delta f^{(1)}_{\parallel k}}\right)\delta f^{(1)}_{\parallel k}. \qquad (351)$$

As before,

$$\frac{\delta \xi_k}{\delta f^{(1)}_{\parallel k}} = \frac{4\pi e}{ik} \widetilde{\mathcal{G}}. \tag{352}$$

From the definition of ξ_{NM},

$$\begin{aligned} \xi_{NM} &= -\pi \frac{4\pi e}{ik_0} \left[\chi^{-1} \overline{f^{(1)}_{\parallel k_0}}(u_0) - \zeta^{-1} f^{(1)}_{\parallel k_0}(u_0) \right] \\ &= -\pi \frac{4\pi e}{ik_0} \int du \, f^{(1)}_{\parallel k_0} \left[\chi^{-1} \frac{1}{\pi} \frac{1}{u - u_0} - \zeta^{-1} \delta(u - u_0) \right], \end{aligned} \tag{353}$$

we find

$$\frac{\delta \xi_{NM}}{\delta f^{(1)}_{\parallel k}} = -\pi \frac{4\pi e}{ik} \delta_{k k_0} \left[\chi^{-1} \frac{1}{\pi} \frac{1}{u - u_0} - \zeta^{-1} \delta(u - u_0) \right] \tag{354}$$

and

$$\frac{\delta \xi^*_{NM}}{\delta f^{(1)}_{\parallel k}} = -\pi \frac{4\pi e}{ik} \delta_{k - k_0} \left[\chi^{-1} \frac{1}{\pi} \frac{1}{u - u_0} - \zeta^{-1} \delta(u - u_0) \right]. \tag{355}$$

Since we are considering an inflection point neutral mode, $\alpha'(u_0) = 0$ and consequently $\zeta^{-1} = 0$ in which case the above simplifies to become

$$\frac{\delta \xi_{NM}}{\delta f^{(1)}_{\parallel k}} = -\frac{4\pi e}{ik} \delta_{k k_0} \chi^{-1} \frac{1}{u - u_0} \tag{356}$$

and

$$\frac{\delta \xi^*_{NM}}{\delta f^{(1)}_{\parallel k}} = -\frac{4\pi e}{ik} \delta_{k - k_0} \chi^{-1} \frac{1}{u - u_0}. \tag{357}$$

Using this in the expression for δF, we see that

$$\begin{aligned} \delta F = \sum_{k=-\infty}^{\infty} \frac{4\pi e}{ik} \int du \, \widetilde{\mathcal{G}} \left[\frac{\delta F}{\delta \xi_k} \right] \delta f^{(1)}_{\parallel k} - \frac{4\pi e}{ik_0} \frac{\partial F}{\partial \xi_{NM}} \chi^{-1} \int du \, \frac{1}{u - u_0} \delta f^{(1)}_{\parallel k_0} \\ + \frac{4\pi e}{ik_0} \frac{\partial F}{\partial \xi^*_{NM}} \chi^{-1} \int du \, \frac{1}{u - u_0} \delta f^{(1)}_{\parallel - k_0}. \end{aligned} \tag{358}$$

Comparing our expressions for δF, we find

$$\frac{\delta F}{\delta f^{(1)}_{\parallel k}} = \frac{4\pi e}{ik}\left\{\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_k}\right] - \chi^{-1}\frac{1}{u-u_0}\left[\delta_{k\,k_0}\frac{\partial F}{\partial \xi_{NM}} + \delta_{k\,-k_0}\frac{\partial F}{\partial \xi^*_{NM}}\right]\right\}. \tag{359}$$

Upon substitution of the above, the longitudinal part of the bracket becomes

$$\begin{aligned}\{F,G\}_{\parallel} = -\frac{16i}{V}\sum_{k=-\infty}^{\infty} k\Bigg\{ &\int du\,\alpha\,\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_k}\right]\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \xi_{-k}}\right] \\ &+ \pi\left[\delta_{k\,k_0}\frac{\partial F}{\partial \xi_{NM}} + \delta_{k\,-k_0}\frac{\partial F}{\partial \xi^*_{NM}}\right]\chi^{-1}\int du\,\eta_0(u)\,\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta \xi_{-k}}\right] \\ &+ \pi\left[\delta_{-k\,k_0}\frac{\partial G}{\partial \xi_{NM}} + \delta_{-k\,-k_0}\frac{\partial G}{\partial \xi^*_{NM}}\right]\chi^{-1}\int du\,\eta_0(u)\,\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta \xi_{-k}}\right] \\ &- \pi\left[\delta_{k\,k_0}\frac{\partial F}{\partial \xi_{NM}}\frac{\partial G}{\partial \xi^*_{NM}} + \delta_{k\,-k_0}\frac{\partial F}{\partial \xi^*_{NM}}\frac{\partial G}{\partial \xi^*_{NM}}\right] \\ &\qquad\times(\chi^{-1})^2\int du\,\frac{\eta_0(u)}{u-u_0}\Bigg\}. \end{aligned} \tag{360}$$

Consider

$$\int du\,\eta_0(u)\,\widetilde{\mathcal{G}}^{\dagger}[\psi] = \int du\,\psi\,\widetilde{\mathcal{G}}[\eta_0] = 0\,, \tag{361}$$

which follows since η_0 is a null vector of $\widetilde{\mathcal{G}}$. Furthermore,

$$(\chi^{-1})^2\int du\,\frac{\eta_0(u)}{u-u_0} = -(\chi^{-1})^2\,\overline{\alpha}'(u_0) = -\frac{1}{\beta'(u_0)}\,, \tag{362}$$

where the last step follows from the definition of χ^{-1} and the fact that $\alpha'(u_0) = 0$. Carrying out the sum and using the above and (109), we find

$$\begin{aligned}\{F,G\}_{\parallel} = \frac{16i}{V}\sum_{k=-\infty}^{\infty} k\int du\,\zeta\,\frac{\delta F}{\delta \xi_k}\frac{\delta G}{\delta \xi_{-k}} \\ -\frac{16i}{V}\frac{\pi k_0}{\beta'(u_0)}\left[\frac{\partial F}{\partial \xi_{NM}}\frac{\partial G}{\partial \xi^*_{NM}} - \frac{\partial F}{\partial \xi^*_{NM}}\frac{\partial G}{\partial \xi_{NM}}\right]. \end{aligned} \tag{363}$$

Writing the bracket in terms of the dielectric function gives

$$\{F,G\}_{\parallel} = -\frac{16i}{V}\sum_{k=-\infty}^{\infty} k \int du \, \frac{\epsilon_L^I}{|\epsilon_L|^2} \frac{\delta F}{\delta \xi_k} \frac{\delta G}{\delta \xi_{-k}}$$
$$-\frac{16i}{V}\frac{\pi k_0}{\epsilon_L^{R\prime}(u_0)}\left[\frac{\partial F}{\partial \xi_{NM}}\frac{\partial G}{\partial \xi_{NM}^*} - \frac{\partial F}{\partial \xi_{NM}^*}\frac{\partial G}{\partial \xi_{NM}}\right]. \quad (364)$$

This is very much like our result of Chapter 5 except for the addition new term involving the neutral mode amplitudes. Further, up to a simple scaling, the neutral mode amplitudes ξ_{NM} and ξ_{NM}^* are canonical conjugate variables.

Hamilton's equations give the expected results:

$$\dot{\xi}_k = \{\xi\,, H_{\parallel}^{(2)}\}_{\parallel} = -iku\xi_k, \quad (365a)$$

$$\dot{\xi}_{NM} = \{\xi_{NM}\,, H_{\parallel}^{(2)}\}_{\parallel} = -ik_0 u_0 \xi_{NM}, \quad (365b)$$

which is identical to the equations of motion found in the previous section.

7

Unstable Modes

We now move on to consider equilibria that support unstable modes. The treatment of unstable modes, while algebraically more complex that of neutral modes, is more straightforward as it does not involve generalized functions but otherwise will closely parallel the neutral mode case.

I. Integral Transforms and Unstable Modes

We now consider the case where the equilibrium supports an unstable mode; that is, the dispersion relation has a root in the upper half-plane. Let $\beta + i\alpha$ have a simple root at $z_0 = x_0 + iy_0$. As before we wish to study the transformation

$$\mathcal{G}[\phi] = \alpha\overline{\phi} + \beta\phi \tag{95}$$

and its inverse

$$\widetilde{\mathcal{G}}[\psi] = \zeta\overline{\psi} + \chi\psi, \tag{101}$$

where

$$\chi = \frac{\beta}{\alpha^2 + \beta^2}, \tag{102a}$$

$$\zeta = -\frac{\alpha}{\alpha^2 + \beta^2}. \tag{102b}$$

Let

$$\Xi(z) = \beta(z) + i\,\alpha(z) = \Xi^{\infty} + \frac{1}{\pi}\int dt\,\frac{\alpha(t)}{t - z}. \tag{366}$$

As z approaches the real axis Ξ has a well defined limit:

$$\lim_{\operatorname{Im} z \to 0^{\pm}} \Xi(z) = \beta \pm i\alpha. \tag{367}$$

Since $\alpha(t)$ is real for real t, $\Xi^*(z) = \Xi(z^*)$ and we see that $\Xi(x_0 - iy_0) = 0$ in addition to $\Xi(x_0 + iy_0) = 0$. Since Ξ has a zero in the upper half-plane

$$\chi + i\zeta = \frac{1}{\Xi} \tag{368}$$

will have a pole at $x_0 \pm iy_0$. Thus even though χ and ζ are well defined functions on the real axis and $\overline{\zeta}$ is well defined, $\chi \neq \chi^{\infty} + i\overline{\zeta}$; were this true then Hilbert's theorem would tell us that $\chi + i\zeta$ is the boundary value of a function analytic in the upper half-plane. As we have just seen, this is not so.

One might think that this is entirely different from the neutral mode case since χ and ζ are non-singular on the real axis and there is no difficulty in computing $\widetilde{\mathcal{G}}[\phi]$ for any ϕ. In the neutral mode case the singularity in $\widetilde{\mathcal{G}}$ was not the real difficulty, it was a manifestation of the true problem: namely that $\chi \neq \chi^{\infty} + \overline{\zeta}$. Viewed in this light, we see that all cases of discrete modes (roots of $\beta + i\alpha$) are closely related.

As a means to determine the relationship between χ and $\overline{\zeta}$, define

$$\begin{aligned} \widehat{\chi} + i\widehat{\zeta} &= \frac{1}{\beta + i\alpha} - \frac{1}{x - x_0 - iy_0} \frac{1}{\Xi'(x_0 + iy_0)} \\ &= \chi + i\zeta - \frac{\chi^{-1} + i\zeta^{-1}}{x - x_0 - iy_0}, \end{aligned} \tag{369}$$

where $\chi^{-1} + i\zeta^{-1} = \Xi'(x_0 + iy_0)^{-1}$. Separating (369) into real an imaginary parts we obtain explicit expression for $\widehat{\chi}$ and $\widehat{\zeta}$:

$$\widehat{\chi} = \chi - \frac{(x - x_0)\chi^{-1} - y_0\zeta^{-1}}{(x - x_0)^2 + y_0^2}, \tag{370a}$$

$$\widehat{\zeta} = \zeta - \frac{(x-x_0)\zeta^{-1} + y_0\chi^{-1}}{(x-x_0)^2 + y_0^2}. \tag{370b}$$

Notice that $\widehat{\chi} + i\widehat{\zeta}$ is the boundary value of

$$\left\{\Xi(z) - \frac{1}{z-z_0}\Xi'(z_0)\right\}^{-1} \tag{371}$$

which is analytic in the upper half-plane and therefore $\widehat{\chi} = \chi^{\infty} + \overline{\widehat{\zeta}}$. From this, we will find the relationship between χ and $\overline{\zeta}$.

Before proceeding we pause to compute a Hilbert transform that we will need frequently. Consider, for $y_0 > 0$,

$$H\left[\frac{1}{x-x_0-iy_0}\right] = \frac{1}{\pi}\int dx' \, \frac{1}{x'-x} \frac{1}{x'-x_0-iy_0}. \tag{372}$$

While is is possible to evaluate this integral directly in terms of elementary functions, it is considerably simpler to compute this by contour integration. Let the contour Γ be as shown in Figure 1 then

$$\frac{1}{\pi}\oint_{\Gamma} dz \, \frac{1}{z-x} \frac{1}{z-x_0-iy_0} = -2i\frac{1}{x-x_0-iy_0}, \tag{373}$$

since the only pole enclosed by the contour is at $z = x_0 + iy_0$. The integral over the large arc tends to zero as its radius becomes large. Thus we have

$$\begin{aligned}
\frac{-2i}{x-x_0-iy_0} &= \frac{1}{\pi}\lim_{\epsilon\to 0}\Bigg[i\int_0^{\pi} d\theta \, \frac{1}{\epsilon e^{i\theta} + x - x_0 - iy_0} \\
&\qquad + \left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty}\right) dx' \, \frac{1}{x'-x} \frac{1}{x'-x_0-iy_0}\Bigg] \\
&= \frac{1}{\pi}\frac{1}{x-x_0-iy_0} + \frac{1}{\pi}\mathrm{P}\!\int dx' \frac{1}{x'-x} \frac{1}{x'-x_0-iy_0},
\end{aligned} \tag{374}$$

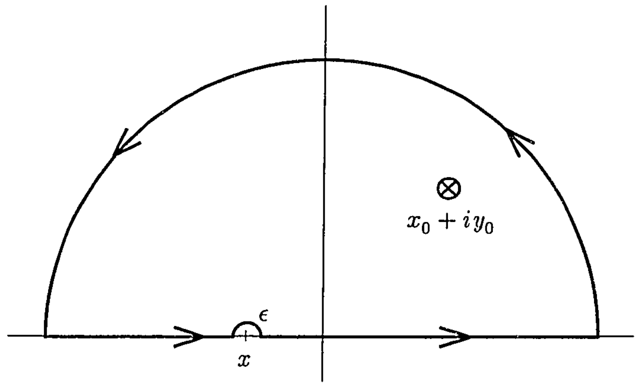

Figure 1: The contour Γ used to evaluate the integral in (373)

implying

$$H\left[\frac{1}{x - x_0 - iy_0}\right] = \frac{-i}{x - x_0 - iy_0}, \quad y_0 > 0 \tag{375}$$

and

$$H\left[\frac{1}{x - x_0 + iy_0}\right] = \frac{i}{x - x_0 + iy_0}, \quad y_0 > 0, \tag{376}$$

since H is a real operator. Furthermore, since

$$\frac{x - x_0}{(x - x_0)^2 + y_0^2} = \frac{1}{2}\left[\frac{1}{x - x_0 - iy_0} + \frac{1}{x - x_0 + iy_0}\right] \tag{377}$$

and

$$\frac{y_0}{(x - x_0)^2 + y_0^2} = \frac{1}{2i}\left[\frac{1}{x - x_0 + iy_0} - \frac{1}{x - x_0 - iy_0}\right], \tag{378}$$

we have

$$H\left[\frac{x - x_0}{(x - x_0)^2 + y_0^2}\right] = \frac{y_0}{(x - x_0)^2 + y_0^2} \tag{379}$$

and

$$H\left[\frac{y_0}{(x - x_0)^2 + y_0^2}\right] = -\frac{x - x_0}{(x - x_0)^2 + y_0^2}. \tag{380}$$

We can now compute $\overline{\zeta}$ in terms of $\overline{\overline{\zeta}}$:

$$\overline{\overline{\zeta}} = \overline{\zeta} - \frac{y_0 \zeta^{-1} - (x - x_0)\chi^{-1}}{(x - x_0)^2 + y_0^2}. \tag{381}$$

Since $\widehat{\chi} = \chi^{\infty} + \overline{\overline{\zeta}}$, we have

$$\begin{aligned} \chi &= \widehat{\chi} + \frac{(x - x_0)\chi^{-1} + y_0 \zeta^{-1}}{(x - x_0)^2 + y_0^2} \\ &= \chi^{\infty} + \overline{\zeta} + 2\frac{(x - x_0)\chi^{-1} + y_0 \zeta^{-1}}{(x - x_0)^2 + y_0^2} \\ &= \chi^{\infty} + \overline{\zeta} + \frac{\chi^{-1} + i\zeta^{-1}}{x - x_0 - iy_0} + \frac{\chi^{-1} - i\zeta^{-1}}{x - x_0 + iy_0}. \end{aligned} \tag{382}$$

It is interesting to compare the above with the corresponding result for the neutral mode case, (320), as well as to Hilbert's theorem, (491). It is important to realize that we *cannot* obtain (320) from the above by taking the limit $y_0 \longrightarrow 0^+$. In this limit, the root of $\beta + i\alpha$ is not simple, as we assumed in the neutral mode case, but will have multiplicity two. We will say more about the significance of this below. We see that (382) is the extension of Hilbert's theorem to the instance of a function analytic but for a simple pole in the upper half-plane.

As we have already seen, when the functions parameterizing the transformation are not related in the standard way, the transformation will have a non-trivial null space. Thus (382) implies that $\widetilde{\mathcal{G}}$ will have a non-trivial null space. This null space is of concern since only functions lying in its complement can be represented by $\mathcal{G}[\phi]$. To determine the null space, we need to evaluate $\mathcal{G}\left[\widetilde{\mathcal{G}}[\psi]\right]$. Let

$$\begin{aligned} \phi &= \widetilde{\mathcal{G}}[\psi] = \chi\psi + \zeta\overline{\psi} \\ &= \chi^{\infty}\psi + \overline{\zeta}\psi + \zeta\overline{\psi} + \frac{\chi^{-1} + i\zeta^{-1}}{x - x_0 - iy_0}\psi + \frac{\chi^{-1} - i\zeta^{-1}}{x - x_0 + iy_0}\psi. \end{aligned} \tag{383}$$

To compute $\overline{\phi}$, we need to evaluate

$$H\left[\frac{\psi}{x-x_0-iy_0}\right] = \frac{1}{\pi}\int dx'\,\frac{1}{x'-x}\,\frac{\psi(x')}{x'-x_0-iy_0}$$
$$= \frac{1}{x-x_0-iy_0}\,\frac{1}{\pi}\int dx'\,\psi(x')\left[\frac{1}{x'-x}-\frac{1}{x'-x_0-iy_0}\right]$$
$$= \frac{1}{x-x_0-iy_0}\left[\overline{\psi}(x)-\overline{\psi}(x_0+iy_0)\right]. \tag{384}$$

Similarly we find

$$H\left[\frac{\psi}{x-x_0+iy_0}\right] = \frac{1}{x-x_0+iy_0}\left[\overline{\psi}(x)-\overline{\psi}(x_0-iy_0)\right]. \tag{385}$$

Using these results, we can now compute $\overline{\phi}$:

$$\overline{\phi} = \chi^{\infty}\,\overline{\psi}+\overline{\zeta}\,\overline{\psi}+\frac{\chi^{-1}+i\zeta^{-1}}{x-x_0-iy_0}\,\overline{\psi}+\frac{\chi^{-1}-i\zeta^{-1}}{x-x_0+iy_0}\,\overline{\psi}$$
$$-\zeta\psi-\frac{\chi^{-1}+i\zeta^{-1}}{x-x_0-iy_0}\,\overline{\psi}(x_0+iy_0)-\frac{\chi^{-1}-i\zeta^{-1}}{x-x_0+iy_0}\,\overline{\psi}(x_0-iy_0)$$
$$= \chi\overline{\psi}-\zeta\psi-\frac{\chi^{-1}+i\zeta^{-1}}{x-x_0-iy_0}\,\overline{\psi}(x_0+iy_0)-\frac{\chi^{-1}-i\zeta^{-1}}{x-x_0+iy_0}\,\overline{\psi}(x_0-iy_0). \tag{386}$$

It is a simple matter to calculate $\mathcal{G}[\phi]$. Doing so we find

$$\mathcal{G}\left[\widetilde{\mathcal{G}}[\psi]\right] = \psi+\pi\eta_{+}(\chi^{-1}+i\zeta^{-1})\overline{\psi}(x_0+iy_0)+\pi\eta_{-}(\chi^{-1}-i\zeta^{-1})\overline{\psi}(x_0-iy_0)$$
$$\equiv \mathcal{P}[\psi], \tag{387}$$

where

$$\eta_{\pm} = -\frac{1}{\pi}\,\frac{\alpha}{x-x_0\mp iy_0}. \tag{388}$$

Thus for an arbitrary function ψ only $\mathcal{P}[\psi]$ can be represented by $\mathcal{G}[\phi]$.

We verify that $\mathcal{P}$, defined by (387), is in fact a projection operator by computing $\mathcal{P}^2$:

$$\mathcal{P}[\mathcal{P}[\psi]] = \mathcal{P}[\psi] + \pi(\chi^{-1} + i\zeta^{-1})\overline{\psi}(x_0 + iy_0)\,\mathcal{P}[\eta_+] + \pi(\chi^{-1} - i\zeta^{-1})\overline{\psi}(x_0 - iy_0)\,\mathcal{P}[\eta_-]\,. \quad (389)$$

To compute $\mathcal{P}[\eta_\pm]$ consider

$$\begin{aligned}\overline{\eta_\pm}(x_0 \pm iy_0) &= -\frac{1}{\pi^2}\int dt\,\frac{\alpha(t)}{(t - x_0 \mp iy_0)^2} \\ &= -\frac{1}{\pi}\frac{d}{dz}\int dt\,\frac{\alpha(t)}{t-z}\bigg|_{z\,=\,x_0 \pm iy_0} \\ &= -\frac{1}{\pi}\,\Xi'(x_0 \pm iy_0) \\ &= -\frac{1}{\pi}\frac{1}{\chi^{-1} \pm i\zeta^{-1}}\end{aligned} \quad (390)$$

and

$$\begin{aligned}\overline{\eta_\pm}(x_0 \mp iy_0) &= -\frac{1}{\pi}\int dt\,\frac{\alpha(t)}{(t - x_0 \mp iy_0)(t - x_0 \pm iy_0)} \\ &= \frac{i}{2y_0}\frac{1}{\pi}\int dt\,\alpha(t)\left[\frac{1}{t - x_0 \mp iy_0} - \frac{1}{t - x_0 \pm iy_0}\right] \\ &= \frac{i}{2y_0}\left[\Xi(x_0 \pm iy_0) - \Xi(x_0 \mp iy_0)\right] \\ &= 0\,.\end{aligned} \quad (391)$$

These results can be combined to show

$$\begin{aligned}\mathcal{P}[\eta_\pm] &= \eta_\pm + \pi\eta_+(\chi^{-1} + i\zeta^{-1})\overline{\eta_\pm}(x_0 + iy_0) + \pi\eta_-(\chi^{-1} - i\zeta^{-1})\overline{\eta_\pm}(x_0 - iy_0) \\ &= \eta_\pm\left[1 - (\chi^{-1} \pm i\zeta^{-1})\,\Xi'(x_0 \pm iy_0)\right]\end{aligned}$$

$$= 0 \tag{392}$$

and thus $\mathcal{P}[\mathcal{P}[\psi]] = \mathcal{P}[\psi]$, confirming that $\mathcal{P}$ is a projection operator. Thus we see that any function ψ can be written as

$$\psi = \mathcal{G}[\phi] + \phi_+ \eta_+ + \phi_- \eta_-, \tag{393}$$

where

$$\phi_\pm = -\pi \left(\chi^{-1} \pm i\zeta^{-1}\right) \overline{\psi}\left(x_0 \pm i y_0\right). \tag{394}$$

II. Solution of the Vlasov Equation by Integral Transform

We consider the solution of the longitudinal Vlasov equation when the equilibrium is such that ϵ_L has a simple root at $u_0 \pm i\gamma_0$. Let $k_0 > 0$ be the solution of $\epsilon_L(u_0 \pm i\gamma_0, k_0) = 0$. To solve

$$\dot{f}^{(1)}_{\parallel k} + ikv_\parallel f^{(1)}_{\parallel k} + \frac{e}{m} E^{(1)}_{\parallel k} f^{(0)\prime}_\parallel = 0, \tag{395}$$

we make the change of variables

$$f^{(1)}_{\parallel k} = \frac{ik}{4\pi e}\left\{ \mathcal{G}\left[\xi_k\right] + \delta_{k k_0}\left(\xi_+ \eta_+ + \xi_- \eta_-\right) + \delta_{k -k_0}\left(\xi^*_- \eta_+ + \xi^*_+ \eta_-\right)\right\}, \tag{396}$$

where the discrete amplitudes, $\xi_\pm$ are evaluated at the (positive) wavenumber k_0. Since $\eta^*_\pm = \eta_\mp$, this representation of $f^{(1)}_{\parallel k}$ satisfies our "reality condition" $f^{(1)*}_{\parallel k} = f^{(1)}_{\parallel -k}$. The dielectric function is even in of k_0 thus there are also two roots of ϵ_L corresponding to $-k_0$ for a total of four discrete modes. However, the amplitudes corresponding to negative wavenumbers are determined by the complex conjugate of the positive wavenumber amplitudes:

$$\xi_\pm(k_0)^* = \xi_\mp(-k_0). \tag{397}$$

The inverse transformation is given by

$$\xi_k(u,t) = \frac{4\pi e}{ik}\widetilde{\mathcal{G}}\left[f^{(1)}_{\parallel k}(v,t)\right], \tag{398a}$$

$$\xi_+(t) = -\frac{4\pi e}{ik_0}\frac{\pi}{\epsilon_L{}'(u_0+i\gamma_0)}\overline{f^{(1)}_{\parallel k_0}}(u_0+i\gamma_0,t), \tag{398b}$$

$$\xi_-(t) = -\frac{4\pi e}{ik_0}\frac{\pi}{\epsilon_L{}'(u_0-i\gamma_0)}\overline{f^{(1)}_{\parallel k_0}}(u_0-i\gamma_0,t). \tag{398c}$$

