Physics of Plasmas

RESEARCH ARTICLE | JULY 10 2026

On the Poisson brackets of hybrid plasma models with kinetic ions and massless electrons

Yingzhe Li ; Philip J. Morrison ; Stefan Possanner ; Eric Sonnendrücker

Phys. Plasmas 33, 072103 (2026)
https://doi.org/10.1063/5.0332100

Physics of Plasmas ARTICLE pubs.aip.org/aip/pop

On the Poisson brackets of hybrid plasma models with kinetic ions and massless electrons

Cite as: Phys. Plasmas **33**, 072103 (2026); doi: 10.1063/5.0332100
Submitted: 2 March 2026 · Accepted: 23 June 2026 ·
Published Online: 10 July 2026

Yingzhe Li,[1,a)] Philip J. Morrison,[2] Stefan Possanner,[1] and Eric Sonnendrücker[1,3]

AFFILIATIONS

[1]Max Planck Institute for Plasma Physics, Boltzmannstrasse 2, 85748 Garching, Germany
[2]Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA
[3]Department of Mathematics, Technical University of Munich, Boltzmannstrasse 3, 85748 Garching, Germany

[a)]Author to whom correspondence should be addressed: yingzhe.li@ipp.mpg.de

ABSTRACT

We investigate the conditions under which the Jacobi identity holds for a class of recently introduced anti-symmetric brackets in hybrid plasma models with kinetic ions and massless electrons. In particular, we establish the precise conditions under which the brackets for the vector-potential-based formulations satisfy the Jacobi identity, and demonstrate that these conditions are fulfilled by all physically relevant functionals. Moreover, for the magnetic-field-based formulation, we show that the corresponding anti-symmetric bracket constitutes a Poisson bracket under Gauss's law for magnetism, and we provide a direct proof of the Jacobi identity. These results are further extended to models incorporating electron entropy as well as more general hybrid kinetic-fluid models.

I. INTRODUCTION

Many important physical models can be formulated within a Hamiltonian framework, including Maxwell's equations and the Schrödinger equation, whose dynamics can be derived from appropriate Hamiltonians and Poisson brackets.[1] In plasma physics, Hamiltonian formulations with corresponding Poisson brackets have been developed for a variety of models, including the Vlasov–Maxwell equations,[2,3] magnetohydrodynamics (MHD),[4] the two-fluid model,[5] and extended MHD.[6,7] More recently, Poisson brackets have been proposed for hybrid plasma models.[8]

The focus of this paper is the hybrid model with kinetic ions and massless electrons (HKM). It has become a cornerstone of modern plasma and space-weather simulations.[9] By treating electrons as a fluid while retaining kinetic ions, it provides an effective balance between computational efficiency and physical fidelity. Compared with fully kinetic approaches, the HKM model significantly reduces computational cost, enabling large-scale simulations over physically relevant spatial and temporal scales. At the same time, it captures ion kinetic phenomena that are inaccessible to conventional magnetohydrodynamic models.[10] Consequently, HKM simulations have become an indispensable tool for investigating a broad range of plasma processes, including collisionless shocks, magnetic reconnection, plasma turbulence, and solar wind magnetosphere interactions. Large-scale simulation frameworks such as Vlasiator have demonstrated the power of this approach for studying space-weather phenomena and heliospheric plasma dynamics.[11]

For the HKM model, the Poisson brackets have been proposed in Ref. 8. The magnetic field based Poisson bracket of the HKM model (referred to as the xvB formulation in Ref. 12) is obtained by simply transforming the Poisson bracket [formula (76) in Ref. 8] in terms of the vector potential $\mathbf{A}$ to the Poisson bracket in terms of the magnetic field via the relations[3,8] $\mathbf{B} = \nabla \times \mathbf{A}, \frac{\delta F}{\delta \mathbf{A}} = \nabla \times \frac{\delta F}{\delta \mathbf{B}}$. Also the canonical momentum based Poisson bracket is given in Ref. 8, where the distribution function depends on the canonical momentum $\mathbf{p}$ defined as $\mathbf{p} = \mathbf{v} + \mathbf{A}$. This formulation is called the xpA formulation in Ref. 12. The third formulation, i.e., also referred to as the xvA formulation, involves the distribution function f depending on velocity $\mathbf{v}$, and the vector potential $\mathbf{A}$.