We see that

$$\begin{aligned}
E^{(1)}_{\parallel k} &= \frac{4\pi e}{ik}\int dv_\parallel\, f^{(1)}_{\parallel k} \\
&= \beta^\infty\int du\,\xi_k + \left(\xi_+\delta_{k\,k_0} + \xi_-^*\delta_{k\,-k_0}\right)\int du\,\eta_+ + \left(\xi_+\delta_{k\,k_0} + \xi_-^*\delta_{k\,-k_0}\right)\int du\,\eta_- \\
&= \beta^\infty\left\{\int du\,\xi_k + \xi_+\delta_{k\,k_0} + \xi_-^*\delta_{k-k_0} + \xi_-\delta_{k\,k_0} + \xi_+^*\delta_{k-k_0}\right\},
\end{aligned} \tag{399}$$

where the last step follows from $\Xi(u_0 \pm i\gamma_0) = 0$ which implies

$$\int du\,\eta_\pm = -\overline{\alpha}(u_0 \pm i\gamma_0) = \beta^\infty. \tag{400}$$

Applying this transformation to the Vlasov equation with $\alpha = \epsilon_L^I$, $\beta = \epsilon_L^R$, and using

$$u\,\mathcal{G}[\xi_k] = \mathcal{G}[u\,\xi_k] - \frac{\alpha}{\pi}\int du\,\xi_k, \tag{401}$$

we get

$$0 = \mathcal{G}[\dot{\xi}_k + iku\,\xi_k] + \eta_+\Big\{\delta_{k\,k_0}\left[\dot{\xi}_+ + ik_0u\,\xi_+\right] + \delta_{k\,-k_0}\left[\dot{\xi}^*_- + ik_0u\,\xi^*_-\right]\Big\}$$
$$+ \eta_-\Big\{\delta_{k\,k_0}\left[\dot{\xi}_- + ik_0u\,\xi_-\right] + \delta_{k\,-k_0}\left[\dot{\xi}^*_+ - ik_0u\,\xi^*_+\right]\Big\}$$
$$+ \delta_{k\,k_0}\frac{ik_0}{\pi}\alpha\left(\xi_+ + \xi_-\right) + \delta_{k\,-k_0}\frac{ik_0}{\pi}\alpha\left(\xi^*_+ + \xi^*_-\right). \qquad (402)$$

Using the definition of $\eta_\pm$, this is equivalent to

$$0 = \mathcal{G}[\dot{\xi}_k + iku\,\xi_k]$$
$$+ \eta_+\Big\{\delta_{k\,k_0}\left[\dot{\xi}_+ + k_0(iu_0 - \gamma_0)\xi_+\right] + \delta_{k\,-k_0}\left[\dot{\xi}^*_- - k_0(iu_0 - \gamma_0)\xi^*_-\right]\Big\}$$
$$+ \eta_-\Big\{\delta_{k\,k_0}\left[\dot{\xi}_- + k_0(iu_0 + \gamma_0)\xi_-\right] + \delta_{k\,-k_0}\left[\dot{\xi}^*_+ - k_0(iu_0 + \gamma_0)\xi^*_+\right]\Big\} \qquad (403)$$

which implies

$$\dot{\xi}_k + iku\,\xi_k = 0\,, \qquad (404a)$$

$$\dot{\xi}_+ + ik_0(u_0 + i\,\gamma_0)\,\xi_+ = 0\,, \qquad (404b)$$

$$\dot{\xi}_+ + ik_0(u_0 - i\,\gamma_0)\,\xi_- = 0\,. \qquad (404c)$$

Thus

$$\xi_k(t,u) = \xi_k(u)e^{-ikut}, \qquad (405a)$$

$$\xi_+(t) = \xi_+(0)e^{k_0(\gamma_0 - iu_0)t}, \qquad (405b)$$

$$\xi_-(t) = \xi_-(0)e^{-k_0(\gamma_0 + iu_0)t}, \qquad (405c)$$

where $\xi_k(u)$, $\xi_+(0)$ and $\xi_-(0)$ are determined by the initial perturbation through

$$\xi_k(u) = \frac{4\pi e}{ik}\,\widetilde{\mathcal{G}}\left[f^{(1)}_{\parallel k}(t=0)\right], \tag{406a}$$

$$\xi_+(0) = -\frac{4\pi e}{ik_0}\,\frac{\pi}{\epsilon_L{}'(u_0+i\gamma_0)}\,\overline{f^{(1)}_{\parallel k_0}}(u_0+i\gamma_0, t=0), \tag{406b}$$

$$\xi_-(0) = -\frac{4\pi e}{ik_0}\,\frac{\pi}{\epsilon_L{}'(u_0-i\gamma_0)}\,\overline{f^{(1)}_{\parallel k_0}}(u_0-i\gamma_0, t=0). \tag{406c}$$

III. Canonizing the Hamiltonian and Bracket

Under the change of variables of the previous section, the Hamiltonian becomes

$$\begin{aligned} H^{(2)}_{\parallel} = \frac{V}{32}\sum_{k=-\infty}^{\infty}\Bigg\{ &\int du\,\frac{u}{\alpha}\Big(\mathcal{G}[\xi_k] + \delta_{k\,k_0}\left(\eta_+\xi_+ + \eta_-\xi_-\right) + \delta_{k\,-k_0}\left(\eta_+\xi_-^* + \eta_-\xi_+^*\right)\Big) \\ &\quad\times\Big(\mathcal{G}[\xi_{-k}] + \delta_{-k\,k_0}\left(\eta_+\xi_+ + \eta_-\xi_-\right) + \delta_{-k\,-k_0}\left(\eta_+\xi_-^* + \eta_-\xi_+^*\right)\Big) \\ &+\frac{1}{\pi}\Big(\int du\,\xi_k + \delta_{k\,k_0}\left(\xi_+ + \xi_-\right) + \delta_{k\,-k_0}\left(\xi_+^* + \xi_-^*\right)\Big) \\ &\quad\times\Big(\int du\,\xi_{-k} + \delta_{-k\,k_0}\left(\xi_+ + \xi_-\right) + \delta_{-k\,-k_0}\left(\xi_+^* + \xi_-^*\right)\Big)\Bigg\}. \end{aligned} \tag{407}$$

There are several integrals that must be evaluated to simplify this expression. First consider

$$\begin{aligned} \int du\,\frac{u}{\alpha}\,\eta_\pm\,\mathcal{G}[\phi] &= -\frac{1}{\pi}\int du\,\mathcal{G}[\phi] - \frac{u_0 \pm i\gamma_0}{\pi}\int du\,\frac{1}{u-u_0 \mp i\gamma_0}\,\mathcal{G}[\phi](u_0 \pm i\gamma_0) \\ &= -\frac{1}{\pi}\int du\,\phi - \frac{u_0 \pm i\gamma_0}{\pi}\int du\,\phi\,\mathcal{G}^\dagger\left[\frac{1}{u-u_0 \mp i\gamma_0}\right] \end{aligned} \tag{408}$$

but

$$\mathcal{G}^\dagger\left[\frac{1}{u-u_0\mp i\gamma_0}\right] = \frac{\beta}{u-u_0\mp i\gamma_0} - H\left[\frac{\alpha}{u-u_0\mp i\gamma_0}\right]$$
$$= \frac{\beta-\overline{\alpha}}{u-u_0\mp i\gamma_0} + \frac{1}{u-u_0\mp i\gamma_0}\frac{1}{\pi}\int du'\,\frac{\alpha}{u'-u_0\mp i\gamma_0}$$
$$= \frac{\beta-\overline{\alpha}+\overline{\alpha}(u_0\pm i\gamma_0)}{u-u_0\mp i\gamma_0}$$
$$= 0\,, \tag{409}$$

which follows from $\overline{\alpha}(u_0 \pm i\gamma_0) = -\beta^\infty$ and an obvious generalization of (490*b*). Hence

$$\int du\,\frac{u}{\alpha}\eta_\pm\,\mathcal{G}[\phi] = -\frac{1}{\pi}\int du\,\phi\,. \tag{410}$$

Now consider

$$\int du\,\frac{u}{\alpha}\eta_+\eta_- = -\frac{1}{\pi}\int du\,\eta_- + \frac{u_0+i\gamma_0}{\pi}\overline{\eta_-}(u_0+i\gamma_0)$$
$$= -\frac{1}{\pi}\,, \tag{411}$$

where the last step follows from (391). Lastly consider

$$\int du\,\frac{u}{\alpha}\eta_\pm^2 = -\frac{1}{\pi}\int du\,\eta_\pm - \frac{u_0\pm i\gamma_0}{\pi}\overline{\eta_\pm}(u_0\pm i\gamma_0)$$
$$= -\frac{1}{\pi} + \frac{1}{\pi}\frac{u_0+i\gamma_0}{\chi^{-1}\pm i\zeta^{-1}}\,, \tag{412}$$

where we have used (390). Using these results and (110) the expression for the energy simplifies to become:

$$H_\parallel^{(2)} = -\frac{V}{32}\sum_{k=-\infty}^{\infty}\int du\,\frac{u}{\zeta}\left|\xi_k\right|^2$$

$$-\frac{V}{16\pi}\left\{(\xi_+^* + \xi_-^*)\int du\,\xi_{k_0} + (\xi_+ + \xi_-)\int du\,\xi_{-k_0}\right\}$$

$$-\frac{V}{16\pi}\left\{\xi_+\xi_-^*\left[\frac{u_0 + i\gamma_0}{\chi^{-1} + i\zeta^{-1}} - 1\right]\right.$$

$$\left. + |\xi_+|^2 + |\xi_-|^2 + \xi_-\xi_+^*\left[\frac{u_0 - i\gamma_0}{\chi^{-1} - i\zeta^{-1}} - 1\right]\right\}$$

$$+\frac{V}{16\pi}\left\{(\xi_+ + \xi_-)\int du\,\xi_{-k_0} + (\xi_+^* + \xi_-^*)\int du\,\xi_{k_0} + |\xi_+ + \xi_-|^2\right\}$$

$$= -\frac{V}{32}\sum_{k=-\infty}^{\infty}\int du\,\frac{u}{\zeta}\,|\xi_k|^2$$

$$+\frac{V}{16\pi}\left\{\frac{u_0 + i\gamma_0}{\chi^{-1} + i\zeta^{-1}}\,\xi_+\xi_-^* + \frac{u_0 - i\gamma_0}{\chi^{-1} - i\zeta^{-1}}\,\xi_+^*\xi_-\right\}. \quad (413)$$

Writing this in terms of the dielectric function, we obtain

$$H_{\parallel}^{(2)} = \frac{V}{32}\sum_{k=-\infty}^{\infty}\int du\,u\,\frac{|\epsilon_L|^2}{\epsilon_L^I}\,|\xi_k|^2$$

$$+\frac{V}{16\pi}(u_0 + i\gamma_0)\epsilon_L'(u_0 + i\gamma_0)\,\xi_+\xi_-^*$$

$$+\frac{V}{16\pi}(u_0 - i\gamma_0)\epsilon_L'(u_0 - i\gamma_0)\,\xi_+^*\xi_-. \quad (414)$$

We begin the the task of transforming the bracket from the variables $f_{\parallel k}^{(1)}$ to the variables ξ_k, ξ_+ and ξ_- by using the chain rule to discover the behaviour of the functional derivatives under this change of variables. Now

$$\delta F = \sum_{k=-\infty}^{\infty}\int dv_{\parallel}\,\frac{\delta F}{\delta f_{\parallel k}^{(1)}}\,\delta f_{\parallel k}^{(1)}$$

$$= \sum_{k=-\infty}^{\infty}\int du\,\frac{\delta F}{\delta \xi_k}\,\delta\xi_k + \frac{\partial F}{\partial \xi_+}\,\delta\xi_+ + \frac{\partial F}{\partial \xi_-}\,\delta\xi_- + \frac{\partial F}{\partial \xi_+^*}\,\delta\xi_+^* + \frac{\partial F}{\partial \xi_-^*}\,\delta\xi_-^*$$

$$= \sum_{k=-\infty}^{\infty} \int du \left\{ \frac{\delta F}{\delta \xi_k} \frac{\delta \xi_k}{\delta f^{(1)}_{\|k}} + \frac{\delta F}{\delta \xi_+} \frac{\delta \xi_+}{\delta f^{(1)}_{\|k}} \right.$$

$$\left. + \frac{\delta F}{\delta \xi_-} \frac{\delta \xi_-}{\delta f^{(1)}_{\|k}} + \frac{\delta F}{\delta \xi_+^*} \frac{\delta \xi_+^*}{\delta f^{(1)}_{\|k}} + \frac{\delta F}{\delta \xi_-^*} \frac{\delta \xi_-^*}{\delta f^{(1)}_{\|k}} \right\} \delta f^{(1)}_{\|k}. \quad (415)$$

From the inverse transformation,

$$\xi_k = \frac{4\pi e}{ik} \tilde{\mathcal{G}} \left[f^{(1)}_{\|k} \right], \quad (416a)$$

$$\xi_\pm = -\frac{4\pi e}{ik_0} \pi (\chi^{-1} \pm i\zeta^{-1}) \overline{f^{(1)}_{\|k_0}}(u_0 \pm i\gamma_0), \quad (416b)$$

we see that

$$\frac{\delta \xi_k}{\delta f^{(1)}_{\|k}} = \frac{4\pi e}{ik} \tilde{\mathcal{G}}, \quad (416c)$$

$$\frac{\delta \xi_\pm}{\delta f^{(1)}_{\|k}} = -\delta_{k\,k_0} \frac{4\pi e}{ik} \frac{\chi^{-1} \pm i\zeta^{-1}}{u - u_0 \pm i\gamma_0}, \quad (416d)$$

$$\frac{\delta \xi_\pm^*}{\delta f^{(1)}_{\|k}} = -\delta_{k\,-k_0} \frac{4\pi e}{ik} \frac{\chi^{-1} \mp i\zeta^{-1}}{u - u_0 \mp i\gamma_0}. \quad (416e)$$

Thus we find

$$\frac{\delta F}{\delta f^{(1)}_{\|k}} = \frac{4\pi e}{ik} \left\{ \tilde{\mathcal{G}}^\dagger \left[\frac{\delta F}{\delta \xi_k} \right] - \frac{\chi^{-1} + i\zeta^{-1}}{u - u_0 - i\gamma_0} \left[\delta_{k\,k_0} \frac{\partial F}{\partial \xi_+} + \delta_{k\,-k_0} \frac{\partial F}{\partial \xi_-^*} \right] \right.$$

$$\left. - \frac{\chi^{-1} - i\zeta^{-1}}{u - u_0 + i\gamma_0} \left[\delta_{k\,k_0} \frac{\partial F}{\partial \xi_-} + \delta_{k\,-k_0} \frac{\partial F}{\partial \xi_+^*} \right] \right\}. \quad (417)$$

Using this expression for the functional derivative, the longitudinal part of the bracket becomes

$$\{F, G\}_\| = -\frac{16i}{V} \sum_{k=-\infty}^{\infty} k \int du\, \alpha \Bigg\{$$

$$\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta F}{\delta\xi_k}\right]-\frac{\chi^{-1}+i\zeta^{-1}}{u-u_0-i\gamma_0}\left[\delta_{k\,k_0}\frac{\partial F}{\partial\xi_+}+\delta_{k\,-k_0}\frac{\partial F}{\partial\xi_-^*}\right]$$
$$-\frac{\chi^{-1}-i\zeta^{-1}}{u-u_0+i\gamma_0}\left[\delta_{k\,k_0}\frac{\partial F}{\partial\xi_-}+\delta_{k\,-k_0}\frac{\partial F}{\partial\xi_+^*}\right]\Bigg\}$$
$$\times\Bigg\{\widetilde{\mathcal{G}}^{\dagger}\left[\frac{\delta G}{\delta\xi_{-k}}\right]-\frac{\chi^{-1}+i\zeta^{-1}}{u-u_0-i\gamma_0}\left[\delta_{-k\,k_0}\frac{\partial G}{\partial\xi_+}+\delta_{-k\,-k_0}\frac{\partial G}{\partial\xi_-^*}\right]$$
$$-\frac{\chi^{-1}-i\zeta^{-1}}{u-u_0+i\gamma_0}\left[\delta_{-k\,k_0}\frac{\partial G}{\partial\xi_-}+\delta_{-k\,-k_0}\frac{\partial G}{\partial\xi_+^*}\right]\Bigg\}. \tag{418}$$

Now

$$\int du\,\frac{\alpha}{u-u_0\mp i\gamma_0}\,\widetilde{\mathcal{G}}^{\dagger}[\phi]=-\pi\int du\,\eta_{\pm}\,\widetilde{\mathcal{G}}^{\dagger}[\phi]$$
$$=-\pi\int du\,\phi\,\widetilde{\mathcal{G}}[\eta_{\pm}]=0\,, \tag{419}$$

since $\eta_{\pm}$ are the null vectors of $\widetilde{\mathcal{G}}$. Using this, (109), (390) and (391), the expression for the longitudinal bracket can be simplified considerably to become

$$\{F,G\}_{\parallel}=\frac{16i}{V}\sum_{k=-\infty}^{\infty}k\int du\,\zeta\,\frac{\delta F}{\delta\xi_k}\,\frac{\delta G}{\delta\xi_{-k}}$$
$$-\frac{16i}{V}\pi k_0\,(\chi^{-1}+i\zeta^{-1})\left[\frac{\partial F}{\partial\xi_+}\frac{\partial G}{\partial\xi_-^*}-\frac{\partial F}{\partial\xi_+^*}\frac{\partial G}{\partial\xi_+}\right]$$
$$-\frac{16i}{V}\pi k_0\,(\chi^{-1}-i\zeta^{-1})\left[\frac{\partial F}{\partial\xi_-}\frac{\partial G}{\partial\xi_+^*}-\frac{\partial F}{\partial\xi_+^*}\frac{\partial G}{\partial\xi_-}\right]. \tag{420}$$

Writing this in terms of the dielectric function gives

$$\{F,G\}_{\parallel}=-\frac{16i}{V}\sum_{k=-\infty}^{\infty}k\int du\,\frac{\epsilon_L^I}{|\epsilon_L|^2}\,\frac{\delta F}{\delta\xi_k}\,\frac{\delta G}{\delta\xi_{-k}}$$

$$- \frac{16i}{V} \frac{\pi k_0}{\epsilon_L'(u_0 + i\gamma_0)} \left[\frac{\partial F}{\partial \xi_+} \frac{\partial G}{\partial \xi_-^*} - \frac{\partial F}{\partial \xi_+^*} \frac{\partial G}{\partial \xi_+} \right]$$

$$- \frac{16i}{V} \frac{\pi k_0}{\epsilon_L'(u_0 - i\gamma_0)} \left[\frac{\partial F}{\partial \xi_-} \frac{\partial G}{\partial \xi_+^*} - \frac{\partial F}{\partial \xi_+^*} \frac{\partial G}{\partial \xi_-} \right]. \tag{421}$$

Again we note that the neutral mode result *cannot* be obtained from the above by considering the coalescence of the stable and unstable roots, *i.e.* by taking the limit $\gamma_0 \longrightarrow 0^+$. The inflection point neutral mode studied in the previous chapter required root of ϵ_L^I have multiplicity two and the root of ϵ_L^R to have multiplicity one. The coalescence of a pair of complex roots will result in the same multiplicities for (real) roots of ϵ_L^I and ϵ_L^R and thus such a limit cannot yield an inflection point neutral mode of this kind. The question of what occurs in the limit $\gamma_0 \longrightarrow 0^+$ is quite interesting and deserves further investigation.

Computing Hamilton's equations yields

$$\dot{\xi}_k = \left\{\xi_k \, , H_\parallel^{(2)}\right\}_\parallel = -iku\,\xi_k, \tag{422a}$$

$$\dot{\xi}_+ = \left\{\xi_+ \, , H_\parallel^{(2)}\right\}_\parallel = -ik_0 \left(u_0 + i\gamma_0\right) \xi_+, \tag{422b}$$

$$\dot{\xi}_- = \left\{\xi_- \, , H_\parallel^{(2)}\right\}_\parallel = -ik_0 \left(u_0 - i\gamma_0\right) \xi_-, \tag{422c}$$

which are just the equations of motion we obtained in the previous section.

8

Computations Using Singular Eigenfunctions

In this chapter we discuss the numerical evaluation of the integral transform solution of the Vlasov equation (212). Although we only consider the longitudinal motion, these techniques are equally well suited to the transverse case. To use the transform solution, we need a numerical implementation of $\widetilde{\mathcal{G}}$ and $\mathcal{G}$. It may occur that there is a closed form expression for the transform of the initial condition, in which case it is only necessary to evaluate $\mathcal{G}$ numerically. In general this will not be the case and we will have to compute $\widetilde{\mathcal{G}}$ in addition to $\mathcal{G}$. As we shall see this will not be a limitation. Central to computing $\mathcal{G}$ is the evaluation of the Hilbert transform, *i.e.* numerically computing a Cauchy integral.

A Cauchy integral can be viewed as the sum of two improper integrals (one on each side of the singularity). Neither integral is guaranteed to exist alone; the singularities cancel as the limit is taken. It is this behaviour that makes numerical evaluation somewhat delicate. The usual advice for computing an improper integral is to make use of an "open" quadrature rule, in effect taking the limit numerically. Clearly we will need something more sophisticated. While the singularity in a Cauchy integral is relatively mild, the behaviour of the integrand, in particular its residue, near the singularity is very important. That said, it is evident that we must use something resembling an open method as we surely cannot evaluate the integrand at the singular point.

On top of these difficulties we have the additional requirement of high accuracy in the evaluation of the Hilbert transform otherwise we run the risk that numerically $\mathcal{G}[\widetilde{\mathcal{G}}]$ will differ significantly from the identity.[31] There are two paths to high accuracy: low order with small step size or high order allowing larger steps. This naturally brings us to the question of Gauss-type rules *versus* those based on the Euler-Maclaurin summation formula (*e.g.* Simpson's rule and the like). Some comment on the choice of how to proceed is called for. One often hears (or reads) the warning "High accuracy does not necessarily mean high order." There is, of course, some truth to this statement; functions whose high order derivatives take on large values within the region of integration may well defeat a high order method. This is, however, the *exception*. (See Oliver[32] for an enlightening discussion of this issue.) In our case, we know that the functions under consideration are well behaved, in practice they will very likely be C^∞. Given our high accuracy requirements, it seems reasonable to explore high order Gauss-type rules.

I. Quadrature Formulæ for Principal Value Integrals

We take as our prototypical Cauchy integral

$$I = \mathrm{P}\!\int_{-1}^{1} dx\, \frac{\phi(x)}{x} \equiv \lim_{\epsilon \to 0} \left(\int_{\epsilon}^{1} + \int_{-1}^{-\epsilon} \right) dx\, \frac{\phi(x)}{x}, \tag{423}$$

where ϕ is assumed to belong to $\mathcal{H}^{\mu}$. The idea is to develop a quadrature rule so that we can write

$$I = \sum_{i=0}^{N} w_i\, \phi(x_i) + R_N, \tag{424}$$

where R_N is the (hopefully small) remainder. Thus we expect to approximate I as

$$I \approx \sum_{i=0}^{N} w_i\, \phi(x_i). \tag{425}$$

Note that this approximation is quite general — by simply scaling and translating the variable of integration we find

$$\mathrm{P}\!\int_{a-\xi}^{a+\xi} dx\, \frac{\phi(x)}{x-\xi} = \mathrm{P}\!\int_{-1}^{1} dx\, \frac{\varphi(x)}{x} \approx \sum_{i=0}^{N} w_i\, \varphi(x_i)\,, \tag{426}$$

where

$$\varphi(x) = \phi\left(\frac{x-\xi}{a}\right). \tag{427}$$

The requirement that the singularity be in the center of the region of integration is both essential and not particularly restricting; it is always possible to divide an integral over an arbitrary interval into sum of a principal value integral over (sub) interval centered on the singularity and a non-singular over the remainder of the original interval.

The basic idea is that if we can find a good polynomial approximation for ϕ then (423) reduces to a sum of simple integrals that can be explicitly evaluated. It is well known that interpolation based on Chebyshev polynomials is very nearly equivalent to using the "minimax" polynomial. This makes Chebyshev polynomials a logical choice. Let $T_n(x)$ be the n-th Chebyshev polynomial with roots x_i. Using Lagrange's interpolation formula[33] we find

$$\phi(x) = \sum_{i=0}^{n} \frac{T_n(x)}{(x-x_i)T_n'(x_i)}\, \phi(x_i) + r_n(x)\,, \tag{428}$$

where the remainder r_n is given by

$$r_n(x) = T_n(x)\, \frac{\phi^{(n+1)}(\zeta)}{(n+1)!}\,, \quad -1 \le \zeta \le 1\,. \tag{429}$$

Since $|T_n(x)|$ is bounded by 1 for $x \in [-1\,,1]$,

$$\left|r_n(x)\right| \le \frac{1}{(n+1)!} \max_{x\in[-1,1]} \left|\phi^{(n+1)}(\xi)\right| \tag{430}$$

and thus the quality of our approximation depends on how rapidly ϕ varies and on the degree of the polynomial used. For the moment, we will ignore the remainder term returning to it later.

Using this approximation for ϕ in (423), we find

$$I \approx \sum_{i=0}^{n} \frac{\phi(x_i)}{T_n'(x_i)} \mathrm{P} \int_{-1}^{1} dx \, \frac{T_n(x)}{x(x - x_i)}. \tag{431}$$

Notice that only the even part of the integrand (which is the odd part of ϕ) will contribute to I. We can take advantage of this and write

$$I \approx \sum_{i=0}^{n} \frac{\phi(x_i)}{T_n'(x_i)} \int_{0}^{1} dx \, \frac{1}{x} \left\{ \frac{T_n(x)}{x - x_i} + (-1)^n \frac{T_n(x)}{x + x_i} \right\}, \tag{432}$$

where we have made use of the parity of the Chebyshev polynomials. The integral in the above now has a removable singularity at $x = 0$ and thus exists in the usual sense. Furthermore, this is exactly the form we were seeking in (425). Let

$$w_i \equiv \frac{1}{T_n'(x_i)} \int_{0}^{1} dx \, \frac{1}{x} \left\{ \frac{T_n(x)}{x - x_i} + (-1)^n \frac{T_n(x)}{x + x_i} \right\}. \tag{433}$$

The evaluation of these integrals is not as bad as it first appears. The key to this is that we have a closed form expression for the roots, x_i:

$$x_i = \cos\left[\frac{2i+1}{2n-1}\pi\right], \quad i = 0, 1 \ldots n. \tag{434}$$

While one would prefer an algebraic expression for w_i, the number of terms involved, even for moderately large n, makes this impractical. We have computed weights and coordinates for orders up to 29 to 30 places in Maple directly from (433) using a combination of symbolic manipulation and arbitrary precision arithmetic.

In principle, we can determine the quadrature remainder R_n by integration of the remainder term of the approximation, r_n. This is not entirely straightforward, but in the case where n is odd it is possible to obtain a generous upper bound:[34]

$$|R_n| \leq \max_{x\in[-1,1]} \left|\phi^{(n+1)}(\xi)\right| \frac{n}{(n+1)!}. \tag{435}$$

In practice, it turns out that the even order rules give consistently better performance than those of odd order. This is in part due to the odd order polynomials having at root at $x = 0$ for which the weight is zero. Hence for a rule of order $2n+1$ only $2n$ points contribute.

As an illustration of the performance of this quadrature method, consider the Sine-Integral function

$$\mathrm{Si}(x) = \int_0^x dx' \, \frac{\sin(x')}{x'} = \frac{1}{2} \mathrm{P}\!\int_{-x}^{x} dx' \, \frac{\sin(x')}{x'} \tag{436}$$

which has the power series

$$\mathrm{Si}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}. \tag{437}$$

Using the quadrature described above, we have evaluated $\mathrm{Si}(x)$ with several different orders and compared with the exact value.[35] Define ε_n as

$$\varepsilon_n = \left|\mathrm{Si}_{\mathrm{exact}}(x) - \mathrm{Si}_{\mathrm{quad}}(x)\right|, \tag{438}$$

where the n-point quadrature rule was used to compute $\mathrm{Si}_{\mathrm{quad}}$. Based on the remainder expressions we expect $\varepsilon \sim x^{n+1}$. In Figure 3, $\log(\varepsilon_n)$ is plotted *versus* x for several different values of n.

The error curves exhibit two distinct behaviours. For small x, ε_n is essentially oscillatory, while for larger x is grows as a power (linearly on a logarithmic scale).

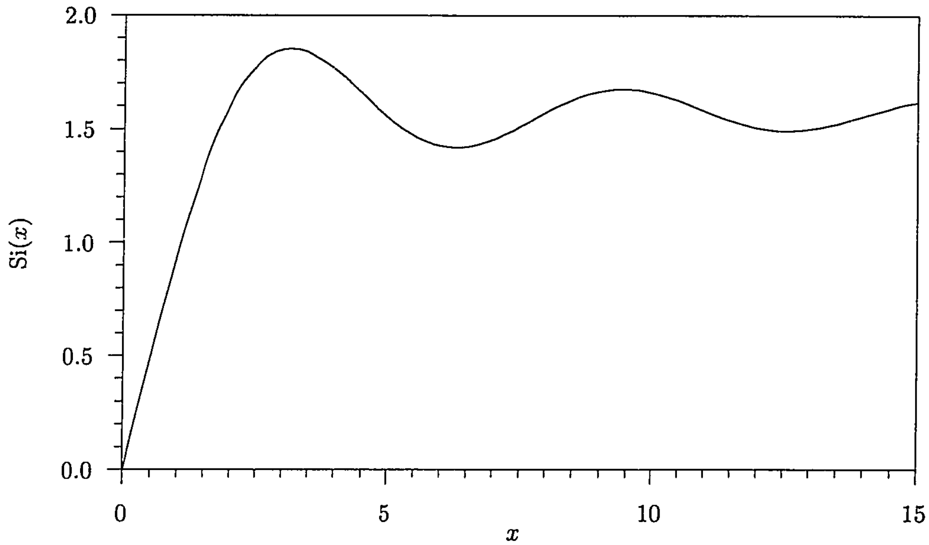

Figure 2: A plot of Si(x). This function is used as a test of the various principal value quadrature methods.

The oscillatory behaviour is a result of the finite word size of the computer (in this case approximately 20 decimal digits) — in this region the exact result and the quadrature result are virtually indistinguishable. As x becomes larger, the error terms grows, finally becoming significant. The power law scaling is exactly what we would expect based on our expression for the remainder. This interpretation is borne out by fitting

$$\varepsilon_n = A_n (2x)^{B_n}. \tag{439}$$

The results of this fit are tabulated below (See Figure 4):

Order	A_n	B_n
8	1.0×10^{-12}	8.4
10	4.6×10^{-16}	10.4
16	3.8×10^{-25}	14.7
20	2.5×10^{-33}	19.2
mid-point	1.4×10^{-6}	2.8

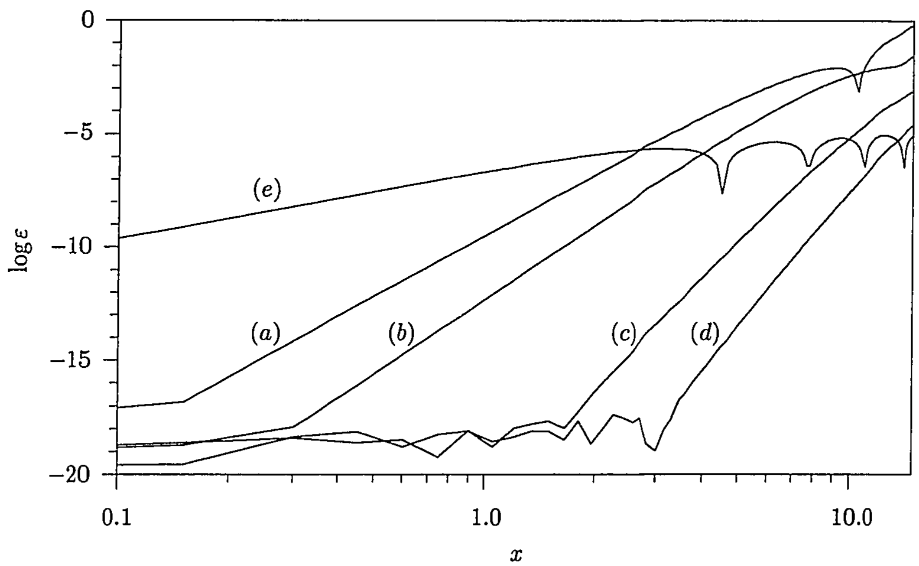

Figure 3: Absolute error (base ten logarithm) in calculating Si(x) using Chebyshev based quadrature: (a) 8-point rule; (b) 10-point rule; (c) 16-point rule; (d) 20-point rule and (e) open mid-point rule using 486 evenly spaced points.

The dependence of ε_n on the interval size matches very nicely with the remainder expression, indicating that this quadrature formula is performing as expected. For these calculations, $\phi = \sin(x)$ and so all of the derivatives are bounded by unity. There seems to be no simple dependence of A_n on n — this is of no real importance — it is sufficient to see that A_n decreases rapidly with increasing n. In practice, we will not use the remainder formula as it will turn out that we have a more reliable error estimate available. Also shown in Figure 3 is a calculation using an adaptive mid-point rule whose error is seen to be somewhat better than quadratic. Note that this method requires significantly more than an order of magnitude more integrand evaluations while yielding an accuracy several orders of magnitude worse than the other methods. When the interval becomes large enough, the mid-point rule is superior, but by this point

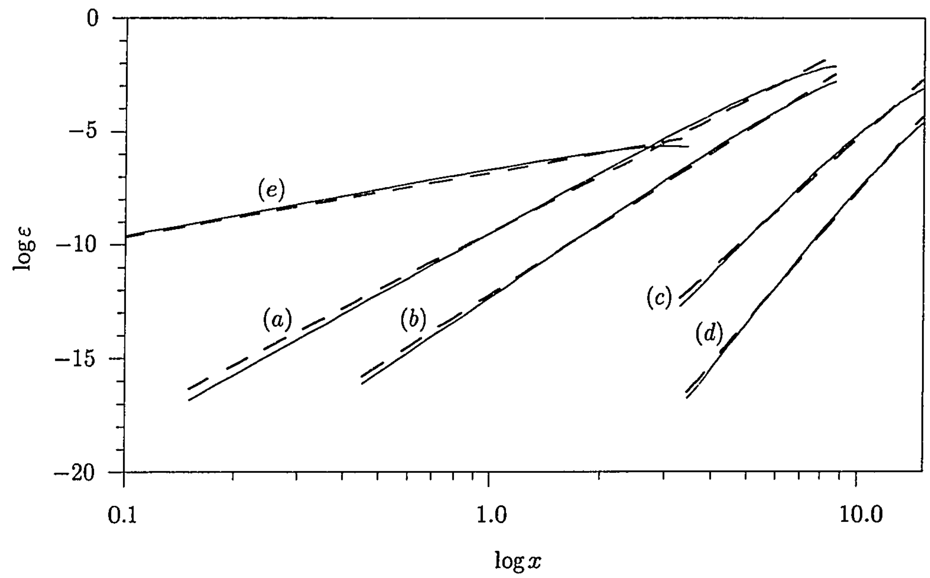

Figure 4: Absolute error (base ten logarithm) in calculating Si(x) using Chebyshev based quadrature: (a) 8-point rule; (b) 10-point rule; (c) 16-point rule; (d) 20-point rule and (e) open mid-point rule using 486 evenly spaced points. The linear region of Figure 3 has been fit to a straight line (dashed line). See text for details.

the accuracy of all methods is so low as to be useless. The point of including this calculation is to demonstrate that, in this application, series based rules are simply not competitive with the high order rules we have been considering.

We have been using the term "order" rather loosely. By a quadrature rule of order n one means a rule that will exactly integrate a polynomial of degree n. Strictly speaking, although these rules make use of non-uniformly spaced abscissa, they are *not* of the Gauss type[36] but rather more closely related to Newton-Cotes formulæ. Similar rules, also based on polynomial interpolation, have been developed by Price.[37] There is also a connection to the Clenshaw-Curtis method which also utilizes Chebyshev based approximation[38–41] but requires the computation of a cosine transform. The method of Clenshaw and Curtis is also sometimes used to compute singular integrals but in general the quadrature schemes that we have discussed here appear to be superior.

Having made the observation that only the odd part of ϕ contributes to the integral in (423), it is tempting to define

$$x\, g(x) = \phi(x) - \phi(-x)\,, \tag{440}$$

enabling us to write

$$I = \int_{-1}^{1} dx\, g(x)\,. \tag{441}$$

The difficulty here is that we cannot guarantee that $g(x)$ is regular as $x \longrightarrow 0$, only that $x\, g(x) \longrightarrow 0$ as $x \longrightarrow 0$. However, as long as we do not evaluate $g(x)$ at a point too close to $x = 0$ this is not likely to be a problem.[42] In Gauss-Legendre quadrature, the points are symmetric about 0. Thus for rules with an even number of points, $x = 0$ is *not* one of the abscissa values. In effect, these Gauss-Legendre rules are "open" with respect to the origin. Even for the 40 point rule, the closest coordinate to the origin is approximately 0.0045 and so the danger of loss of significance, even if $g(x)$ is singular at $x = 0$, is not great. In practice, the functions that we will be interested in are in $\mathcal{H}^1$, in which case we know that $g(x)$ will be well behaved at $x = 0$.

We now use a standard Gauss-Legendre formula to compute (441), keeping in mind there are some functions for which this will not be successful. The advantage of this over the quadrature rule developed above is that Gauss rules using n points are of order $2n$. Using the integral representation of $\mathrm{Si}(x)$ we have made a comparison of the error associated with different order Gauss-Legendre quadrature rules. In Figure 5, $\log \varepsilon_n$ is plotted *versus* x for the same values of n as were used in Figure 3. The overall behaviour of the error term is qualitatively very

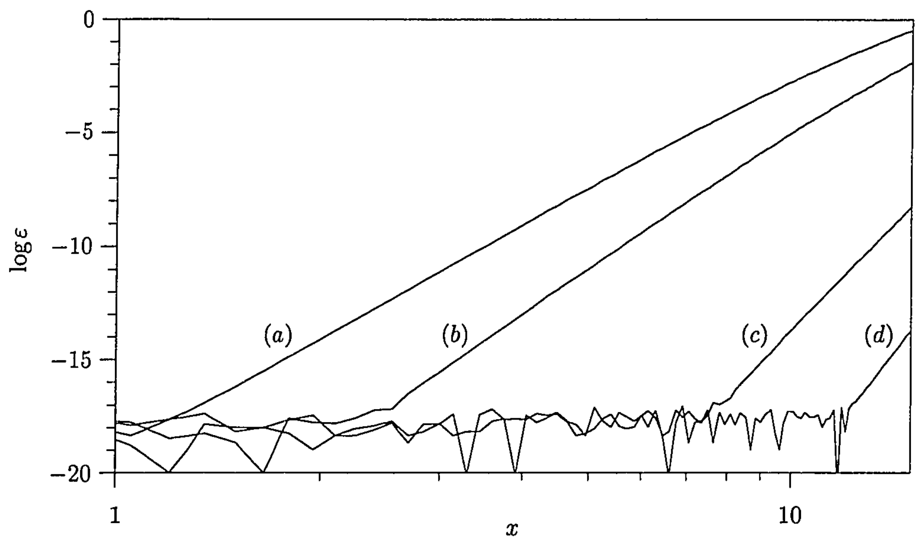

Figure 5: Absolute error (base ten logarithm) in calculating Si(x) using Gauss-Legendre quadrature: (a) 8-point rule; (b) 10-point rule; (c) 16-point rule and (d) 20-point rule.

similar to the Chebyshev methods. The significant difference is in dependence of the error on the interval size; the exponent is approximately twice as large. Fitting the error to a power law as before, we find:

Points	A_n	B_n
8	4.1×10^{-24}	15.8
10	2.4×10^{-31}	19.6
16	5.1×10^{-55}	31.2
20	5.3×10^{-71}	38.3

As we expect with Gaussian quadratures, the n-point rule is of order $2n$. Furthermore, A_n decreases even more rapidly with n than in the Chebyshev case.

In Gaussian quadratures, the abscissa are chosen to ensure that the resulting rule will be of order $2n$. In developing our quadrature rule, we choose the abscissa

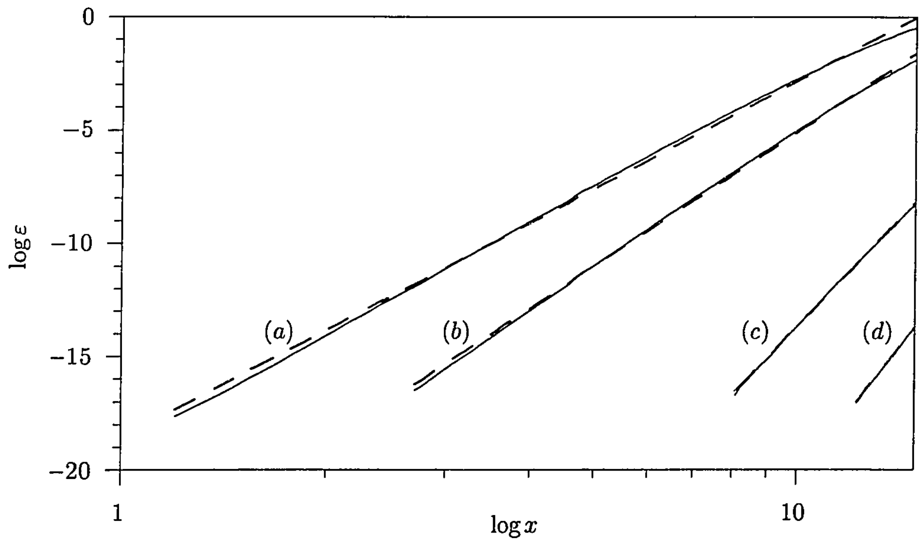

Figure 6: Absolute error (base ten logarithm) in calculating Si(x) using Gauss-Legendre quadrature: (a) 8-point rule; (b) 10-point rule; (c) 16-point rule and (d) 20-point rule. The linear region of Figure 5 has been fit to a straight line (dashed line).

to minimize the interpolation error. Although this yielded a lower order rule (for a given number of integrand evaluations), the advantage is that the Chebyshev based rule will be able to compute some integrals that will defeat the Gauss-Legendre method. The use of Gauss-Legendre quadrature for computing Cauchy integrals has also been suggested by Price.[37] The algorithm for computing the Hilbert transform, that we will describe below, will allow for the use of either method as appropriate.

II. Numerical Evaluation of Hilbert Transforms

Having found a satisfactory method for computing a principal value integral, we are now in a position to construct an algorithm to compute the Hilbert transform.

Recall the definition of the Hilbert transform of ϕ:

$$\begin{aligned}\overline{\phi}(x) &= \frac{1}{\pi}\mathrm{P}\!\int dx'\,\frac{\phi}{x'-x}\\ &= \frac{1}{\pi}\lim_{\substack{\epsilon\longrightarrow 0\\ X\longrightarrow\infty}}\left\{\int_{-X}^{x-\epsilon}dx'\,\frac{\phi(x')}{x'-x}+\int_{x+\epsilon}^{X}dx'\,\frac{\phi(x')}{x'-x}\right\}.\end{aligned} \tag{442}$$

Notice that we treat the integration about infinity as a Cauchy principal value. We introduce points $A \gg B > 0$ and partition the real axis into six intervals, giving

$$\begin{aligned}\overline{\phi}(x) = \frac{1}{\pi}\lim_{\epsilon\rightarrow 0}&\left\{\int_{x-B}^{x-\epsilon}dx'\,\frac{\phi(x')}{x'-x}+\int_{x+\epsilon}^{x+B}dx'\,\frac{\phi(x')}{x'-x}\right\}\\ &+\frac{1}{\pi}\lim_{X\rightarrow\infty}\left\{\int_{A}^{X}dx'\,\frac{\phi(x')}{x'-x}+\int_{-X}^{-A}dx'\,\frac{\phi(x')}{x'-x}\right\}\\ &+\frac{1}{\pi}\int_{-A}^{x-B}dx'\,\frac{\phi(x')}{x'-x}+\frac{1}{\pi}\int_{x+B}^{A}dx'\,\frac{\phi(x')}{x'-x}.\end{aligned} \tag{443}$$

By a simple change of variables, $x' \longrightarrow 1/u$, we can map the integral about infinity to a principal value integral about the origin. Doing so gives

$$\begin{aligned}\overline{\phi}(x) &= \frac{1}{\pi}\mathrm{P}\!\int_{x-B}^{x+B}dx'\,\frac{\phi(x')}{x'-x}\\ &\quad+\frac{1}{\pi}\lim_{u\rightarrow 0}\left\{\int_{-1/A}^{-u}du\,\frac{\phi(1/u)}{u(1-ux)}+\int_{u}^{1/A}du\,\frac{\phi(1/u)}{u(1-ux)}\right\}\\ &\quad+\frac{1}{\pi}\int_{-A}^{x-B}dx'\,\frac{\phi(x')}{x'-x}+\frac{1}{\pi}\int_{x+B}^{A}dx'\,\frac{\phi(x')}{x'-x}\\ &= \frac{1}{\pi}\mathrm{P}\!\int_{x-B}^{x+B}dx'\,\frac{\phi(x')}{x'-x}+\frac{1}{\pi}\mathrm{P}\!\int_{-1/A}^{1/A}du\,\frac{\phi(1/u)}{u(1-ux)}\\ &\quad+\frac{1}{\pi}\int_{-A}^{x-B}dx'\,\frac{\phi(x')}{x'-x}+\frac{1}{\pi}\int_{x+B}^{A}dx'\,\frac{\phi(x')}{x'-x}.\end{aligned} \tag{444}$$

The two principal value integrals are in the form of (426) and can be directly evaluated with either of the quadrature rules described above. Provided that B is not too small *i.e.* larger than say 1/2, the integrands in the two remaining integrals are going to be well behaved and can be evaluated using standard (Gaussian) quadrature techniques. The choice of A is governed by ϕ; it should be chosen such that ϕ has no significant features for $x > A$. Ideally, we would like to choose A such that $\phi \approx 0$ for $x > A$. We have implemented an algorithm[34], essentially as described here and found extremely good performance, yielding accuracies of approximately 1 part in 10^{18}, for functions that approach 0 rapidly for large x. We will return to the issue of the nature of ϕ below.

As outlined above, accuracy is of primary importance to us, however, efficiency is also important since we have a great number of calculation to perform. From a standpoint of efficiency, one would like to find some way to "re-use" integrand evaluations when computing successive transforms. The basic idea is to recognize that the integral we wish to evaluate is of the form

$$\overline{\phi}(x) = \int dx'\, f(x')\, g(x', x)\,, \tag{445}$$

where $f = \phi$ which is, in general, computationally expensive, while $g = 1/(x' - x)$ can be computed quickly. The point is that the expensive part, f, *does not* depend on x and thus does not need to be re-computed when we evaluate the transform at different points.

Adopting this notion means that we have to be more systematic in picking A and B. The procedure is as follows: We divide the real axis into N panels such that the n-th panel, denoted by P_n, covers the interval $[\mu_{n-1}, \mu_n]$ and $\mu_0 = -\mu_N$. It is convenient to pick μ_N such that one is only interested in evaluating $\overline{\phi}$

for $|x| < \mu_N$. This restriction is not essential but greatly simplifies the code and is reasonable in the case of computing $\mathcal{G}$ and $\widetilde{\mathcal{G}}$.

Assuming this restriction, we can write

$$\overline{\phi}(x) = \frac{1}{\pi}\,\mathrm{P}\!\int_{P_{n_s}} dx'\,\frac{\phi(x')}{x'-x} + \frac{1}{\pi}\,\mathrm{P}\!\int_{P_0} du\,\frac{\varphi(u)}{1-ux} + \frac{1}{\pi}\sum_{\substack{n=1\\ n\neq n_s}}^{N}\int_{P_n} dx'\,\frac{\phi(x')}{x'-x}, \quad (446)$$

where n_s is the label of the panel containing x, P_0 denotes the interval $[-\mu_0^{-1}, \mu_0^{-1}]$ and

$$\varphi(x) = \frac{1}{x}\,\phi\left(\frac{1}{x}\right). \quad (447)$$

In each panel, we introduce a local coordinate, ξ, that ranges from -1 to 1. Explicitly, for $x \in P_n$

$$x = x_n \equiv d_n\xi + s_n, \quad (448)$$

where

$$d_n = \frac{1}{2}\left(\mu_n - \mu_{n-1}\right), \quad n \neq 0, \quad (449a)$$

$$s_n = \frac{1}{2}\left(\mu_n + \mu_{n-1}\right), \quad n \neq 0, \quad (449b)$$

$$d_0 = \frac{1}{\mu_0}, \quad (449c)$$

$$s_0 = 0. \quad (449d)$$

We can now write

$$\overline{\phi}(x) = \frac{1}{\pi}\,\mathrm{P}\!\int_{P_{n_s}} dx'\,\frac{\phi(x')}{x'-x} + \frac{d_0}{\pi}\,\mathrm{P}\!\int_{-1}^{1} d\xi\,\frac{\varphi(\xi)}{1-\xi x} + \frac{d_n}{\pi}\sum_{n\neq n_s}\int_{-1}^{1} d\xi\,\frac{\phi(x_n)}{x_n-x}. \quad (450)$$

For each panel, we evaluate the integrals using a quadrature rule with M_n abscissa ξ_m^n and corresponding weights w_m^n. The expression for $\overline{\phi}$ now reads

$$\overline{\phi}(x) = \frac{1}{\pi}\,\mathrm{P}\!\int_{P_{n_s}} dx'\,\frac{\phi(x')}{x'-x} + \frac{1}{\pi}\sum_{m=1}^{M_0} d_0\,w_m^0\,\frac{\varphi(\xi_m^0)}{1-\xi_m^0\,x} + \frac{1}{\pi}\sum_{n\neq n_s}\sum_{m=1}^{M_n} d_n\,w_m^n\,\frac{\phi(x_n)}{x_n - x}, \tag{451}$$

where x_n is understood to be shorthand for $d_n\,\xi_m^n + s_n$. Defining

$$W_m^0 = d_0\,w_m^0\,\varphi(\xi_m^0), \tag{452a}$$

$$W_m^n = d_n\,w_m^n\,\phi(x_n), \quad n \neq 0, \tag{452b}$$

$$\Xi_m^n = d_n\,\xi_m^n + s_n, \tag{452c}$$

we can write

$$\overline{\phi}(x) = \frac{1}{\pi}\,\mathrm{P}\!\int_{P_{n_s}} dx'\,\frac{\phi(x')}{x'-x} + \frac{1}{\pi}\sum_{m=1}^{M_0} W_0^n\,\frac{1}{1-\Xi_m^0\,x} + \frac{1}{\pi}\sum_{n\neq n_s}\sum_{m=1}^{M_n} W_m^n\,\frac{1}{\Xi_m^n - x}. \tag{453}$$

Viewing the x dependence in the right-hand side of the above as parametric, we see that these sums have the form of a standard quadrature rule for the functions

$$g(\Xi) = \frac{1}{\Xi - x}, \tag{454a}$$

$$h(\Xi) = \frac{1}{1-\Xi\,\xi}, \tag{454b}$$

with points Ξ_m^n and weights W_m^n. That is, we can write

$$\overline{\phi}(x) = \frac{1}{\pi}\,\mathrm{P}\!\int_{P_{n_s}} dx'\,\frac{\phi(x')}{x'-x} + \frac{1}{\pi}\sum_{m=1}^{M_0} W_0^n\,h(\Xi_m^0) + \frac{1}{\pi}\sum_{n\neq n_s}\sum_{m=1}^{M_n} W_m^n\,g(\Xi_m^n). \tag{455}$$

The key point here is that W_m^n depends on ϕ but *not* on x, thus they need only be calculated once, no matter how many times $\overline{\phi}$ is to be evaluated. There still remains the matter of the principal value integral. As we saw above, we require the singular point to be in the center of interval of integration thus we must divide P_{n_s} into at least two regions. Since this division will depend on x, integrations over P_{n_s} for different values of x will have no evaluations of f in common (even for those values of x that have the same value of n_s).