For the HKM model, the Poisson brackets proposed in Ref. 8 involve the ion and electron charge densities, which are assumed to be equal under the quasi-neutrality condition. Preserving quasi-neutrality numerically requires special attention to both charge densities during simulations, particularly when particle-in-cell methods are used for ions while grid-based methods are employed for electrons, as discussed in Refs. 10 and 12. From a theoretical perspective, the quasi-neutrality condition can be viewed as a constraint in non-dissipative plasma models. The Dirac theory of constraints has been applied and further developed in recent works Refs. 13–15.

These three formulations, i.e., xpA, xvA, and xvB formulations, are mathematically equivalent descriptions of kinetic plasma dynamics, which employ different sets of variables; each offers distinct numerical advantages. In the xpA formulation, the particle position and canonical momentum form a pair of variables in a canonical Hamiltonian system. This structure enables the use of symplectic particle pushers and provides favorable long-term behavior.[12] Moreover, the use of the vector potential facilitates the numerical preservation of $\nabla\cdot\mathbf{B}=0$, a property that is also shared by the xvA formulation. In contrast, in the particle discretization of the xvB formulation, the particle kinetic energy is independent of particle position, which makes it straightforward to construct energy-conserving methods.[10] From the perspective of Hamiltonian structure, the xpA formulation possesses a considerably simpler Poisson bracket, involving significantly fewer terms than those of the xvA and xvB formulations.

In this work, we show that the quasi-neutrality condition of the HKM model corresponds to a momentum map in the canonical momentum–based formulation, arising from gauge symmetry, and to a Casimir functional in the velocity–based formulations. In Refs. 10 and 12, simplified antisymmetric brackets were derived by neglecting terms associated with the functional derivatives of the electron charge density in the Poisson brackets introduced in Ref. 8. Here, we investigate the Jacobi identities for these simplified antisymmetric brackets.[10,12] Specifically, we determine the conditions under which the Jacobi identities hold for the vector-potential-based antisymmetric brackets proposed in Ref. 12, showing that they are satisfied by all physically meaningful functionals. Furthermore, we demonstrate that the magnetic-field-based antisymmetric bracket proposed in Ref. 10 satisfies the Jacobi identity as a Poisson bracket under Gauss's law for magnetism, $\nabla\cdot\mathbf{B}=0$, and provide a proof via direct calculation.

First, we present the dimensionless HKM model[10] as follows:

$$
\begin{gathered}
\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}}=0\,,\\
\frac{\partial \mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}\,,\qquad \nabla\cdot\mathbf{B}=0\,,\\
\frac{\partial n_e}{\partial t}+\nabla\cdot(n_e\mathbf{u}_e)=0,\qquad \mathbf{u}_e=\mathbf{u}-\frac{\mathbf{J}}{ne}\,,\\
\frac{\partial s}{\partial t}+\mathbf{u}_e\cdot\nabla s=0,\\
\mathbf{E}=-\frac{\nabla p_e}{n_e}-\left(\mathbf{u}-\frac{\mathbf{J}}{n_e}\right)\times\mathbf{B}.
\end{gathered}\tag{1}
$$