In practise, it is convenient (primarily for testing purposes) to use a fixed-size interval for the principal value integration and so we need to divide P_{n_s} into three pieces. Furthermore, it is not desirable to integrate over too small an interval. Therefore, whenever x is too close to one of the end points of P_{n_s}, it is necessary to merge P_{n_s} with the adjacent panel before dividing into three sub-intervals. These additional considerations merely add to the "bookkeeping" requirement of the implementation and do not alter the basic algorithm.

The motivation for this more complex algorithm was efficiency. The price for this is complexity and storage. Storage is simply not an issue — typically the storage requirement are on the order of several thousand floating-point numbers. The bookkeeping requirements are likewise reasonably modest. The overall performance increase depends on many factors but a crude estimate (ignoring the time needed for bookkeeping operations etc.) is the ratio the computational times for g and f. For computing the transform of a simple function this turns out to be a serious over estimate of the improvement since the "extra" computations associated with the integration over P_{n_s} are significant and the overall computation takes about 70% of the time of the first algorithm. Below, we will see that this idea can be used even more effectively in calculating $\mathcal{G}$, where the savings are more than an order of magnitude.

We have implemented an algorithm based on the method described above performed extensive testing. Shown in Figure 7 is the result of using this algorithm to compute the Hilbert transform of

$$\phi(x) = \sqrt{\pi}\, e^{-x^2} \tag{456}$$

which is given exactly by

$$\overline{\phi}(x) = -2\,\mathrm{daw}\left(\frac{x}{\sqrt{2}}\right), \tag{457}$$

where $\mathrm{daw}(x)$ is Dawson's integral.[43] This computation was done with 10 uniformly sized panels covering $[-7.5, 7.5]$. The singular integrations we done over an interval of size 0.6. All integral we evaluated with 28-th order Gauss-Legendre quadrature. A comparison with the exact result (see Figure 8) shows the high accuracy possible with this method.

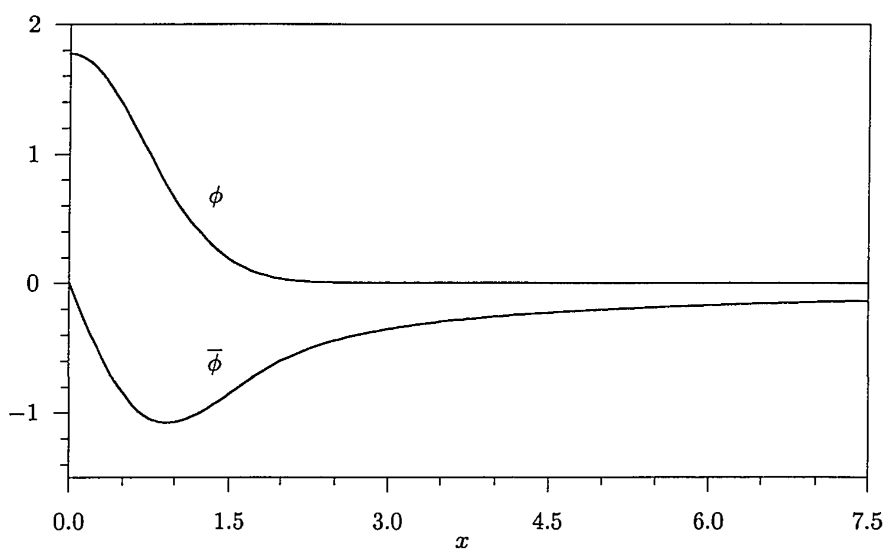

Figure 7: Plot of $\overline{\phi}(x)$ for $\phi = \sqrt{\pi}e^{-x^2}$, computed using the algorithm described in the text. The exact result is $\overline{\phi} = -2\,\mathrm{daw}(x/\sqrt{2})$.

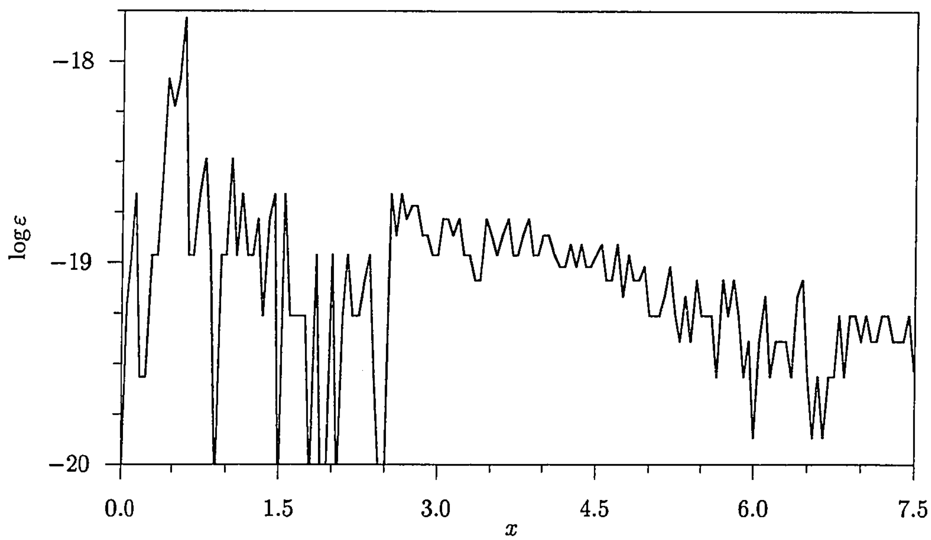

Figure 8: Absolute error (base ten logarithm) in the numerical evaluation $\overline{\phi}(x)$ for $\phi = e^{-x^2}$, computed using the algorithm described in the text.

In part the high accuracy seen in Figure 8 is due to ϕ vanishing rapidly as x increases. Thus one finds that using this algorithm to compute transforms of functions that vanish slowly for large x to be considerably less satisfactory. This is largely caused by the loss of significance in subtracting nearly equal quantities. This will be of little concern to us since we will only be computing transforms of functions that contain Gaussian factors.

A further (and more interesting) test can be found in connection with the parameter functions for the longitudinal transforms $\mathcal{G}$ and $\widetilde{\mathcal{G}}$. Recall that if $\beta + i\alpha \neq 0$ and $\beta = \beta^{\infty} + \overline{\alpha}$, then χ and ζ defined by

$$\chi + i\zeta = \frac{1}{\beta + i\alpha} \tag{458}$$

are related by $\chi = \chi^{\infty} + \overline{\zeta}$.

Consider the function

$$\Xi(z) = 1 + a\left[1 + f_a z Z(z) + f_b b(z-b)Z(z)\right] = \beta + i\alpha\,, \tag{459}$$

where $z = x + iy$, Z is the plasma dispersion function,[44] and a, b, f_a and f_b are constants. As $y \longrightarrow 0^+$, we obtain

$$\alpha(x) = a\sqrt{\pi}\left[f_a x\, e^{-x^2} + f_b\,(x-b)\, e^{-(x-b)^2}\right], \tag{460a}$$

$$\beta(x) = 1 + a\left[1 - 2f_a x\,\mathrm{daw}(x) - 2f_b\,(x-b)\,\mathrm{daw}(x-b)\right]. \tag{460b}$$

The character of this function, which is essentially the dielectric function for a two stream plasma, depends rather strongly on the parameters a, b, f_a and f_b. In particular for $a = 7$, $b = 1.75$, $f_a = 0.6$ and $f_b = 0.4$, $\Xi(z)$ has no roots in the upper half-plane. (This can be easily seen using Nyquist's method.) In this case Ξ^{-1} is analytic in the upper half plane and has boundary values

$$\chi + i\zeta = \frac{1}{\Xi(x + i0^+)}\,. \tag{461}$$

Using the values of the parameters given above, χ and ζ were computed (see Figure 9) and $1+\overline{\zeta}$ was computed and compared to χ (see Figure 10). The calculation was done with 16 uniformly sized panels covering [−8,8] and the singular integrations were evaluated over an interval of size 0.5. All integrals were evaluated with 24-th order Gauss-Legendre quadrature. The function ζ has some rather sharp features which contribute to the structure of the error shown in Figure 10. These features required the smaller panel sizes than in the previous calculation. Clearly, the efficiency of this method could be further increased if the panel sizes were adaptively chosen based on the structure of the function under consideration.

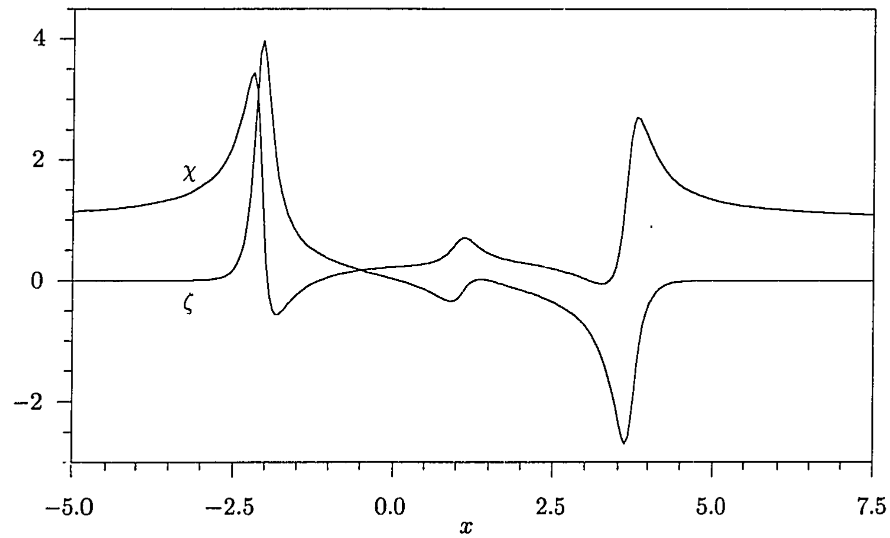

Figure 9: Plot of χ and ζ for $a = 7$, $b = 1.75$, $f_a = 0.6$ and $f_b = 0.4$

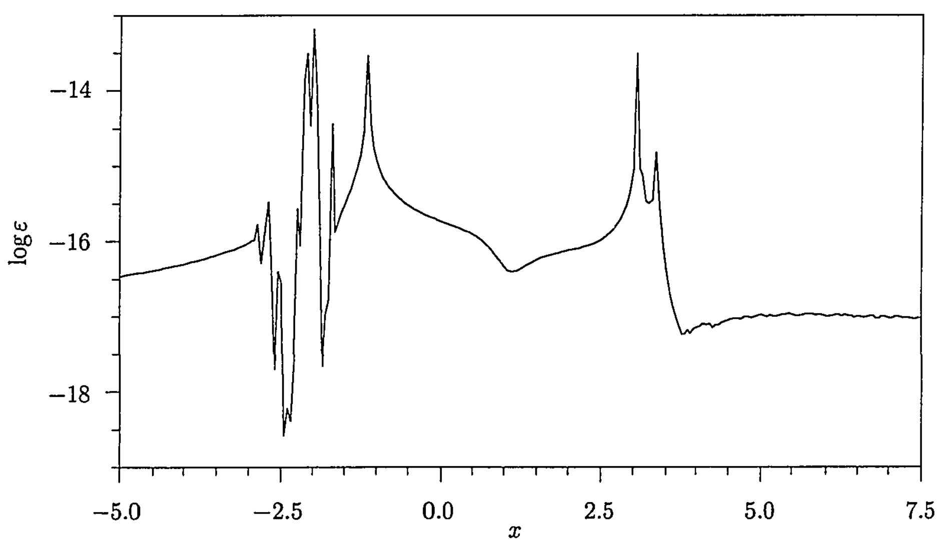

Figure 10: Absolute difference (base ten logarithm) between the numerical evaluation of $\overline{\zeta}(x) + \chi^{\infty}$ and ζ for $a = 7$, $b = 1.75$, ${f^{(0)}}_a = 0.6$ and $f_b = 0.4$

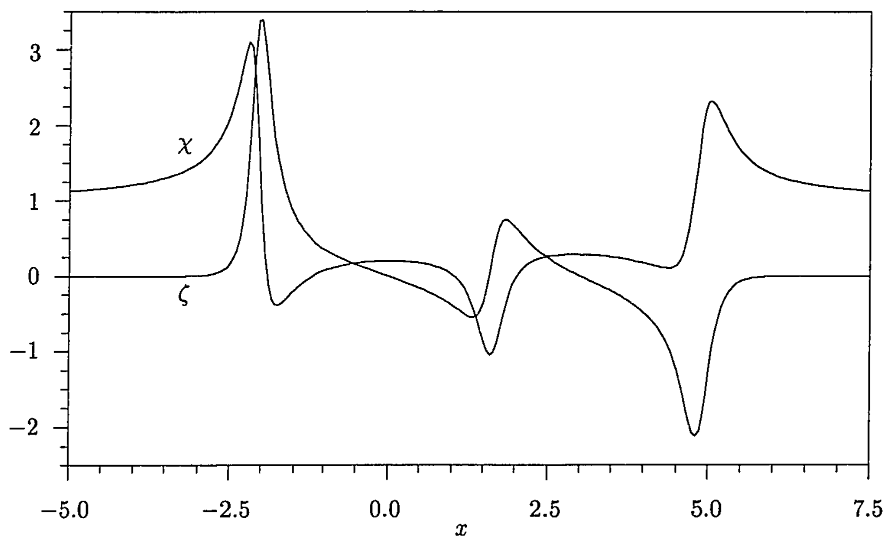

Figure 11: Plot of χ and ζ, for $a = 7$, $b = 3$, $f^{(0)}{}_a = 0.6$ and $f_b = 0.4$

If we change the parameters slightly, taking $b = 3$, we find that Ξ now has a root in the upper half-plane. As we saw in Chapter 7, this root alters the relationship between χ and $\overline{\zeta}$ such that

$$\chi = \chi^{\infty} + \overline{\zeta} + 2\frac{(x - x_0)\chi^{-1} + y_0\zeta^{-1}}{(x - x_0)^2 + y_0^2}, \tag{382}$$

where $z_0 = x_0 + iy_0$ is the root of Ξ and $\chi^{-1} + i\zeta^{-1} = \Xi'(x_0 + iy_0)^{-1}$. For these values of the parameters, χ and ζ are shown in Figure 11 and $z_0 \cong 1.632 + 0.268i$. In Figure 12 we compare the right-hand side of (382) with $1 + \overline{\zeta}$ and see that they are substantially different, with the former agreeing with the value of χ to better that 1 part in 10^{13} over the entire interval $[-5, 7.5]$. The transforms were evaluated with 16 uniformly sized panels covering $[-8, 8]$ with the singular integrations computed on an interval of size 0.4. All integrals were evaluated using 24-th order Gauss-Legendre quadrature. This example provides a dramatic

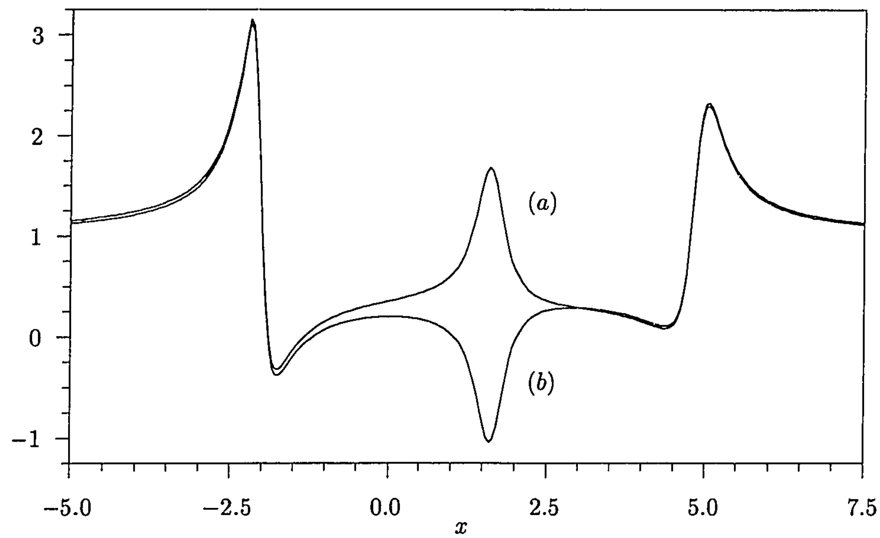

Figure 12: Showing the importance of the discrete eigenfunction when Ξ has a root in the upper half-plane. Here the root is located at $1.63226070416668 + 0.268383680182842i$. Plotted are (a) $1+\overline{\zeta}$ and (b) $1+\overline{\zeta}+2\,\mathrm{Re}(\chi^{-1}+i\,\zeta^{-1})/(x-z_0)$ as in (382). The curve (b) differs from χ by less than 10^{-13} over the entire range of the plot. The parameter as have the same values as in Figure 11

demonstration of the importance of the discrete eigenfunctions as well as an indication of the correctness of the regularizing procedure developed in Chapter 7.

III. Evaluating the Integral Transform Solutions of the Longitudinal Vlasov Equation

We now move on to the numerical evaluation the solution of the longitudinal Vlasov equation given in (212) which is valid for stable equilibria.[45] We consider the equilibrium

$$f_{\parallel}^{(0)} = \frac{1}{\sqrt{\pi}} n_0 e^{-v^2}, \tag{462}$$

where n_0 is the particle density. Note that we measure velocity in units of the equilibrium thermal velocity. For this equilibrium the dielectric function is given by

$$\begin{aligned}\epsilon_L(k,u) &= 1 + \frac{\omega_p^2}{k^2}\left[2 - 4u\,\mathrm{daw}\left(\frac{u}{\sqrt{\pi}}\right)\right] + 2i\,\frac{\omega_p^2}{k^2}\sqrt{\pi}\,u\,e^{-u^2} \\ &= \beta + i\,\alpha\,. \end{aligned} \tag{463}$$

For large u, $\alpha \approx 0$ and $\beta = 0$ can be solved for u provided

$$\frac{k}{\omega_p} \gtrsim 0.5336\,. \tag{464}$$

While ϵ_L is not exactly zero for any u in the upper half-plane, for sufficiently large u, it will not be possible (numerically) to distinguish ϵ_L from zero — thus we will have the equivalent of a neutral mode. To avoid the complication of including neutral modes into the code, we restrict k such $k\,v_{\mathrm{th}} < 0.5336\,\omega_p$. If we take a box of size L to be our spatial domain; the above restriction on k is equivalent to a Debye length of approximately 17% L.[46] Thus this restriction on possible wave numbers is physically acceptable.

Given an initial condition $F_k(v)$, we compute ξ_k as above (dropping the factor of $-4i\pi e/k$):

$$\xi_k(u) = \widetilde{\mathcal{G}}\left[F_k\right] = \chi F_k + \zeta \overline{F}_k, \tag{465}$$

from which we can compute

$$f^{(1)}_{\parallel k}(v,t) = \mathcal{G}\left[\xi_k \cos(kut)\right] - i\,\mathcal{G}\left[\xi_k \sin(kut)\right]. \tag{466}$$

We can easily calculate the electric field:

$$E^{(1)}_{\parallel k}(t) = \frac{4\pi e}{ik}\int dv_{\parallel}\, f^{(1)}_{\parallel k}(v_{\parallel},t)$$

$$= \frac{4\pi e}{ik} \int du \, e^{-ikut} \, \xi_k(u) \,. \tag{467}$$

One of the significant advantages of this method is that it is possible to compute $E^{(1)}_{\parallel k}(t)$ directly from the initial condition *without* computing $f^{(1)}_{\parallel k}$. This results in considerable efficiency — not having to perform a high accuracy calculation of $f^{(1)}_{\parallel k}$ to obtain the electric field — as well as in great convenience as often one is only interested in the behaviour of the perturbed field. Note that this is not the case for the transverse motion; there it is necessary to compute (at least indirectly) $\boldsymbol{f}^{(1)}_{\perp}$ to determine the evolution of the fields.

In our Hilbert transform algorithm, we factored the integrand into two pieces: one that is computationally expensive and one that is easily computed. The calculation of $\mathcal{G}$ can make further use of this idea by absorbing the trigonometric terms into the "fast" function, g, which then has a parametric dependence on t. This is advantageous since the weights, W^n_m, which are functions of χ and ζ, need only be computed *once*. These weights can also be used in the calculation of the electric field.

Since $\mathcal{G}$ is linear we have an explicit expression for $\dot{f}^{(1)}_k$:

$$\dot{f}^{(1)}_k(v,t) = -k \, \mathcal{G}\left[\xi_k \, u \sin(kut)\right] - ik \, \mathcal{G}\left[\xi_k \, u \cos(kut)\right]. \tag{468}$$

This is computationally no more intensive than computing $f^{(1)}_k$. Therefore it seems reasonable to make use of this as an independent error estimation by computing the left-hand side of the Vlasov equation. Define

$$\begin{aligned} \varepsilon_R + i\varepsilon_I = 4e\alpha \int du \, e^{-ikut} \xi_k(u) \\ -\mathcal{G}\left[\xi_k \cos(kut)\{1 - iku\}\right] + i \, \mathcal{G}\left[\xi_k \sin(kut)\{1 + iku\}\right]. \end{aligned} \tag{469}$$

Assuming that the extra factor of u does not greatly affect the accuracy of our calculation of $\mathcal{G}$, we can use the magnitude of ε as an indication of the numerical error in the computed $f_k^{(1)}$.[47] Having this error estimate is a significant advantage of our method. Since we are evaluating an exact solution, as opposed to explicitly solving the differential equation, the differential equation is available as a test of the quality of the solution. This permits long time evolution calculations since we can, whenever necessary — based on the *actual* numerical error, re-initialize (at modest computational cost) the panels used in the Hilbert transform algorithm to cope with the fine structure produced by velocity filamentation. This ability effectively circumvents the computational problems associated with velocity filamentation.

As a simple test of our methods, we consider a single, dynamically accessible Fourier mode

$$F(v) = -v\, e^{-v^2/b^2}, \tag{470}$$

corresponding to $k = 1$. We readily find

$$\overline{F}(v) = \frac{2}{\sqrt{\pi}}\, v \,\mathrm{daw}\left(\frac{v}{b}\right) - \frac{b}{\sqrt{\pi}} \tag{471}$$

and thus

$$\xi(u) = \widetilde{\mathcal{G}}\left[\overline{F}\right] = -u\,\chi(u)\, e^{-u^2/b^2} + \frac{1}{\sqrt{\pi}}\,\zeta(u)\left[2u\,\mathrm{daw}\left(\frac{u}{b}\right) - b\right]. \tag{472}$$

As we alluded above, both terms in ξ have exponential factors and thus our Hilbert transform algorithm should work nicely.

As a test on the quality of $\mathcal{G}$, we can compute $\mathcal{G}[\xi]$ and compare with F (see Figure 13). As seen from the figure, the implementation of $\mathcal{G}$ is extremely good. The structure of the error arises from two sources. The large scale features

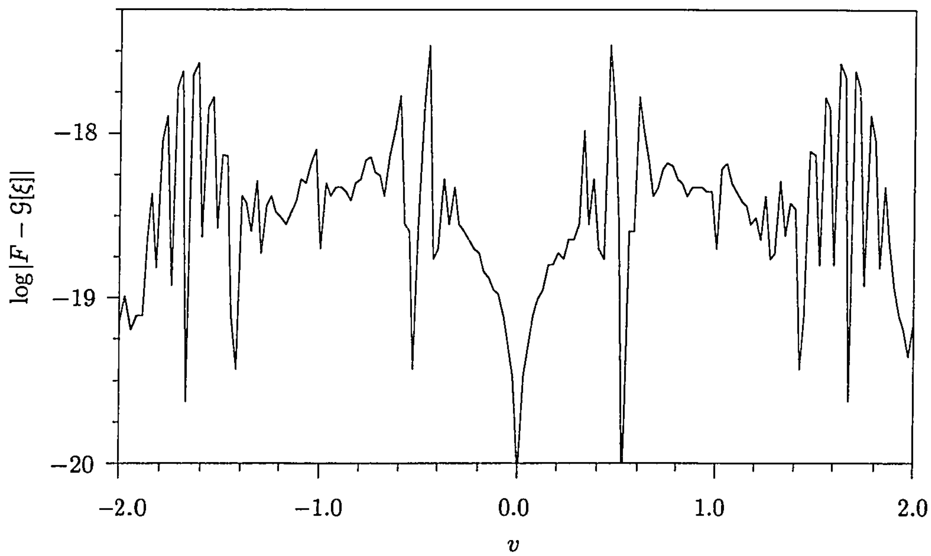

Figure 13: Absolute error (base ten logarithm) in calculating the initial condition with $b = 1$ using $\mathcal{G}$. This is a measure of the extend to which $\mathcal{G}\left[\tilde{\mathcal{G}}\right] = 1$ numerically.

are due to variations in the derivatives of χ and ζ. The finer details are an artifact of the quadrature formula. Recall that that remainder term is an *upper* bound on the error — thus for certain values of the transformed variable the actual error can be significantly less. These fine details amount to an aliasing of the points where $\mathcal{G}[\xi]$ is evaluated and these special values.

Setting $b = 1$, $L = 4$ and $\omega_p = 1$, we calculate the perturbed electric field directly from the initial condition. (See Figure 14.) Figures 15 and 16 show the real and imaginary parts of $f^{(1)}_{\parallel k}$, respectively, obtained from evaluating $\mathcal{G}\left[\xi(u,t)\right]$ using the methods discussed above. For these calculations, 14 uniformly sized panels spanning $[-7, 7]$ were used. The singular integrals were computed over an interval of size 1.0. All integrals were evaluated using 26-th order Gauss-Legendre quadrature. Shown in Figures 17 and 18 are the real and imaginary parts of ε, respectively. As we can see, the overall error in the solution is quite small, growing with time due the oscillatory nature of $\xi_k(t)$. As noted above, the initial condition was dynamically accessible. For our equilibrium, (462), whose derivative has a simple root at $v = 0$, the condition of dynamical accessibility requires that the perturbation have at least a first order zero at $v = 0$. From Figures 15 and 16, this condition is seen to be satisfied over the time interval considered. This behaviour is exactly that shown in Section 4-I. Note that with our choice of parameters, we are in the regime of strong damping as is evidenced by the rapid decay of $E^{(1)}_{\parallel k}$. This can be confirmed by observing that $f^{(1)}_{\parallel k}$ becomes highly oscillatory and thus, due to phase mixing, $E^{(1)}_{\parallel k}$ dies away.

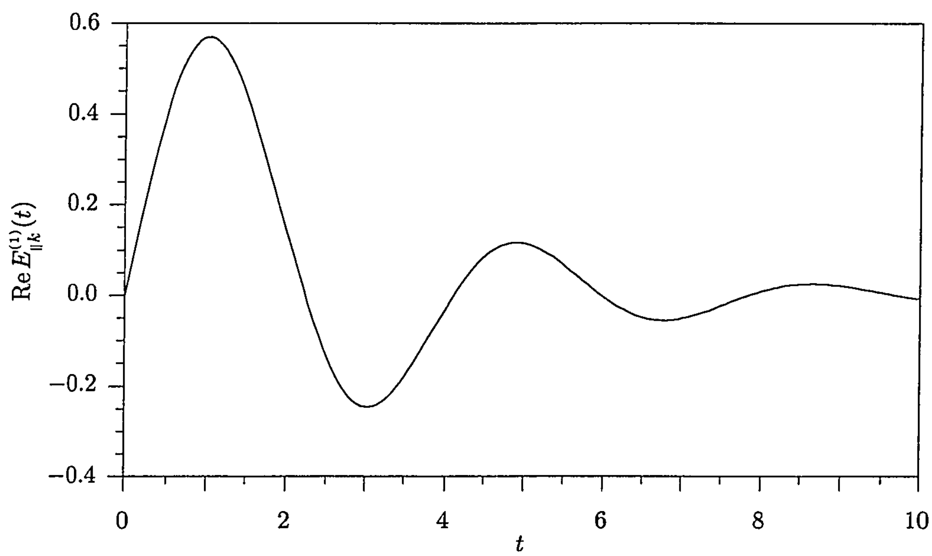

Figure 14: The real part of $E^{(1)}_{\parallel k}(t)$ compute from the initial condition (470). For this initial condition, $\operatorname{Im} E^{(1)}_{\parallel k}(t) \equiv 0$.

Notice that not only does $f^{(1)}_{\parallel k}$ become more oscillatory with time, but the real and imaginary parts develop a nearly $\pi/2$ phase shift. One then wonders how an initial condition having these same characteristics would evolve and what would

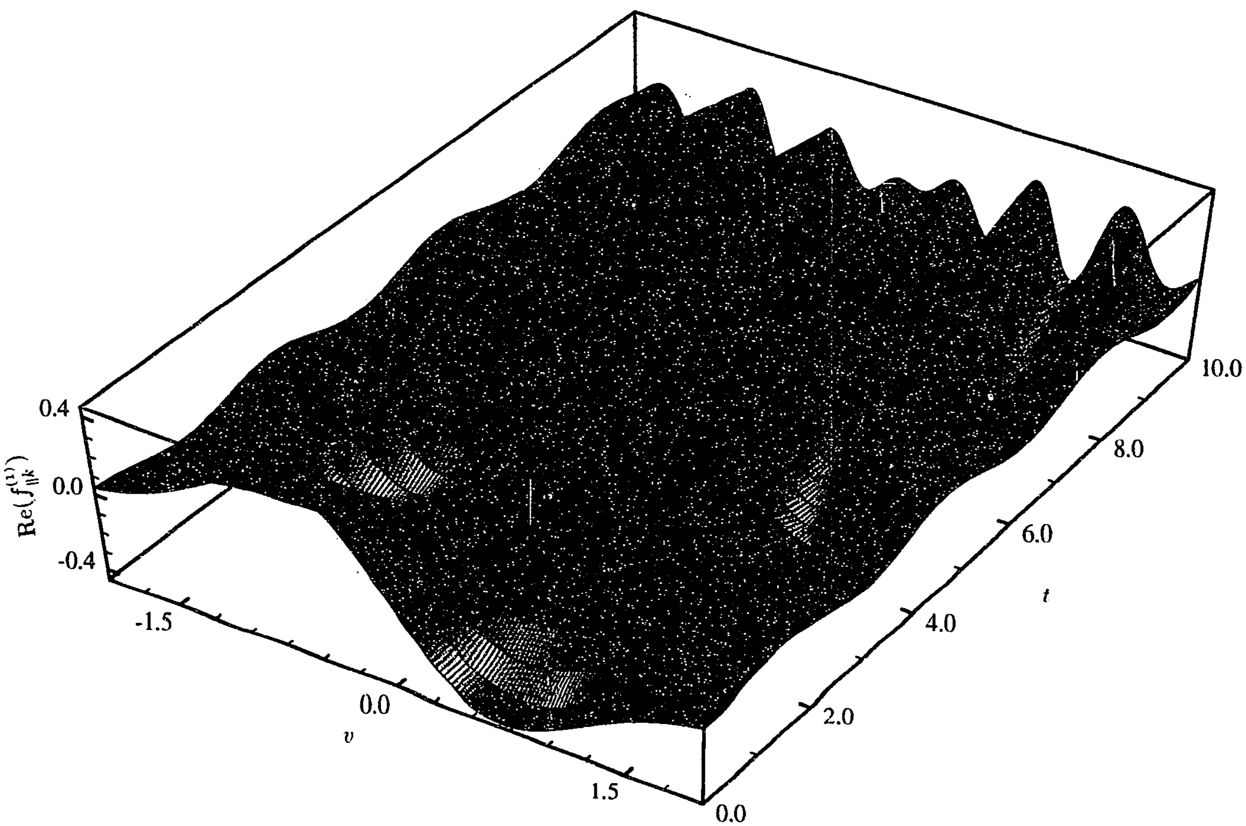

Figure 15: Re($f^{(1)}_{\|k}$) computed using a singular eigenfunction expansion of the initial condition (470).

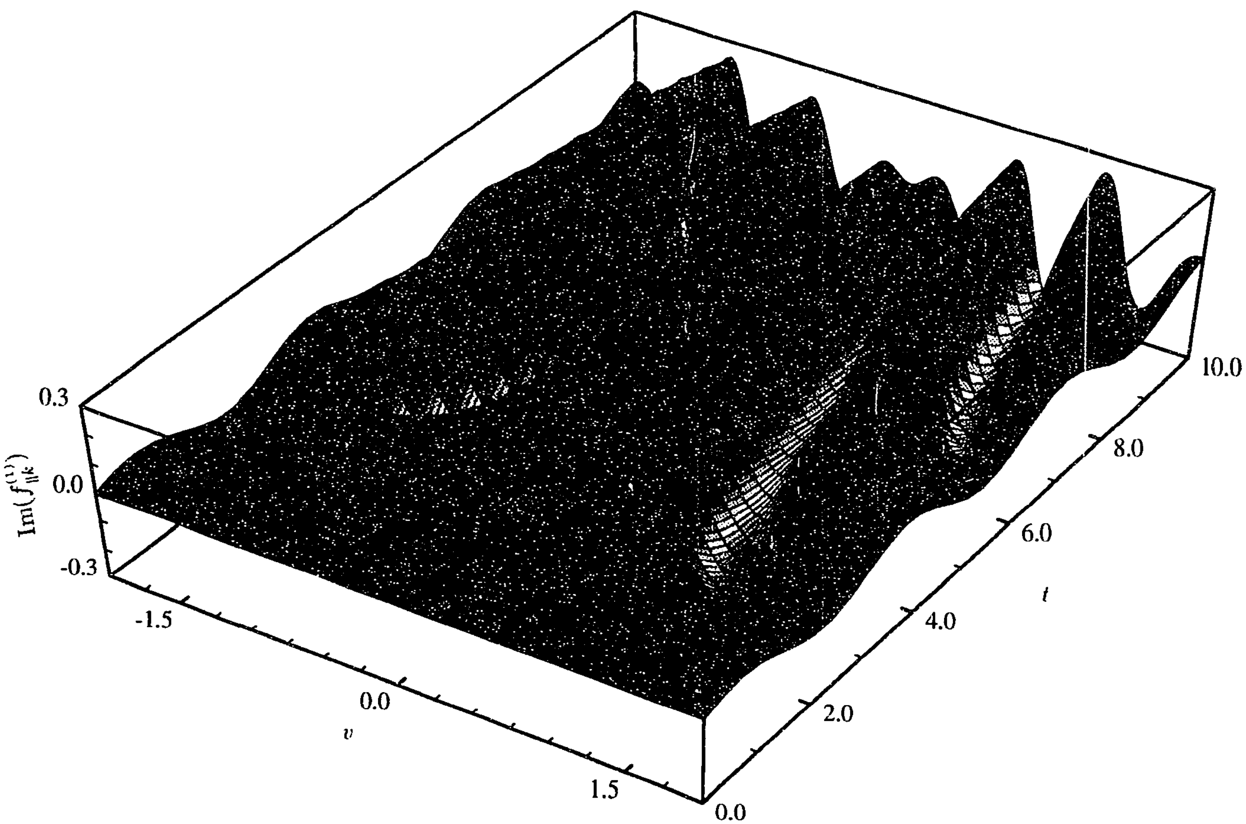

Figure 16: $\mathrm{Im}(f_{\parallel k}^{(1)})$ computed using a singular eigenfunction expansion of the initial condition (470).

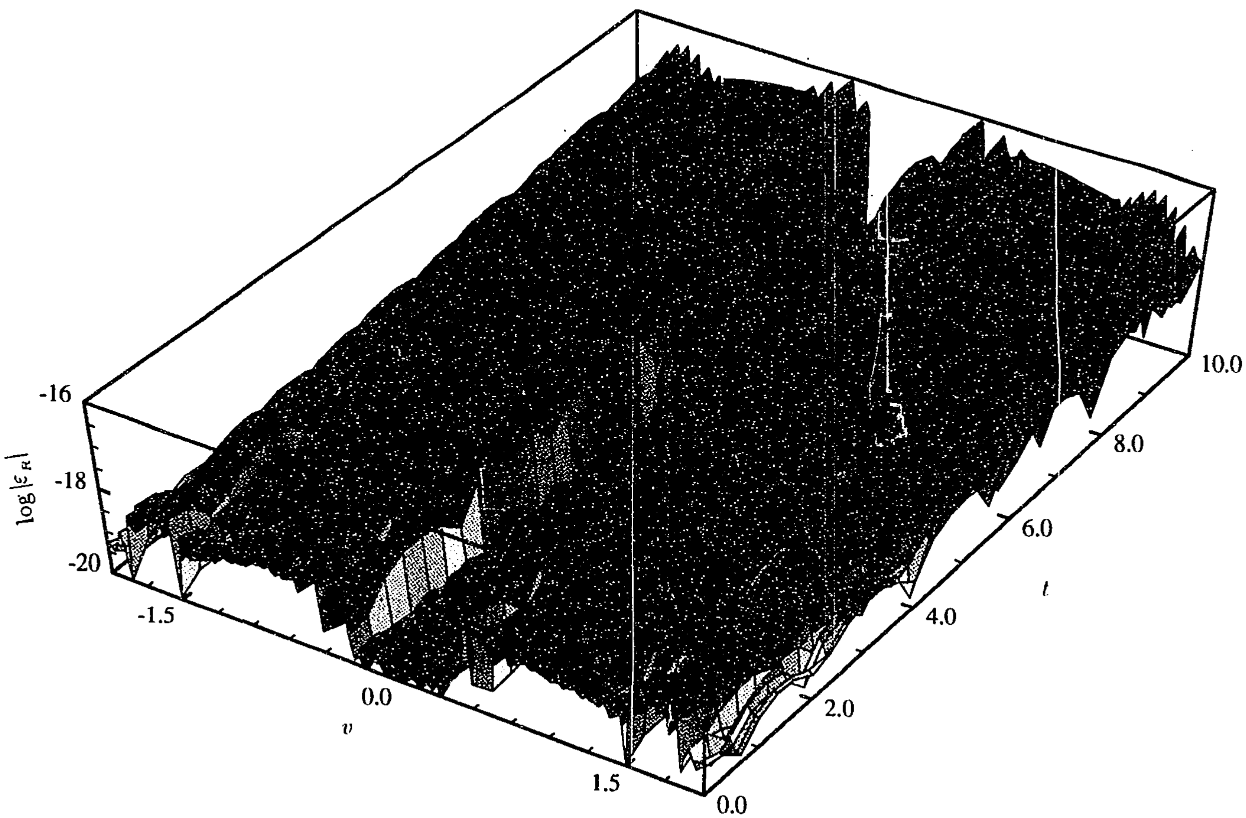

Figure 17: Real part of the absolute error (base ten logarithm). See text for details.

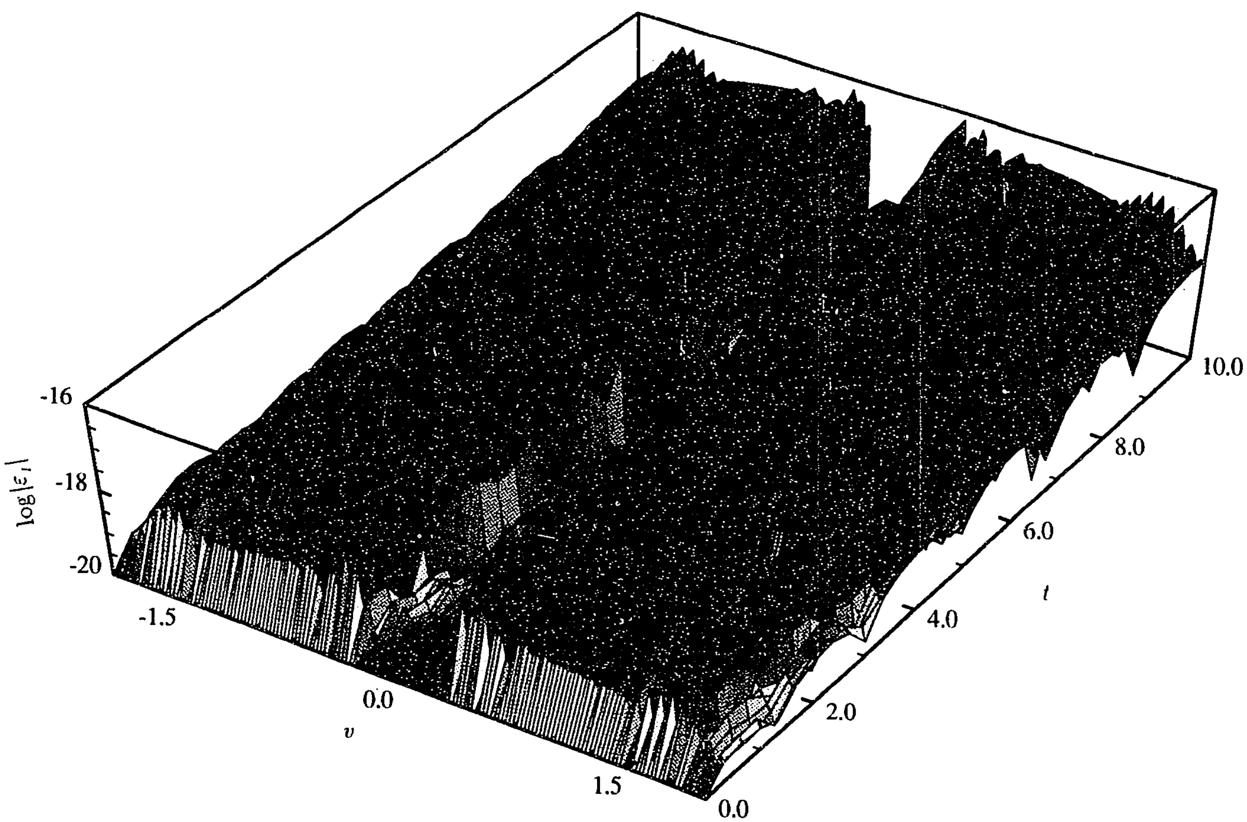

Figure 18: Imaginary part of the absolute error (base ten logarithm). See text for details.

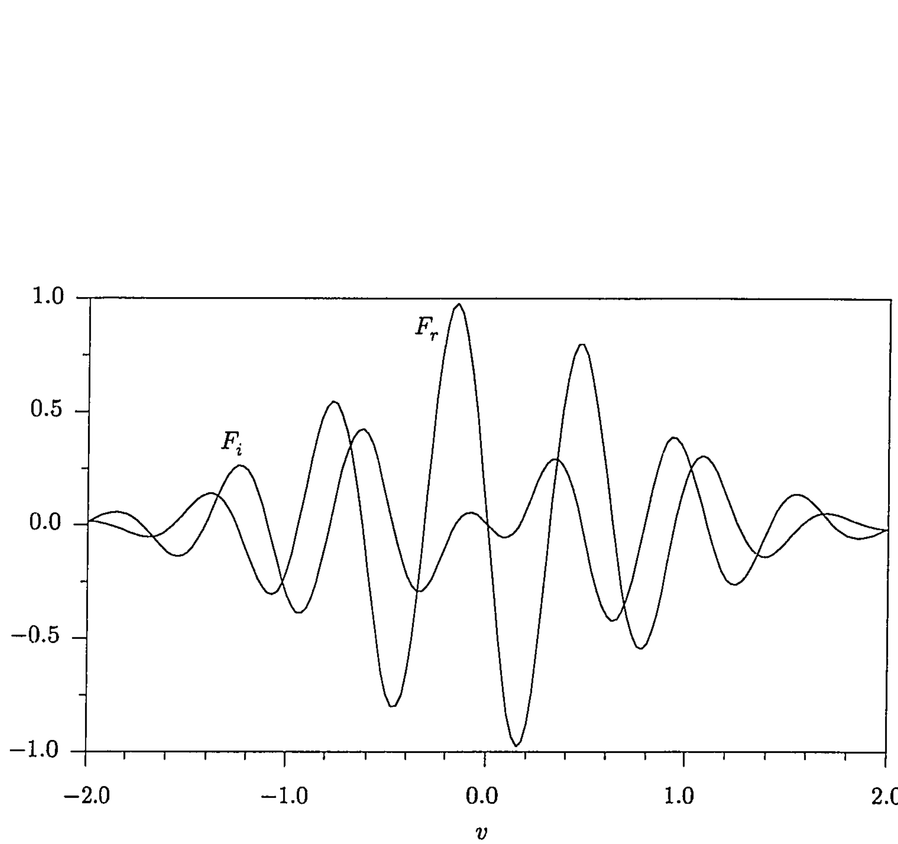

Figure 19: A plot of the initial condition given in (473).

be the behaviour of the resulting perturbed electric field. To this end we repeat the above calculation, using

$$F_r = \sin(10v)\, e^{-v^2}, \tag{473a}$$

$$F_i = -v\cos(10v)\, e^{-v^2}. \tag{473b}$$

This initial condition is plotted in Figure 19. Unlike the previous example, we cannot compute $\overline{F}$ in closed form and hence we will not have a closed form expression for $\xi(u)$. This will be an even more stringent test of the Hilbert transform algorithm as we are now using it to compute ξ. Again we compute $\mathcal{G}\,[\xi]$ and compare with F as a check (see Figure 20) and we find that the algorithm is performing satisfactorily.

The time evolution of this initial condition is somewhat surprising. One might expect that, since it closely resembles the large time behaviour of a smooth

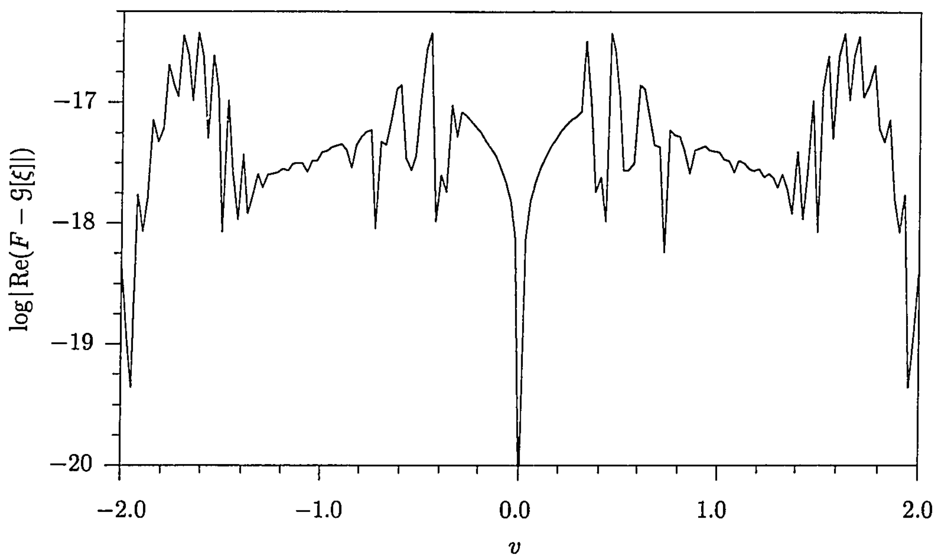

Figure 20: The real part of the absolute error (base ten logarithm) in calculating the initial condition using $\mathcal{G}$. The imaginary part of this error is similar. We use this is a measure of the extend to which $\mathcal{G}\left[\tilde{\mathcal{G}}\right] = 1$ numerically.

condition, the solution would continue to become more oscillatory creating no electric field. The real and imaginary parts of $f^{(1)}_{\parallel k}$ corresponding to this initial condition are shown in Figures 21 and 22 respectively. Shown in Figures 23 and 24 are the real and imaginary parts of ε, respectively. The actually behaviour is considerably different from what we had (naïvely) expected; $f^{(1)}_{\parallel k}$ develops a significant area of slow oscillation which gives rise (quite quickly) to a large electric field (see Figure 25). While the initial condition may *look* like the result of time-evolving a smooth initial condition, it is not exactly thus. The behaviour of $f^{(1)}_{\parallel k}$ and $E^{(1)}_{\parallel k}$ prior to $t = 10$ is the *initial* transient that precedes Landau damping. Had our initial condition been *exactly* the been the long-time state of a smooth initial condition, we would not have seen this transient stage; the solution would have continued to become every more oscillatory. This example serves to illustrate

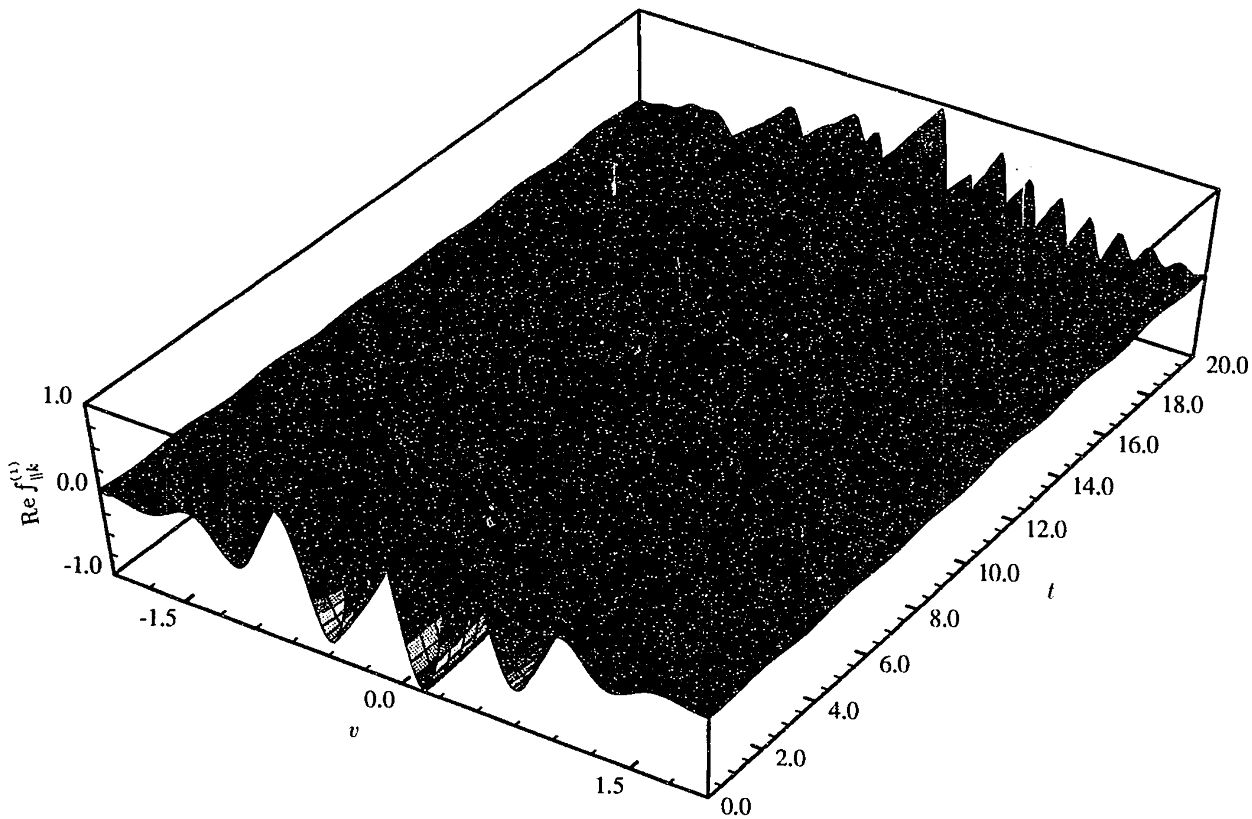

Figure 21: Re($f^{(1)}_{\|k}$) computed using a singular eigenfunction expansion of the initial condition (473).

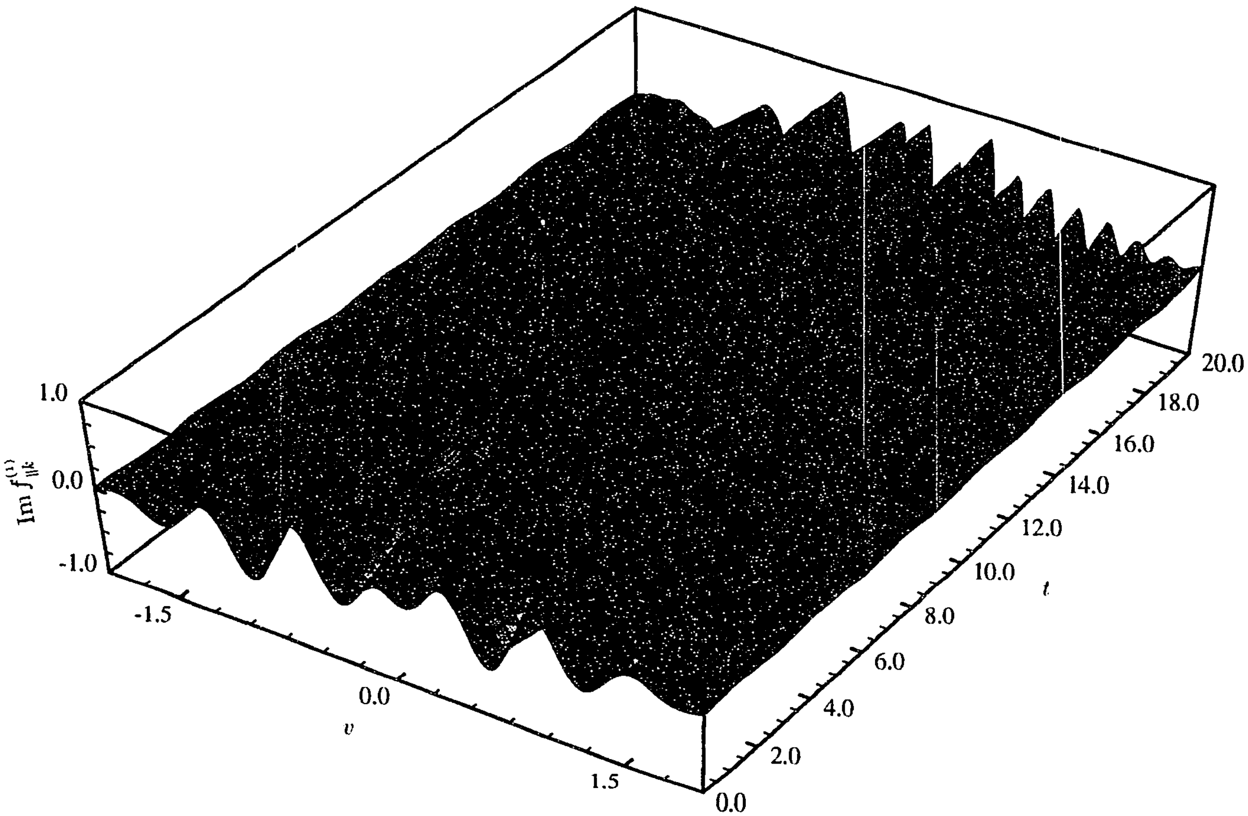

Figure 22: $\mathrm{Im}(f_{\|k}^{(1)})$ computed using a singular eigenfunction expansion of the initial condition (473).

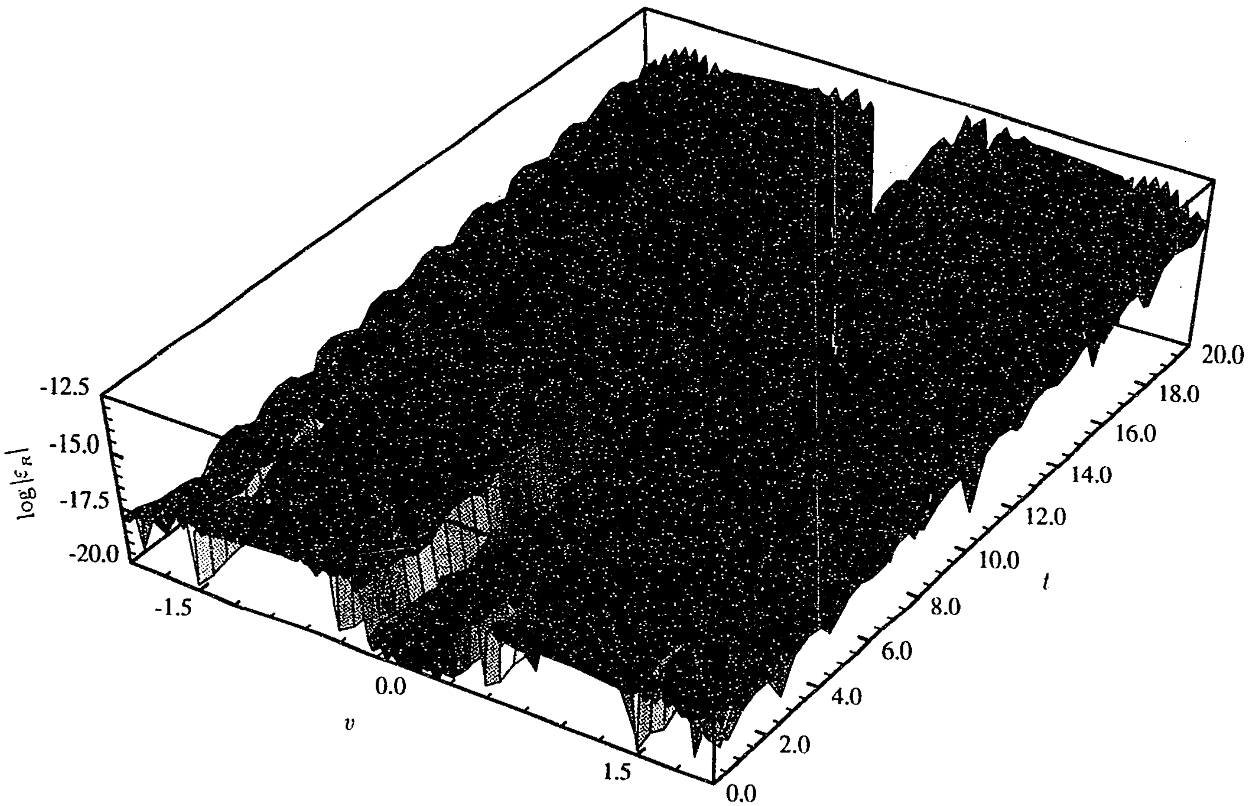

Figure 23: Real part of the absolute error (base ten logarithm). See text for details.

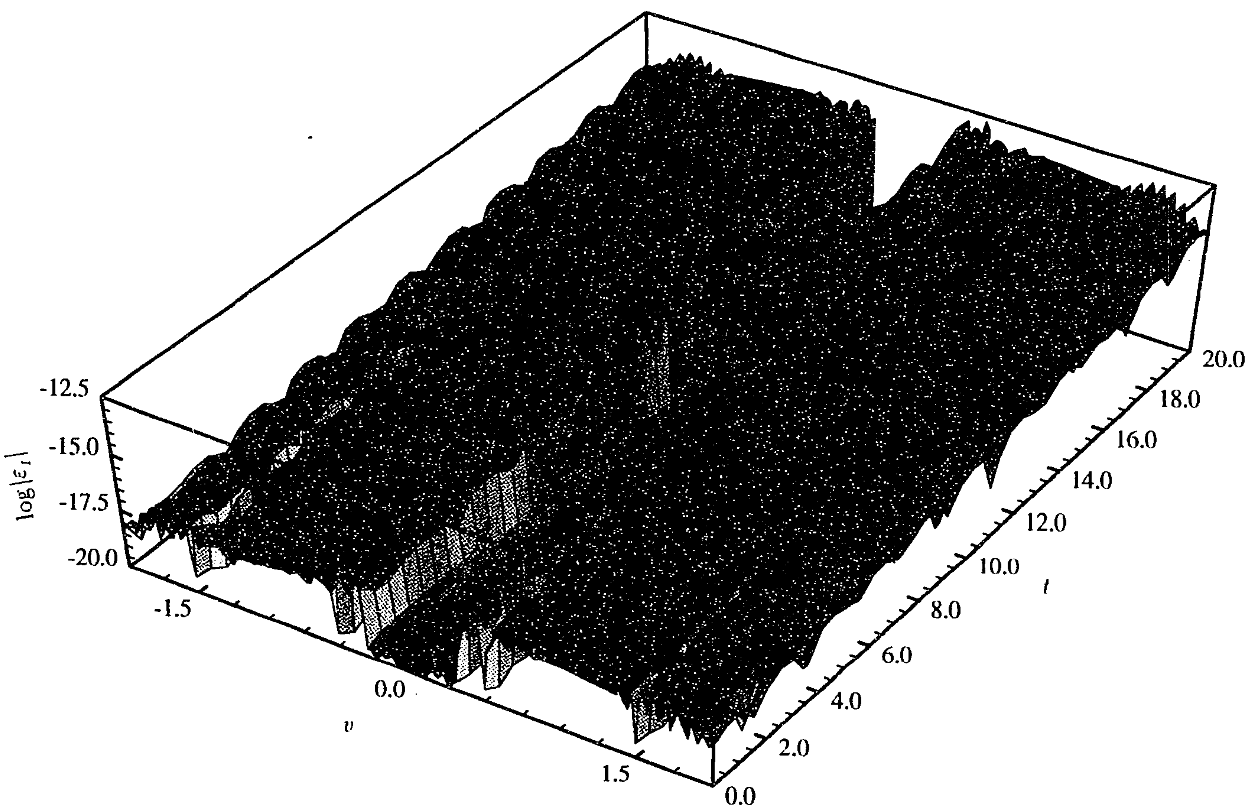

Figure 24: Imaginary part of the absolute error (base ten logarithm). See text for details.

the special nature of the structure of the solutions of the Vlasov equation; given a particular initial condition, the Vlasov equation "knows" whether it is the result of time evolving a smooth initial condition (and thus should continue to become more oscillatory) or whether it is still in the pre-Landau damping transient phase.

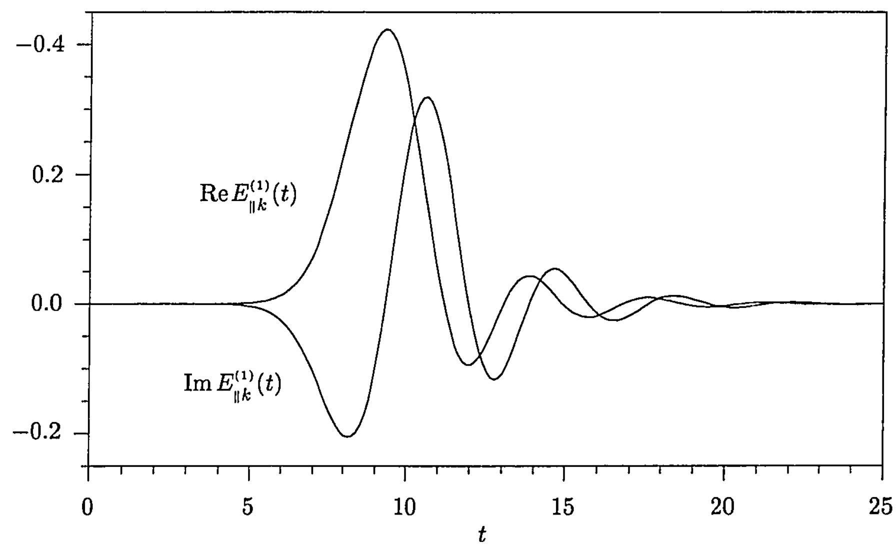

Figure 25: The perturbed electric field arising from the initial condition (473).

The results of the chapter not demonstrate the computational efficacy of using singular eigenfunctions. Using singular eigenfunction expansions, we are able to efficiently solve the initial value problem with the added feature of having a reliable error bound. These numerical techniques open the door to exploring numerous phenomena associated with the linearized Vlasov equations, not the least of which is transient behaviour prior to the on set of Landau damping.

9

Conclusions

In the preceding chapters we have seen that by constructing an appropriate integral transform, the Maxwell-Vlasov system, linearized about a homogeneous equilibrium, can be reduced to a simple set of ordinary differential equations. For both the longitudinal and transverse degrees of freedom, we were able to construct a general solution and identify a class of initial conditions, $\mathcal{D}^{\mu}_{L}$ and $\mathcal{D}^{\mu}_{T}$ respectively, for which a solution can be obtained. In addition we were able to show that as time evolved, the solution remained in the function space of the initial condition.

The same integral transforms that we used to solve the Maxwell-Vlasov equations were also used to change coordinates resulting in a diagonal form for the bracket and Hamiltonian. A further scaling yielded action-angle variables for this system. Thus we were able to take a noncanonical field theory and, by means of a coordinate transformation obtain, a *canonical* Hamiltonian system. These techniques have been generalized by Balmforth and Morrison[48] and applied to problems in fluid mechanics.

The diagonal form of the Hamiltonian was quadratic in the Fourier amplitudes of the electric field, however, this expression was not equivalent to electromagnetic energy stored in a dielectric. The source of this discrepancy is the presence in a plasma of resonant particles — a characteristic *not* shared by dielectrics.

We extended our formalism for the longitudinal case, to include both neutral and unstable modes. It was seen that the necessary discrete eigenfunctions appear as a natural consequence of regularizing certain singular quantities. The key ingredient in the neutral mode analysis was the introduction of generalized functions and their Hilbert transforms which allowed for a natural regularization procedure. Notably for the neutral mode case only *one* discrete eigenfunction was necessary to restore completeness. We have a general (constructive) prescription for the treatment of discrete modes. While we only explicitly computed the eigenfunctions for a simple root of the dielectric function, it is clear that the process of regularizing the inverse transform by canceling the singular terms in the Laurent expansion will generalize to multiple roots. The neutral mode contribution to the energy *was* equal to the dielectric energy reinforcing the importance of resonant particles in the distinction between a plasma and a dielectric. Using the neutral modes as a model, the treatment of unstable modes was straightforward not requiring generalized functions.

We considered various numerical algorithms for the evaluation of the Hilbert transform. An algorithm based on Gaussian quadrature was implemented and proved to be quite effective. This algorithm was then used to evaluate the integral transforms used to solve the longitudinal equations. The algorithm was tested on a simple Maxwellian initial condition and the resulting perturbed electric field was seen to Landau damp due to phase mixing. A further test was performed with an initial condition which already exhibited significant phase mixing. This configuration in a significant transient period prior to the onset of Landau damping. After a long period of quiescence, a large transient electric field built up which subsequently damped away. The performance of this algorithm was extremely good and gives a clear indication of the computational value of singular mode expansions.

There are a number of future directions that this work can take. The analytical techniques that we have developed for solving the Riemann-Hilbert problem are applicable to a wide range of singular integral equations. Since the Hilbert transform commutes with the derivative operator, these techniques could be successfully applied to more complicated intergo-differential equations.

A logical next step would be to apply these methods to second order perturbation theory. The approach would be to use the linear solutions as a basis for representing the next higher order effects. Since we know that the linear solutions are complete over a broad class of functions, one hopes that they would be a suitable basis for describing weakly nonlinear behaviour. This is a particularly intriguing idea since we already have the necessary numerical tools to evaluate singular mode expansions.

We have specifically ignored the effects of the finite value of the speed of light on the transverse motion. As Felderhof points out, the distribution function must vanish identically for velocities greater than the speed of light and as a result the dielectric functions has infinitely many (two for each wavenumber) discrete solutions with superluminal phase velocity. This effect is more important in the transverse case than in the longitudinal due to the way that the speed of light enters Maxwell's equations. One can view this as the effect of choosing a particular boundary condition for the perturbed distribution function. A proper treatment of this effect would require the extension of our formalism to include step-function initial conditions. One expects that a procedure similar to the method used to handle the poles in the neutral mode case would be effective here. This extension is also of interest since one can envision a variety of physically realistic initial conditions that have discontinuities in velocity.

Experimental techniques have recently been developed that use laser-induced fluorescence to perform non-perturbing measurements of the phase space distribution functions[49–51] in collisionless ion plasmas. Experimental conditions are such that linear theory applies well and the spatial variation can be restricted to one dimension. These experiments appear to be well suited to test our various calculational methods as well as verify the correctness of the energy expressions.

There are similar singular integral equations in other areas of physics that are equally interesting, in particular the Lippmann-Schwinger equation which describes quantum mechanical scattering. In this case the structure of the continuum is slightly different since the discrete modes are not embedded in the continuum as they are in Vlasov theory, however the parallels are sufficiently strong to make a serious study of the relationship worthwhile.

Having developed a considerable set of analytic and corresponding numerical tools one wants to find a broad a range of applications as possible — this is fuelled by the feeling that at the deepest levels many areas of physics have a great deal in common.

Appendix A

Some Properties of the Hilbert Transform

Here we review some aspects of the theory of Cauchy integrals with application to Hilbert transforms. This subject is often formulated in terms of L^p function spaces.[52,53] An alternative formulation, based upon the concept of Hölder continuity, is due to several Soviet mathematician[19,20] whose work was motivated in part by problems in elasticity theory. Although, for our purposes, this latter formulation has certain important advantages, we are not able to completely dispose of the notion of L^p spaces. Thus we will make use of a hybrid of these approaches.

1. Preliminaries

A function $\phi : R \mapsto R$ is said to satisfy the Hölder condition of index μ if

$$|\phi(x) - \phi(y)| \leq A|x - y|^{\mu}, \qquad \forall x, y \in R, \tag{474}$$

where A, $\mu > 0$. If $\mu > 1$, satisfying this condition would imply that $\phi' = 0$ for all x, *i.e.* that ϕ is a constant, thus we only consider μ in the range $0 < \mu \leq 1$. (For $\mu = 1$ the Hölder condition is identical to the Lipschitz condition.) We denote the class of functions satisfying the Hölder condition by $\mathcal{H}^{\mu}$. Note that if $\phi \in \mathcal{H}^{\mu}$ then $\phi \in \mathcal{H}^{\nu}$ for $\nu < \mu$ and thus when we say that a function is in $\mathcal{H}^{\mu}$, it will be understood that μ is to be taken as the largest number such that the Hölder condition is satisfied.

There is a sub-class of Hölder function that will be of special interest to us. Namely those functions which, in addition to belonging to $\mathcal{H}^{\mu}$, possess a limit ϕ^{∞}, as $|x| \longrightarrow \infty$ and, for sufficiently large $|x|$, satisfy

$$|\phi(x) - \phi^{\infty}| \leq \frac{A'}{|x|^{\alpha}}, \qquad (475)$$

where A', $\alpha > 0$. Denote this restricted class by $\mathcal{H}^{\mu}_{*}$.

Consider $\phi \in \mathcal{H}^{\mu}_{*}$ and $\psi \in \mathcal{H}^{\nu}_{*}$ then the following results hold:[20]

$$\phi \pm \psi \in \mathcal{H}^{\rho}_{*}, \qquad (476a)$$

$$\phi\psi \in \mathcal{H}^{\rho}_{*}, \qquad (476b)$$

where $\rho = \min(\mu, \nu)$. Further, if ϕ/ψ is bounded for all x we have

$$\frac{\phi}{\psi} \in \mathcal{H}^{\rho}_{*}. \qquad (476c)$$

In this context, the Hölder condition is essentially a *local* statement of the properties of ϕ. For x and y arbitrarily far apart (474) reduces to a statement of boundedness and the index is meaningless. When x and y are taken to be close together, the index is related to the series expansions of ϕ and in this way, clearly separates functions into different categories.

Consider the integral

$$\Phi(x) = \mathrm{P}\int_{a}^{b} dx' \, \frac{\phi(x')}{x' - x}, \qquad (477)$$

where P indicates the Cauchy principal value:

$$\mathrm{P}\int_a^b dx' \, \frac{\phi(x')}{x'-x} \equiv \lim_{\epsilon \to 0}\left(\int_{x+\epsilon}^b + \int_a^{x-\epsilon}\right) dx' \, \frac{\phi(x')}{x'-x}, \tag{478}$$

For $\phi(x) \in \mathcal{H}^\mu$ the integral in (477) is guaranteed to exist and $\Phi(x)$ satisfies the Hölder condition[19] for $x \in (a,b)$. It is important to note that the endpoints of the interval have been excluded. An important feature of Cauchy integrals arises when one compares

$$\mathrm{P}\int_a^b dx'' \, \frac{1}{x''-x} \mathrm{P}\int_a^b dx' \, \frac{\phi(x',x'')}{x'-x} \tag{479}$$

and

$$\mathrm{P}\int_a^b dx' \, \mathrm{P}\int_a^b dx'' \, \frac{\phi(x',x'')}{(x'-x)(x''-x)}. \tag{480}$$

Although the only difference between (479) and (480) is the order of integration, the two expressions are, in general, not equal but are related by the Poincaré–Bertrand formula:[54]

$$\begin{aligned} \mathrm{P}\int dx' \frac{1}{x'-x} \mathrm{P}\int dx'' \frac{\phi(x',x'')}{x''-x} &= -\pi^2 \phi(x,x) \\ &\quad + \mathrm{P}\int dx'' \mathrm{P}\int dx' \frac{\phi(x',x'')}{(x'-x)(x''-x)}. \end{aligned} \tag{481}$$

As we mentioned above, there is an alternative formulation of this theory in terms of L^p spaces. The central concept of that formulation[53] is the result: if $\phi \in L^p$, for $p > 1$ then Φ belongs to L^p. Unfortunately there is no general inclusion relationship between L^p spaces and classes of Hölder functions.

2. Hilbert Transforms

Of particular interest to us is the limit $-a = b \longrightarrow \infty$. This is equivalent to treating the integration about infinity in the Cauchy sense. Provided that we only consider functions belonging to $\mathcal{H}_*^\mu$, the results of the previous section apply.

We define the Hilbert transform of $\phi \in \mathcal{H}_*^\mu$ as[55]

$$\overline{\phi}(x) \equiv H\left[\phi\right](x) = \frac{1}{\pi}\mathrm{P}\!\int dx' \, \frac{\phi(x')}{x'-x}, \tag{482}$$

where the limits of integration have been taken to infinity as above. Henceforth we will drop the P with the implicit understanding that, where necessary, integrals are to be interpreted in the sense of the Cauchy principal values. As we saw above, for $\phi(x) \in \mathcal{H}_*^\alpha$ the integral is guaranteed to exist and $\overline{\phi} \in \mathcal{H}_*^\alpha$ and thus the Hilbert transform can be thought of as a linear functional that maps $\mathcal{H}_*^\alpha$ to $\mathcal{H}_*^\alpha$.

Using the Poincaré-Bertrand formula it is possible to establish two useful properties of (482). Firstly it thus seems natural to consider $H\left[H\left[\phi\right]\right]$ which clearly exists for $\phi \in \mathcal{H}_*^\alpha$. Explicitly we have

$$H\left[H\left[\phi\right]\right](x) = \frac{1}{\pi^2}\int dx' \, \frac{1}{x'-x}\int dx'' \, \frac{\phi(x'')}{x''-x}. \tag{483}$$

Using (481)

$$\begin{aligned} H\left[H\left[\phi\right]\right](x) &= -\phi(x) + \frac{1}{\pi^2}\int dx'' \, \phi(x'')\int dx' \, \frac{1}{(x'-x)(x''-x')} \\ &= -\phi(x) + \frac{1}{\pi^2}\int dx'' \, \frac{\phi(x'')}{x''-x}\int dx' \, \left[\frac{1}{x'-x} - \frac{1}{x'-x''}\right] \\ &= -\phi(x), \end{aligned} \tag{484}$$

since $H\left[1\right] = 0$. Thus we see that the inverse of the Hilbert transform is simply $-H$.

Secondly, we also use the Poincaré-Bertrand formula to obtain a "convolution" formula for the Hilbert transform. Let $\phi \in \mathcal{H}_*^\alpha$ and $\psi \in \mathcal{H}_*^\nu$. Consider

$$\begin{aligned} H\left[\phi\psi\right](x) &= \frac{1}{\pi}\int dx' \, \frac{\phi(x')\psi(x')}{x'-x} \\ &= -\frac{1}{\pi^2}\int dx' \, \frac{\psi(x')}{x'-x}\int dx'' \, \frac{\overline{\phi}(x'')}{x''-x'} \,. \end{aligned} \tag{485}$$

Using (481) to interchange the order of integration gives

$$\begin{aligned} H\left[\phi\psi\right](x) &= \psi(x)\,\overline{\phi}(x) - \frac{1}{\pi^2}\int dx'' \int dx' \, \frac{\psi(x')\,\overline{\phi}(x'')}{(x'-x)(x''-x')} \\ &= \psi(x)\,\overline{\phi}(x) - \frac{1}{\pi^2}\int dx'' \, \frac{\overline{\phi}(x'')}{x''-x}\left\{\int dx' \, \frac{\psi(x')}{x'-x} - \int dx' \, \frac{\psi(x')}{x''-x'}\right\} \\ &= \psi(x)\overline{\phi}(x) - \frac{1}{\pi}\int dx'' \, \frac{\overline{\phi}(x'')}{x''-x}\left\{\overline{\psi}(x) - \overline{\psi}(x'')\right\}. \end{aligned} \tag{486}$$

Thus we conclude that

$$\overline{\phi\psi} = \phi\overline{\psi} + \overline{\phi}\psi + \overline{\overline{\phi}\overline{\psi}}\,. \tag{487}$$

A generalization of Parseval's formula can also be derived. Let ϕ and ψ be as above with the further requirement $\phi \in L^p$ and $\psi \in L^q$ where p, $q > 1$ and $1/p + 1/q = 1$. Then $\phi\psi$, $\phi\overline{\psi}$, $\overline{\phi}\psi$ and $\overline{\psi}\overline{\psi}$ are all in L^1 and

$$\int dx \, \phi(x)\,\psi(x) = \int dx \, \overline{\phi}(x)\overline{\psi}(x)\,, \tag{488a}$$

$$\int dx \, \phi(x)\,\overline{\psi}(x) = -\int dx \, \overline{\phi}(x)\,\psi(x)\,. \tag{488b}$$

We can also derive forumlæ which are analogous of the derivative and integral formulæ for Fourier transforms:

$$\overline{x^n\,\phi} = x\,\overline{x^{n-1}\,\phi} + \overline{x^n\,\phi}(0)\,, \quad n > 0\,, \tag{489a}$$

$$\overline{x^n\,\phi} = \frac{1}{x}\,\overline{x^{n-1}\,\phi} - \frac{1}{x}\,\overline{x^{n+1}\,\phi}(0)\,, \quad n < 0\,, \tag{489b}$$

where, in the case of $n < 0$, $x^n \phi$ must be bounded at $x = 0$ and for $n > 0$, $x^n \phi$ must be bounded as $x \longrightarrow \infty$. For $n = 1$ we have

$$\overline{x\phi} = x\,\overline{\phi} + \frac{1}{\pi}\int dx'\, \phi(x')\,, \tag{490a}$$

$$\overline{\left(\frac{\phi}{x}\right)} = \frac{\overline{\phi}}{x} - \frac{1}{x}\frac{1}{\pi}\int dx'\, \frac{\phi(x')}{x'}\,. \tag{490b}$$

In 1904, D. Hilbert proved the *Hilbert Inversion Theorem* which, through Cauchy integrals over the unit circle, provided a connection between the real and imaginary parts of an analytic function. This theorem can be applied to the upper half-plane where the relevant Cauchy integrals are over the real axis. In this case, the theorem is as follows: Let $F(z) = u(x,y) + i\,v(x,y)$ be analytic in the upper half-plane and let $u(x)$ and $v(x)$ be the limiting values of $u(x,y)$ and $v(x,y)$ as $y \longrightarrow 0^+$ respectively. Then

$$v(x) = -\overline{u}(x) + v^{\infty}, \tag{491a}$$

$$u(x) = \overline{v}(x) + u^{\infty}, \tag{491b}$$

where u^{∞} and v^{∞} are the values of u and v at infinity. The converse also holds: If $\alpha(x) \in \mathcal{H}_*^{\mu}$ then, to within an arbitrary (complex) constant, $\overline{\alpha} + i\,\alpha$ is the boundary value of a uniquely determined function analytic in the upper half plane. Further, this function is given by

$$F(z) = \frac{1}{\pi}\int dt\, \frac{\alpha(t)}{t - z} + F^{\infty}. \tag{492}$$

This result can also be applied to the limit $y \longrightarrow 0^-$ of a function analytic in the lower half-plane, in which case the boundary value is $\overline{\alpha} - i\,\alpha$ and the formula (492) still holds.

These results will be of considerable value in our analysis of integral transform. As an example of the power of this theorem, compare the boundary value of $F(z)$:

$$F(x+i0^{+}) = \overline{\alpha} + i\alpha, \tag{493}$$

with that of $F'(z) = iF$:

$$F'(x+i0^{+}) = \overline{\alpha'} + i\alpha' = -\alpha + i\overline{\alpha}. \tag{494}$$

Thus

$$-\alpha = \overline{\alpha'} = \overline{\overline{\alpha}}, \tag{495}$$

which is seen to be equivalent to (484). By a similar argument involving the product of analytic functions, one obtains the convolution theorem, (487).

Appendix B

The Dielectric Function

Starting with Ichimaru's definition[56] of the dielectric tensor ϵ_{ij}, we specialize to the case where the equilibrium is isotropic in $\boldsymbol{v}$ and decompose this tensor into its longitudinal and transverse parts, which we find can be expressed entirely in terms of $f^{(0)}_{\parallel}$. For notational convenience, we assume that $f^{(0)}$ is normalized (in velocity space) to unity and the equilibrium density is contained in ω_p.

We take as out staring point the definition:

$$\epsilon_{ij}(k,u) = \delta_{ij} - \frac{\omega_p^2}{k^2}\frac{1}{u^2}\lim_{\mu\to 0+}\int d^3\boldsymbol{v}\,\frac{v_i}{\widehat{\boldsymbol{k}}\cdot\boldsymbol{v} - u - i\mu}\left[u - \widehat{\boldsymbol{k}}\cdot\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_j} + v_j\,\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}_v f^{(0)}\right]. \tag{496}$$

Since $f^{(0)}$ is isotropic in $\boldsymbol{v}$,

$$\boldsymbol{\nabla}_v f^{(0)} = \widehat{\boldsymbol{v}} f^{(0)\prime} = \frac{1}{v_{\parallel}}\frac{\partial f^{(0)}}{\partial v_{\parallel}}. \tag{497}$$

So

$$\begin{aligned} u - \widehat{\boldsymbol{k}}\cdot\boldsymbol{v}\,\frac{\partial f^{(0)}}{\partial v_j} + v_j\,\widehat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}_v f^{(0)} &= \left(\frac{u - v_{\parallel}}{v_{\parallel}}v_j + v_j\right)\frac{\partial f^{(0)}}{\partial v_{\parallel}} \\ &= v_j\,\frac{u}{v_{\parallel}}\frac{\partial f^{(0)}}{\partial v_{\parallel}}. \end{aligned} \tag{498}$$

Thus the dielectric tensor can be written as

$$\begin{aligned} \epsilon_{ij} &= \delta_{ij} - \frac{\omega_p^2}{k^2}\frac{1}{u}\lim_{\mu\to 0+}\int d^3\boldsymbol{v}\,\frac{v_i v_j}{v_{\parallel} - u - i\mu}\frac{1}{v_{\parallel}}\frac{\partial f^{(0)}}{\partial v_{\parallel}} \\ &= \delta_{ij} - \frac{\omega_p^2}{k^2}\frac{1}{u}\lim_{\mu\to 0+}\int dv_{\parallel}\,\frac{1}{v_{\parallel}(v_{\parallel} - u - i\mu)}\int d^2\boldsymbol{v}_{\perp}\,v_i v_j\,\frac{\partial f^{(0)}}{\partial v_{\parallel}}. \end{aligned} \tag{499}$$

Decomposing $\boldsymbol{v}$ into longitudinal and transverse components yields

$$\int d^2\boldsymbol{v}_\perp\, v_i v_j \frac{\partial f^{(0)}}{\partial v_\parallel} = \int d^2\boldsymbol{v}_\perp\, \frac{f^{(0)}}{v_\parallel}\left\{k_i k_j \frac{v_\parallel^2}{k^2} + \frac{v_\parallel}{k}\left[k_i v_{\perp j} + k_j v_{\perp i}\right] + v_{\perp i} v_{\perp j}\right\}. \quad (500)$$

Making use of (570), (573) and (580), the above becomes

$$\int d^2\boldsymbol{v}_\perp\, v_i v_j \frac{\partial f^{(0)}}{\partial v_\parallel} = \frac{k_i k_j}{k^2} v_\parallel^2 \frac{\partial f_\parallel^{(0)}}{\partial v_\parallel} - \Pi_{ij}^{\boldsymbol{k}} v_\parallel f_\parallel^{(0)}, \quad (501)$$

where

$$\Pi_{ij}^{\boldsymbol{k}} = \delta_{ij} - \widehat{k}_i \widehat{k}_j, \quad (502)$$

projects vectors into the plane perpendicular to $\widehat{\boldsymbol{k}}$. Using (502) in the expression for ϵ_{ij}, we obtain

$$\begin{aligned}\epsilon_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}\frac{\omega_p^2}{k^2}\frac{1}{u}\lim_{\mu\to 0^+}\int dv_\parallel\, \frac{v_\parallel}{v_\parallel - u - i\mu}\frac{\partial f_\parallel^{(0)}}{\partial v_\parallel} \\ -\Pi_{ij}^{\boldsymbol{k}}\frac{\omega_p^2}{k^2}\frac{1}{u}\lim_{\mu\to 0^+}\int dv_\parallel\, \frac{1}{v_\parallel - u - i\mu} f_\parallel^{(0)}. \quad (503)\end{aligned}$$

We define ϵ_L and ϵ_T by the decomposition

$$\epsilon_{ij} = \frac{k_i k_j}{k^2}\epsilon_L + \Pi_{ij}^{\boldsymbol{k}}\epsilon_T. \quad (504)$$

Comparing our expressions for ϵ_{ij}, we see that

$$\begin{aligned}\epsilon_L &= 1 - \frac{\omega_p^2}{k^2}\frac{1}{u}\mathrm{P}\int dv_\parallel\, \frac{v_\parallel}{v_\parallel - u} f_\parallel^{(0)\prime} - i\pi\frac{\omega_p^2}{k^2} f_\parallel^{(0)\prime} \\ &= 1 - \frac{\omega_p^2}{k^2}\mathrm{P}\int dv_\parallel\, \frac{1}{v_\parallel - u} f_\parallel^{(0)\prime} - i\pi\frac{\omega_p^2}{k^2} f_\parallel^{(0)\prime} \\ &\equiv \epsilon_L^R + i\epsilon_L^I \quad (505)\end{aligned}$$

and

$$\epsilon_T = 1 + \frac{\omega_p^2}{k^2}\frac{1}{u}\mathrm{P}\!\int dv_\parallel \,\frac{1}{v_\parallel - u} f_\parallel^{(0)} + i\pi \frac{\omega_p^2}{k^2}\frac{1}{u} f_\parallel^{(0)}$$
$$\equiv \epsilon_T^R + i\epsilon_T^I, \tag{506}$$

where P denotes the Cauchy principal value.

Examining (505), it is immediately apparent that $\epsilon_L^R = 1 + \overline{\epsilon_L^I}$. Since $f_\parallel^{(0)}$ is an even function of u, $\overline{f_\parallel^{(0)}/u} = \overline{f_\parallel^{(0)}}/u$. Thus from (506) we see that $\epsilon_T^R = 1 + \overline{\epsilon_T^I}$. This is, of course, exactly what we expect given that ϵ_{ij} is defined as the limiting value of a function analytic in the upper half-plane. This relationship between the real and imaginary parts of the dielectric function is extremely important in the solution of the linearized Maxwell-Vlasov equations.

Appendix C

Review of the Van Kampen Solution

Here we present the essence of the solution of the moment equations due to van Kampen and Felderhof.[57] Our notation differs from theirs to allow easy comparison with our integral transform solutions.

1. Solution of the Moment Equations

The moment equations, derived in Chapter 2, are given by

$$\dot{f}^{(1)}_{\parallel k} + ikv_{\parallel} f^{(1)}_{\parallel k} + \frac{e}{m} E^{(1)}_{\parallel k} f^{(0)\prime}_{\parallel} = 0\,, \tag{57a}$$

$$-\dot{E}^{(1)}_{\parallel k} = 4\pi e \int dv_{\parallel}\, v_{\parallel} f^{(1)}_{\parallel k}\,, \tag{57b}$$

$$ik\, E^{(1)}_{\parallel k} = 4\pi e \int dv_{\parallel}\, f^{(1)}_{\parallel k}\,, \tag{59}$$

and

$$\dot{\boldsymbol{f}}^{(1)}_{\perp k} + ikv_{\parallel} \boldsymbol{f}^{(1)}_{\perp k} - \frac{e}{m} \boldsymbol{E}^{(1)}_{\perp k} f^{(0)}_{\parallel} = 0\,, \tag{58a}$$

$$-\dot{\boldsymbol{E}}^{(1)}_{\perp k} + ck^2 \boldsymbol{A}^{(1)}_{\perp k} = 4\pi e \int dv_{\parallel}\, \boldsymbol{f}^{(1)}_{\perp k}\,, \tag{58b}$$

$$\dot{\boldsymbol{A}}^{(1)}_{\perp k} + c\boldsymbol{E}^{(1)}_{\perp k} = 0\,. \tag{58c}$$

Following van Kampen and Felderhof, we seek a family of solutions parameterized by k and u of the form

$$f^{(1)}_{\parallel k}(v_{\parallel}, t) = \int du\, h_{\parallel k}(u, v_{\parallel})\, e^{-ikut}, \tag{507a}$$

$$\boldsymbol{f}^{(1)}_{\perp k}(v_{\parallel},t) = \int du\, \boldsymbol{h}_{\perp k}(u,v_{\parallel})\, e^{-ikut}. \tag{507b}$$

Given the form of (507), it is clear that we expect u to take on a continuum of values which we can assume to range from $-\infty$ to ∞.

In what follows, it is convenient to retain $E^{(1)}_{\parallel k}$ as a dynamical variable, that is, to retain Poisson's equation as a constraint. Let $\mathcal{E}_{\parallel k}$, $\boldsymbol{\mathcal{E}}_{\perp k}$ and $\boldsymbol{\mathcal{A}}_{\perp k}$ be the temporal Fourier transforms of $E^{(1)}_{\parallel k}$, $\boldsymbol{E}^{(1)}_{\perp k}$ and $\boldsymbol{A}^{(1)}_{\perp k}$ respectively:

$$E^{(1)}_{\parallel k}(t) = \int du\, \mathcal{E}_{\parallel k}(u)\, e^{-ikut}, \tag{508a}$$

$$\boldsymbol{E}^{(1)}_{\perp k}(t) = \int du\, \boldsymbol{\mathcal{E}}_{\perp k}(u)\, e^{-ikut}, \tag{508b}$$

$$\boldsymbol{A}^{(1)}_{\perp k}(t) = \int du\, \boldsymbol{\mathcal{A}}_{\perp k}(u)\, e^{-ikut}, \tag{508c}$$

where $\mathcal{E}_{\parallel k}(u)$ *etc.* correspond to the Fourier mode of frequency $\omega = ku$.

Substituting (507) and (508) into (57) and (58) gives

$$\int du\, e^{-ikut} \left(ik(v_{\parallel} - u)h_{\parallel k} + \frac{e}{m}\mathcal{E}_{\parallel k} f^{(0)\prime}_{\parallel} \right) = 0\,, \tag{509a}$$

$$\int du\, e^{-ikut} \left(ik\,\mathcal{E}_{\parallel k} - 4\pi e \int dv_{\parallel}\, h_{\parallel k} \right) = 0\,, \tag{509b}$$

and

$$\int du\, e^{-ikut} \left\{ ik(v_{\parallel} - u)\boldsymbol{h}_{\perp k} - \frac{e}{m}\boldsymbol{\mathcal{E}}_{\perp k} f^{(0)}_{\parallel} \right\} = 0\,, \tag{510a}$$

$$\int du\, e^{-ikut} \left\{ iku\,\boldsymbol{\mathcal{E}}_{\perp k} + ck^2 \boldsymbol{\mathcal{A}}_{\perp k} - 4\pi e \int dv_{\parallel}\, \boldsymbol{h}_{\perp k} \right\} = 0\,, \tag{510b}$$

$$\int du\, e^{-ikut} \left\{ iku\,\boldsymbol{\mathcal{A}}_{\perp k} - c\boldsymbol{\mathcal{E}}_{\perp k} \right\} = 0\,, \tag{510c}$$

respectively. Since the Fourier modes are linearly independent the above equations are equivalent to

$$ik(v_{\parallel} - u)h_{\parallel k} + \frac{e}{m}\mathcal{E}_{\parallel k} f_{\parallel}^{(0)\prime} = 0, \tag{511a}$$

$$ik\mathcal{E}_{\parallel k} - 4\pi e \int dv_{\parallel}\, h_{\parallel k} = 0, \tag{511b}$$

and

$$ik(v_{\parallel} - u)\boldsymbol{h}_{\perp k} - \frac{e}{m}\boldsymbol{\mathcal{E}}_{\perp k} f_{\parallel}^{(0)} = 0, \tag{512a}$$

$$iku\boldsymbol{\mathcal{E}}_{\perp k} + ck^2\boldsymbol{\mathcal{A}}_{\perp k} - 4\pi e \int dv_{\parallel}\, \boldsymbol{h}_{\perp k} = 0, \tag{512b}$$

$$iku\boldsymbol{\mathcal{A}}_{\perp k} - c\boldsymbol{\mathcal{E}}_{\perp k} = 0. \tag{512c}$$

I. The Longitudinal Solution

We first consider the longitudinal equations (511). The general solution of the kinetic equation, (511a), is

$$h_{\parallel k}(u, v_{\parallel}) = \frac{ie}{mk}\mathcal{E}_{\parallel k} f_{\parallel}^{(0)\prime}\,\mathrm{P}\frac{1}{v_{\parallel} - u} + C_{\parallel}(k,u)\,\delta(u - v_{\parallel}), \tag{513}$$

where $C_{\parallel}$ is to be determined by substituting (513) into (511b). Doing so gives

$$ik\,\mathcal{E}_{\parallel k}(u) - \frac{4\pi i e^2}{mk}\mathcal{E}_{\parallel k}(u)\,\mathrm{P}\!\int dv_{\parallel}\, \frac{f_{\parallel}^{(0)\prime}}{v_{\parallel} - u} - 4\pi e C_{\parallel} = 0. \tag{514}$$

Solving the above for $C_{\parallel}$, we find

$$\begin{aligned} C_{\parallel}(k,u) &= \frac{ik}{4\pi e}\mathcal{E}_{\parallel k}\left[1 - \frac{4\pi e^2}{mk^2}\,\mathrm{P}\!\int dv_{\parallel}\, \frac{f_{\parallel}^{(0)\prime}}{v_{\parallel} - u}\right] \\ &= \frac{ik}{4\pi e}\mathcal{E}_{\parallel k}\epsilon_L^R(k,u). \end{aligned} \tag{515}$$

Combining the expression for $C_\parallel$ with (513) gives the complete expression for $h_{\parallel k}$:

$$h_{\parallel k}(u,v_\parallel) = \frac{ik}{4\pi e}\mathcal{E}_{\parallel k}\left(\frac{1}{\pi}\epsilon_L^I(k,v_\parallel)\,\mathrm{P}\frac{1}{u-v_\parallel} + \epsilon_L^R(k,v_\parallel)\,\delta(u-v_\parallel)\right)$$

$$\equiv \frac{ik}{4\pi e}\mathcal{E}_{\parallel k}\,\mathcal{G}_L(u,v_\parallel). \tag{516}$$

So we find

$$f^{(1)}_{\parallel k}(v_\parallel,t) = \frac{ik}{4\pi e}\int du\,\mathcal{E}_{\parallel k}(u)\,\mathcal{G}_L(u,v_\parallel)\,e^{-ikut}. \tag{517}$$

Notice that the van Kampen mode, $\mathcal{G}_L$, is the same as the kernel of our transform that was used to diagonalize the longitudinal motion however, in the later no time dependence is assumed.

II. The Transverse Solution

We now solve to transverse equations using the same method. Here we begin by solving (512c) for $\boldsymbol{\mathcal{A}}_{\perp k}$:

$$\boldsymbol{\mathcal{A}}_{\perp k} = -\frac{ic}{ku}\boldsymbol{\mathcal{E}}_{\perp k}, \tag{518}$$

and then eliminating $\boldsymbol{\mathcal{A}}_{\perp k}$ in the remaining field equation giving

$$ik\left(u - \frac{c^2}{u}\right)\boldsymbol{\mathcal{E}}_{\perp k} = 4\pi e\int dv_\parallel\,\boldsymbol{h}_{\perp k}. \tag{519}$$

The most general solution of the transverse kinetic equation, (512a), is

$$\boldsymbol{h}_{\perp k}(u,v_\parallel) = \frac{ie}{mk}\boldsymbol{\mathcal{E}}_{\perp k}\,f^{(0)}_\parallel\mathrm{P}\frac{1}{u-v_\parallel} + \boldsymbol{C}_\perp(k,u)\,\delta(u-v_\parallel). \tag{520}$$

We determine $\boldsymbol{C}_\perp$ by substituting (520) into (519) giving

$$ik\left(u - \frac{c^2}{u}\right)\boldsymbol{\mathcal{E}}_{\perp k} = \frac{4\pi i e^2}{mk}\boldsymbol{\mathcal{E}}_{\perp k}\,\mathrm{P}\int dv_\parallel\,\frac{f^{(0)}_\parallel}{u-v_\parallel} + 4\pi e\,\boldsymbol{C}_\perp. \tag{521}$$

Solving for $C_\perp$, we obtain

$$\begin{aligned} C_\perp &= \frac{ik}{4\pi e}\mathcal{E}_{\perp k}\left[u - \frac{c^2}{u} + \frac{4\pi e^2}{mk^2}\mathrm{P}\int dv_\parallel \frac{f_\parallel^{(0)}}{v_\parallel - u}\right] \\ &= \frac{ik}{4\pi e}\mathcal{E}_{\perp k}\left[u\,\epsilon_T^R - \frac{c^2}{u}\right]. \end{aligned} \tag{522}$$

Substituting the above into (520) and using the definition of the transverse dielectric function, (506), we find

$$\begin{aligned} h_{\perp k}(u, v_\parallel) &= \frac{ik}{4\pi e}\mathcal{E}_{\perp k}\left(v_\parallel\,\epsilon_T^I(k, v_\parallel)\frac{1}{\pi}\mathrm{P}\frac{1}{u - v_\parallel} + \left[v_\parallel\,\epsilon_T^R(k, v_\parallel) - \frac{c^2}{v_\parallel}\right]\delta(u - v_\parallel)\right) \\ &\equiv \frac{ik}{4\pi e}\mathcal{E}_{\perp k}\,\mathcal{G}_T(u, v_\parallel). \end{aligned} \tag{523}$$

Thus we have

$$f_{\perp k}^{(1)}(v_\parallel, t) = \frac{ik}{4\pi e}\int du\,\mathcal{E}_{\perp k}(u)\,\mathcal{G}_T(u, v_\parallel). \tag{524}$$

Again notice that the van Kampen mode, $\mathcal{G}_T$, is the same as the kernel of the transform that was used to diagonalize the transverse motion.

2. Completeness of the Solutions

The last step it to show that the general solution is given by a sum over the van Kampen modes, that is to show that the van Kampen modes are a complete basis for expanding a general solution. With $\mathcal{G}_L$ and $\mathcal{G}_T$ given by

$$\mathcal{G}_L(u, v_\parallel) = \epsilon_L^I(k, v_\parallel)\frac{1}{\pi}\mathrm{P}\frac{1}{u - v_\parallel} + \epsilon_L^R(k, v_\parallel)\,\delta(u - v_\parallel), \tag{525a}$$

$$\mathcal{G}_T(u, v_\parallel) = v_\parallel\,\epsilon_T^I(k, v_\parallel)\frac{1}{\pi}\mathrm{P}\frac{1}{u - v_\parallel} + \left[v_\parallel\,\epsilon_T^R(k, v_\parallel) - \frac{c^2}{v_\parallel}\right]\delta(u - v_\parallel), \tag{525b}$$

we interpret $\mathcal{E}_{\parallel k}(u)$ and $\boldsymbol{\mathcal{E}}_{\perp k}(u)$ as the coefficients of the basis functions, $\mathcal{G}_L$ and $\mathcal{G}_T$ in the expansion. Thus we have to show that the general solution, $f^{(1)}_{\parallel k}$ and $\boldsymbol{f}^{(1)}_{\perp k}$ can be written as

$$f^{(1)}_{\parallel k}(x, v_\parallel, t) = \frac{1}{2}\sum_k \int du\, \frac{ik}{4\pi e}\, e^{ikx-ikut}\, \mathcal{G}_L(u, v_\parallel)\, \mathcal{E}_{\parallel k}(u), \tag{526a}$$

$$\boldsymbol{f}^{(1)}_{\perp k}(x, v_\parallel, t) = \frac{1}{2}\sum_k \int du\, \frac{ik}{4\pi e}\, e^{ikx-ikut}\, \mathcal{G}_T(u, v_\parallel)\, \boldsymbol{\mathcal{E}}_{\perp k}(u). \tag{526b}$$

In both cases, these completeness requirements amount to the ability to expand the appropriate initial condition. Thus we need to show that

$$f_{Lk}(v_\parallel) = \frac{ik}{4\pi e}\int du\, \mathcal{G}_L(u, v_\parallel)\, \mathcal{E}_{\parallel k}(u), \tag{527a}$$

$$\boldsymbol{f}_{Tk}(v_\parallel) = \frac{ik}{4\pi e}\int du\, \mathcal{G}_T(u, v_\parallel)\, \boldsymbol{\mathcal{E}}_{\perp k}(u), \tag{527b}$$

where $f_{Lk}(v_\parallel)$ and $\boldsymbol{f}_{Tk}(v_\parallel)$ are the values of $f^{(1)}_\parallel$ and $\boldsymbol{f}^{(1)}_\perp$ at $t = 0$ respectively, can be solved for $\mathcal{E}_{\parallel k}$ and $\boldsymbol{\mathcal{E}}_{\perp k}$.

I. Sokhotski Formulæ

In solving these equations, we make use of the Sokhotski[58] formulæ. Let $\phi \in \mathcal{H}^\mu_*$ for some μ. Then the functions Φ^+ and Φ^- defined by

$$\Phi^+(x) = \frac{1}{2}\phi(x) + \frac{1}{2\pi i}\,\mathrm{P}\!\int dx'\, \frac{\phi(x')}{x'-x}, \tag{528a}$$

$$\Phi^-(x) = -\frac{1}{2}\phi(x) + \frac{1}{2\pi i}\,\mathrm{P}\!\int dx'\, \frac{\phi(x')}{x'-x}, \tag{528b}$$

are limiting values of functions analytic in the upper (Φ^+) or lower (Φ^-) half plane as the real axis is approached from above or below respectively. The functions Φ^+ and Φ^- also belong to the class $\mathcal{H}^\mu_*$.

Two immediate consequences of (528) are

$$\Phi^{+}(x) - \Phi^{-}(x) = \phi(x), \tag{529a}$$

$$\Phi^{+}(x) + \Phi^{-}(x) = \frac{1}{\pi i}\,\mathrm{P}\!\int dx\,'\,\frac{\phi(x')}{x'-x} = -i\,\overline{\phi}. \tag{529b}$$

The splitting of an arbitrary Hölder function into the into "+" and "−" parts as defined by (529) is unique.[52]

II. The Longitudinal Case

We start with the longitudinal case. To solve (524) let

$$\sigma_{\parallel}(u) = \frac{ik}{4\pi e}\,\mathcal{E}_{\parallel k}(u), \tag{530}$$

where the k dependence in $\sigma_{\parallel}$ will be temporarily suppressed. Then (524) now reads

$$\begin{aligned} f_{Lk}(v_{\parallel}) &= \int du\, \mathcal{G}_{L}(u, v_{\parallel})\,\sigma_{\parallel}(u) \\ &= \epsilon^{R}_{L}(v_{\parallel})\,\sigma_{\parallel}(v_{\parallel}) + \epsilon^{I}_{L}(v_{\parallel})\,\frac{1}{\pi}\,\mathrm{P}\!\int du\,\frac{\sigma_{\parallel}(u)}{u - v_{\parallel}}. \end{aligned} \tag{531}$$

Using the notation introduced in (529a) and (529b) an assuming $f_{L} \in \mathcal{H}^{\alpha}_{\mu}$, this becomes

$$\begin{aligned} F^{+}_{L} - F^{-}_{L} &= \epsilon^{R}_{L}\left(\Sigma^{+}_{\parallel} - \Sigma^{-}_{\parallel}\right) + i\epsilon^{I}_{L}\left(\Sigma^{+}_{\parallel} + \Sigma^{-}_{\parallel}\right) \\ &= \epsilon_{L}\,\Sigma^{+}_{\parallel} - \epsilon^{*}_{L}\,\Sigma^{-}_{\parallel} \\ &= \epsilon^{+}_{L}\,\Sigma^{+}_{\parallel} - \epsilon^{-}_{L}\,\Sigma^{-}_{\parallel}, \end{aligned} \tag{532}$$

where the last line follows from the definition of $\epsilon_{L}(v)$ as the limiting value as v approaches the real axis of a function analytic in the upper half-plane. Since the splitting is unique this can be separated into two independent equations:

$$F_L^+ = \epsilon_L \Sigma_\parallel^+, \tag{533a}$$

$$F_L^- = -\epsilon_L^* \Sigma_\parallel^-. \tag{533b}$$

Provided ϵ_L has no root in the upper half-plane (including the real axis) then the same is true of ϵ_L^* in the lower half-plane and (533) can be readily solved to give

$$\Sigma_\parallel^+ = \frac{F_L^+}{\epsilon_L}, \tag{534a}$$

$$\Sigma_\parallel^- = \frac{F_L^-}{\epsilon_L^*}, \tag{534b}$$

and

$$\begin{aligned}
\sigma_\parallel(u) &= \frac{F_L^+}{\epsilon_L} - \frac{F_L^-}{\epsilon_L^*} \\
&= \frac{1}{|\epsilon_L|^2}\left[\epsilon_L^* F_L^+ + \epsilon_L F_L^-\right] \\
&= \frac{1}{|\epsilon_L|^2}\left[\epsilon_L^R(F_L^+ - F_L^-) + i\epsilon_L^I(F_L^- + F_L^+)\right] \\
&= \frac{\epsilon_L^R(u)}{|\epsilon_L(u)|^2} f_{Lk}(u) + \frac{\epsilon_L^I(u)}{|\epsilon_L(u)|^2}\frac{1}{\pi}\,\mathrm{P}\!\int du'\,\frac{f_{Lk}(u')}{u-u'} \\
&\equiv \int du'\,\widetilde{\mathcal{G}}_L(u',u)\, f_{Lk}(u'),
\end{aligned} \tag{535}$$

giving

$$\mathcal{E}_{\parallel k}(u) = \frac{4\pi e}{ik}\int du'\,\widetilde{\mathcal{G}}_L(k,u',u)\, f_{Lk}(u'). \tag{536}$$

III. The Transverse Case

The transverse case proceeds along virtually identical lines. Put

$$\sigma_{\perp}(u) = \frac{ik}{4\pi e}\, u\, \mathcal{E}_{\perp k}(u)\,, \tag{537}$$

where again the k dependence has again been suppressed. From (519), we see that $\boldsymbol{\sigma}_{\perp}(u)$ and $\boldsymbol{\sigma}_{\perp}(u)/u^2$ are well-defined for all u and consequently so are $\boldsymbol{\Sigma}_{\perp}^{\pm}$ and $\boldsymbol{\Sigma}_{\perp}^{\pm}/u^2$.

Substituting this into (525) gives

$$\boldsymbol{f}_{Tk}(v_{\parallel}) = \left[\epsilon_T^R(v_{\parallel}) - \frac{c^2}{v_{\parallel}}\right] \boldsymbol{\sigma}_{\perp}(v_{\parallel}) + \epsilon_T^I(v_{\parallel})\,\frac{1}{\pi}\,\mathrm{P}\!\int du\, \frac{\boldsymbol{\sigma}_{\perp}(u)}{u - v_{\parallel}}\,. \tag{538}$$

Using the Sokhotski formulæ, we write the above as

$$\boldsymbol{F}_T^{+} - \boldsymbol{F}_T^{-} = \left[\epsilon_T - \frac{c^2}{u^2}\right] \boldsymbol{\Sigma}_{\perp}^{+} - \left[\epsilon_T^{*} - \frac{c^2}{u^2}\right] \boldsymbol{\Sigma}_{\perp}^{-}\,. \tag{539}$$

Once again appealing to the uniqueness of the splitting we see that

$$\boldsymbol{F}_T^{+} = \left[\epsilon_T - \frac{c^2}{u^2}\right] \boldsymbol{\Sigma}_{\perp}^{+}\,, \tag{540a}$$

$$\boldsymbol{F}_T^{-} = -\left[\epsilon_T^{*} - \frac{c^2}{u^2}\right] \boldsymbol{\Sigma}_{\perp}^{-}\,. \tag{540b}$$

Provided that $\epsilon_T - c^2/u^2$ has no roots in the upper half-plane or on the real axis, the same will be true of $\epsilon_T^{*} - c^2/u^2$; and we can solve (540) giving

$$\begin{aligned} \boldsymbol{\sigma}_{\perp}(u) &= \boldsymbol{\Sigma}_{\perp}^{+} - \boldsymbol{\Sigma}_{\perp}^{-} \\ &= \frac{\boldsymbol{F}_T^{+}}{\epsilon_T - c^2/u} - \frac{\boldsymbol{F}_T^{-}}{\epsilon_T^{*} - c^2/u} \\ &= \int du'\, \widetilde{\mathcal{G}}_T(u', u)\, \boldsymbol{f}_{Tk}(u')\,, \end{aligned} \tag{541}$$

which in turn gives

$$\boldsymbol{\mathcal{E}}_{\perp k}(u) = \frac{4\pi e}{ik} \int du' \, \widetilde{\mathcal{G}}_T(u', u) \, \boldsymbol{f}_{Tk}(u') . \tag{542}$$

We have shown that the van Kampen mode solution for both the longitudinal and transverse equations are complete in the sense that they can be used to represent an arbitrary (Hölder) initial condition and hence the general solution can be constructed by summing these modes.

Appendix D

Functional Derivatives and Change of Variables

The goal here is to examine the behaviour of the functional derivative under a change of variables. This is of central importance in determining how a Poisson bracket will transform when a new set of field variables is introduced. Let ϕ be some function of x and let F be a functional of ϕ. Consider a general transformation

$$\psi(x) = \mathcal{F}[\phi](x). \tag{543}$$

That is, we consider the case where the new field ψ is a functional of the original field.[10] We do *not* require that this transformation be invertible. The reason for this is independent of the invertibility of $\mathcal{F}$, for every variation in F due to a variation in ψ, there is a corresponding (not necessarily unique) variation in ϕ that gives rise to the same variation in F. For simplicity, we assume that the domain of ψ is the same as ϕ. Through (543) we can view F as a functional of ψ, which leads to two equivalent expressions for the variation in F:

$$\delta F = \int dx\, \frac{\delta F}{\delta \phi}\, \delta\phi \tag{544a}$$

$$= \int dx\, \frac{\delta F}{\delta \psi}\, \delta\psi = \int dx\, \frac{\delta F}{\delta \psi}\, \frac{\delta \psi}{\delta \phi}\, \delta\phi. \tag{544b}$$

Taking the functional derivative of the transformation yields:

$$\frac{\delta \psi}{\delta \phi} = \mathcal{F}[\cdot], \tag{545}$$

where we now interpret $\mathfrak{F}[\cdot]$ as an operator. From the expression for δF we find

$$\int dx\, \frac{\delta F}{\delta \phi}\, \delta\phi = \int dx\, \frac{\delta F}{\delta \psi}\, \mathfrak{F}[\delta\phi] = \int dx\, \mathfrak{F}^{\dagger}\left[\frac{\delta F}{\delta \psi}\right] \delta\phi, \tag{546}$$

where $\mathfrak{F}^{\dagger}$ is the adjoint of $\mathfrak{F}$ and is defined by

$$\int dx\, \vartheta\, \mathfrak{F}\,[\varphi] = \int dx\, \varphi\, \mathfrak{F}^{\dagger}\,[\vartheta]. \tag{547}$$

Since $\delta\phi$ is arbitrary, we conclude

$$\frac{\delta F}{\delta \phi} = \mathfrak{F}^{\dagger}\left[\frac{\delta F}{\delta \psi}\right]. \tag{548}$$

In the case where the inverse transformation, $\widetilde{\mathfrak{F}}$, exists we also have

$$\frac{\delta F}{\delta \psi} = \widetilde{\mathfrak{F}}^{\dagger}\left[\frac{\delta F}{\delta \phi}\right]. \tag{549}$$

This has an immediate generalization. Let $\{\phi_a\}$, $a = 1 \ldots N$ be a set of functions of x and F be any functional of $\{\phi_a\}$. Consider the transformation

$$\psi_\mu(x) = \mathfrak{F}_\mu\left[\phi_a\right](x), \qquad \mu = 1 \ldots N, \tag{550}$$

where M and N are not necessarily equal. Following the same reasoning as above, one can show

$$\frac{\delta F}{\delta \psi_\mu} = \sum_{a=1}^{N} \left(\frac{\delta \psi_\mu}{\delta \phi_a}\right)^{\dagger}\left[\frac{\delta F}{\delta \phi_a}\right]. \tag{551}$$

We now use this to establish two results that will be of considerable use.

1. Directional Functional Derivatives

Let $\boldsymbol{X}$ be a vector valued function of $\boldsymbol{r}$, F be a functional of $\boldsymbol{X}$ and $\widehat{\boldsymbol{k}}$ be a unit vector that *may* be depend on $\boldsymbol{r}$. We can decompose $\boldsymbol{X}$ into its projections parallel and perpendicular to $\boldsymbol{k}$, *viz*

$$\boldsymbol{X} = \widehat{\boldsymbol{k}} X_{\parallel} + \boldsymbol{X}_{\perp}, \tag{552}$$

where

$$X_{\parallel} = \widehat{\boldsymbol{k}} \cdot \boldsymbol{X}, \tag{553a}$$

$$(\boldsymbol{X}_{\perp})_i = \Pi^{\boldsymbol{k}}_{ij} X_j, \tag{553b}$$

and

$$\Pi^{\boldsymbol{k}}_{ij} = \delta_{ij} - \widehat{k}_i \widehat{k}_j. \tag{554}$$

This decomposition can be viewed as a change of variables from $\boldsymbol{X}$ to $X_{\parallel}$ and $\boldsymbol{X}_{\perp}$. In this case it is easy to see that the operators

$$\frac{\delta X_{\parallel}}{\delta X_j} = \widehat{k}_j, \tag{555a}$$

$$\frac{\delta \boldsymbol{X}_{\perp i}}{\delta X_j} = \Pi^{\boldsymbol{k}}_{ij}, \tag{555b}$$

are both are self-adjoint. Using (551), we find

$$\frac{\delta F}{\delta X_i} = \widehat{k}_i \frac{\delta F}{\delta X_{\parallel}} + \Pi^{\boldsymbol{k}}_{ij} \frac{\delta F}{\delta \boldsymbol{X}_{\perp j}}, \tag{556}$$

from which, we deduce

$$\frac{\delta F}{\delta X_{\parallel}} = \left(\frac{\delta F}{\delta \boldsymbol{X}}\right)_{\parallel} \tag{557}$$

and

$$\frac{\delta F}{\delta \boldsymbol{X}_{\perp}} = \left(\frac{\delta F}{\delta \boldsymbol{X}}\right)_{\perp}. \tag{558}$$

2. Fourier Transforms and Functional Derivatives

Taking a Fourier transform can be thought of as a change of coordinates.[1] Let ψ be an L^2 function of $\boldsymbol{k}\cdot\boldsymbol{r}$. We define the Fourier amplitudes, ψ_k, by

$$\psi_k = \frac{2}{V}\int dx\, e^{-ikr_{\parallel}}\psi(\boldsymbol{r}) = \mathcal{F}_k[\psi], \tag{559}$$

where $kr_{\parallel} = \boldsymbol{k}\cdot\boldsymbol{r}$. The inverse transformation is given by

$$\psi(\boldsymbol{r}) = \frac{1}{2}\sum_{k=-\infty}^{\infty}\psi_k e^{ikr_{\parallel}}. \tag{560}$$

If ψ is also a function of $\boldsymbol{v}$ or t, then ψ_k will depend on these variables as well. Using the chain rule, (548), we have

$$\frac{\delta F}{\delta\psi} = \sum_{k=-\infty}^{\infty}\mathcal{F}_k^{\dagger}\left[\frac{\delta F}{\delta\psi_{k'}}\right] = \sum_{k=-\infty}^{\infty} e^{ikr_{\parallel}}\frac{\delta F}{\delta\psi_k}. \tag{561}$$

Applying the inverse transform to (561) we obtain

$$\frac{\delta F}{\delta\psi_k} = \frac{2}{V}\int dx\, e^{-ikr_{\parallel}}\frac{\delta F}{\delta\psi}. \tag{562}$$

Appendix E

Velocity Moments

In taking velocity moments of the equations of motion and in the brackets, there are certain integrals that must be evaluated. There are three basic integrals that arise:

$$\int d^2\boldsymbol{v}_{\perp} \frac{\partial f^{(0)}}{\partial v_{\parallel}}, \tag{563}$$

$$\int d^2\boldsymbol{v}_{\perp} \boldsymbol{v}_{\perp} \frac{\partial f^{(0)}}{\partial v_{\parallel}}, \tag{564}$$

and

$$\int d^2\boldsymbol{v}_{\perp} v_{\perp i} v_{\perp j} \frac{\partial f^{(0)}}{\partial v_{\parallel}}. \tag{565}$$

Recall that $f^{(0)}$ is isotropic, that is

$$f^{(0)}(v) = f^{(0)}\left(\sqrt{v_{\parallel}^2 + \left|\boldsymbol{v}_{\perp}\right|^2}\right). \tag{566}$$

Thus

$$\frac{\partial f^{(0)}}{\partial v_{\parallel}} = f^{(0)\prime} \frac{\partial v}{\partial v_{\parallel}} = \frac{v_{\parallel}}{v} f^{(0)\prime} \tag{567}$$

and

$$\frac{\partial f^{(0)}}{\partial v_{\perp}} = f^{(0)\prime} \frac{\partial v}{\partial v_{\perp}} = \frac{v_{\perp}}{v} f^{(0)\prime}. \tag{568}$$

Hence

$$\frac{1}{v} f^{(0)\prime} = \frac{1}{v_{\parallel}} \frac{\partial f^{(0)}}{\partial v_{\parallel}} = \frac{1}{v_{\perp}} \frac{\partial f^{(0)}}{\partial v_{\perp}}. \tag{569}$$

The first integral is easily evaluated:

$$\int d^2\boldsymbol{v}_{\perp} \frac{\partial f^{(0)}}{\partial v_{\parallel}} = \frac{\partial}{\partial v_{\parallel}} \int d^2\boldsymbol{v}_{\perp} f^{(0)} = f_{\parallel}^{(0)\prime}. \tag{570}$$

The second integral is also straightforward to compute:

$$\int d^2\boldsymbol{v}_\perp \boldsymbol{v}_\perp \frac{\partial f^{(0)}}{\partial v_\parallel} = \int_0^{2\pi} dv_\theta \int_0^\infty dv_\perp\, v_\perp \boldsymbol{v}_\perp \frac{v_\parallel}{v_\perp}\frac{\partial f^{(0)}}{\partial v_\perp}$$

$$= v_\parallel \int_0^{2\pi} dv_\theta\, \widehat{\boldsymbol{v}}_\perp \int_0^\infty dv_\perp\, v_\perp \frac{\partial f^{(0)}}{\partial v_\perp}, \tag{571}$$

where v_θ is the polar angle between $\boldsymbol{v}_\perp$ and the x-direction. Since

$$\int_0^{2\pi} dv_\theta\, \widehat{\boldsymbol{v}}_\perp = 0, \tag{572}$$

we obtain

$$\int d^2\boldsymbol{v}_\perp \boldsymbol{v}_\perp \frac{\partial f^{(0)}}{\partial v_\parallel} = 0. \tag{573}$$

Evaluating the third integral is somewhat more involved. We start by choosing the orientation of our coordinate system so that $\widehat{\boldsymbol{z}}$ is parallel to $\widehat{\boldsymbol{k}}$. Further, we can orient the remaining directions so that

$$v_{\perp 1} = v_\perp \cos v_\theta \quad \text{and} \quad v_{\perp 2} = v_\perp \sin v_\theta. \tag{574}$$

Then

$$v_{\perp i} v_{\perp j} = v_\perp^2 \begin{pmatrix} \cos^2 v_\theta & \sin v_\theta \cos v_\theta & 0 \\ \sin v_\theta \cos v_\theta & \sin^2 v_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \equiv v_\perp^2\, \Theta_{ij}(v_\theta). \tag{575}$$

Thus we have

$$\int d^2\boldsymbol{v}_\perp v_{\perp i} v_{\perp j} \frac{\partial f^{(0)}}{\partial v_\parallel} = \int dv_\theta \int dv_\perp\, v_{\perp i} v_{\perp j}\, v_\perp \frac{\partial f^{(0)}}{\partial v_\parallel}$$

$$= v_\parallel \int dv_\theta \int dv_\perp\, v_{\perp i} v_{\perp j} \frac{\partial f^{(0)}}{\partial v_\perp}$$

$$= v_\parallel \int dv_\theta\ \Theta_{ij} \int dv_\perp\, v_\perp^2 \frac{\partial f^{(0)}}{\partial v_\perp}$$

$$= -2v_{\parallel} \int dv_\theta \; \Theta_{ij} \int dv_{\perp} \, v_{\perp} f^{(0)}$$

$$= -2v_{\parallel} \int dv_\theta \; \Theta_{ij} \, \frac{1}{2\pi} \int d^2\boldsymbol{v}_{\perp} \, f^{(0)}$$

$$= -v_{\parallel} \frac{1}{\pi} f_{\parallel}^{(0)} \int dv_\theta \; \Theta_{ij} \, . \qquad (576)$$

The integral over v_θ can be easily computed. Doing so, one finds

$$\int dv_\theta \; \Theta_{ij} = \pi \, \delta_{\perp ij} \, , \qquad (577)$$

where $\delta_{\perp ij}$ is the perpendicular projection of δ_{ij}:

$$\delta_{\perp ij} = \Pi^{\boldsymbol{k}}_{in} \, \Pi^{\boldsymbol{k}}_{mj} \, \delta_{nm} = \Pi^{\boldsymbol{k}}_{ij} \, , \qquad (578)$$

where

$$\Pi^{\boldsymbol{k}}_{ij} = \delta_{ij} - \widehat{k}_i \, \widehat{k}_j \, . \qquad (579)$$

Thus we find

$$\int d^2\boldsymbol{v}_{\perp} \, v_{\perp i} \, v_{\perp j} \, \frac{\partial f^{(0)}}{\partial v_{\parallel}} = -\Pi^{\boldsymbol{k}}_{ij} \, v_{\parallel} \, f_{\parallel}^{(0)} . \qquad (580)$$

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in which $\boldsymbol{k} \cdot \boldsymbol{A}_k$ is not necessarily 0, there exists a gauge transformation,

$$\boldsymbol{A}'_k = \boldsymbol{A}_k + ik\Lambda_k \quad \text{and} \quad \varphi'_k = \varphi_k + \frac{1}{c}\frac{\partial \Lambda_k}{\partial t}, \tag{581}$$

such that $\boldsymbol{k} \cdot \boldsymbol{A}'_k = 0$. This is, of course, precisely what it meant by a valid gauge-fixing condition. Naturally the existence of this gauge transformation is subject to our (implicit) assumptions about the existence of spatial fourier transforms of the potentials and fields.

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[24] Deriving equations (174) and (175) are extremely tedious computations that were actually carried out symbolically using *Mathematica*™.

[25] The reader might, as this point, rightly question the choice of definitions of ζ and χ in the transverse case. See V. Gates *et al.*, Physica 15 **D**, 289 (1985) for an interesting discussion of notational difficulties.

[26] Suppose we had decided the physical argument for choosing $\gamma = c$. In this case the two terms in (240) would be

$$\dot{\boldsymbol{\xi}}_{2k} - \frac{ik}{c}\dot{\boldsymbol{A}}^{(1)}_{\perp k}, \tag{582a}$$

$$\boldsymbol{\xi}_{2k} - \frac{ikc}{\gamma^2}\boldsymbol{A}^{(1)}_{\perp k}. \tag{582b}$$

Thus in addition to taking $\boldsymbol{\xi}_{2k} = ik/c\boldsymbol{A}^{(1)}_{\perp k}$, we would have been forced to set $\gamma = c$. This goes along way towards vindicating our original choice for γ.

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over [0,3]. Thus the influence of this error on the evaluation of the performance of the quadrature rules is insignificant.

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[55] To see why (475) is required instead of the weaker condition

$$\lim_{x \to \infty} f(x) = 0, \tag{583}$$

consider a function $f(x) = 1/\log(x)$. Clearly $f \longrightarrow 0$ as $x \longrightarrow \infty$ but

$$\lim_{x \to \infty} x^{\mu} f(x) = \infty, \qquad \forall \mu > 0, \tag{584}$$

which tells us that (475) does not hold for any μ. Now consider

$$\int_X^\infty dx \frac{f(x)}{x} = \int_{\log X}^\infty du \frac{1}{u} = [\log\log(x)]_X^\infty = \infty . \tag{585}$$

Thus we see that (475) is at least necessary.

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