Here, we take the single ion species case for an example, have assumed ion has an unit charge, $f(\mathbf{x},\mathbf{v},t)$ denotes the ion distribution function depending on time $t\in\mathbb{R}$, position $\mathbf{x}\in\mathbb{R}^3$ and velocity $\mathbf{v}\in\mathbb{R}^3$, quasi-neutrality number density is $n_e=n$ with $n=\int f\,\mathrm{d}\mathbf{v}$, where n and n_e are for ions and electrons, respectively, $\mathbf{u}=\int \mathbf{v}f\,\mathrm{d}\mathbf{v}/n$ is the ion current carrying drift velocity, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ stand for the electromagnetic fields, $\mathbf{J}=\nabla\times\mathbf{B}$ denotes the plasma current, and p_e is the electron pressure determined by the internal energy function[16,17] $\mathscr{U}(n_e,s,|\mathbf{B}|)$ as $p_e=n_e^2\frac{\partial\mathscr{U}}{\partial n_e}$ with s denoting the entropy per unit mass. The hybrid model (1) contains the Vlasov equation, Faraday's law, continuity equation for mass-less electrons, electron entropy equation, and Ohm's law. The Hamiltonian for the above hybrid model (1) is given in Refs. 8 and 10 as

$$
H(f,\mathbf{B},n_e)=\frac{1}{2}\int|\mathbf{v}|^2 f\,\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\mathbf{B}|^2\,\mathrm{d}\mathbf{x}+\int n_e\mathscr{U}\,\mathrm{d}\mathbf{x}.
$$

This work is organized as follows. The xpA, xvA, and xvB formulations of the HKM model are investigated in Sec. II. The conditions for the Jacobi identity of the simplified brackets proposed in Refs. 10 and 12 are presented in Secs. II A–II C, respectively. A direct proof of the Jacobi identity for the simplified Poisson bracket of the xvB formulation is given in Sec. II C. Relationships among the simplified Poisson brackets of these three formulations are established using the chain rule for functional derivatives in Sec. II D. Extensions to more general hybrid models are briefly discussed in Sec. III.

II. THE POISSON BRACKETS OF THE HKM MODEL

In this section, we investigate the three formulations, i.e., xpA, xvA, and xvB formulations, of the HKM model (1) with isothermal electrons. The quasi-neutrality condition corresponds to a momentum map in the canonical momentum-based formulation, arising from gauge symmetry, and to a Casimir functional in the velocity-based formulations. We also present the conditions under which the Jacobi identities hold for the three antisymmetric brackets proposed in Refs. 10 and 12.

When the electrons are assumed to be isothermal with a constant temperature T_e, we have $p_e=T_en_e$ and $\mathscr{U}=T_e\ln n_e$ in Eq. (1). The equation of s in Eq. (1) is not needed, as the electrons are isothermal and $\mathscr{U}$ does not depends on s. The isothermal electron case of Eq. (1) is the main focus of the current section, as it already captures the essential structural features relevant to the verification of the Jacobi identity, while avoiding additional technical complications associated with more general electron thermodynamics. Both the Jacobi identity conditions presented in the current section and the techniques for simplifying Poisson brackets are extended to more general electrons in Sec. III.

A. xpA formulation

We begin with the xpA formulation, whose Poisson brackets have a relatively simple structure. The following Poisson bracket was proposed in Ref. 8 (for simplicity, the time variable t is omitted),

$$
\begin{aligned}
&\{F,G\}(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e)\\
&=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}-\int\frac{1}{n_e}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x}\\
&\quad+\int\frac{\delta F}{\delta\mathbf{A}}\cdot\nabla\frac{\delta G}{\delta n_e}-\frac{\delta G}{\delta\mathbf{A}}\cdot\nabla\frac{\delta F}{\delta n_e}\,\mathrm{d}\mathbf{x},
\end{aligned}\tag{2}
$$

where the Lie–Poisson bracket[1] $[\cdot,\cdot]_{xp}$ for functions f and g depending on $\mathbf{x}$ and $\mathbf{p}$ is

$$
[f,g]_{xp}=\nabla f\cdot\frac{\partial g}{\partial\mathbf{p}}-\nabla g\cdot\frac{\partial f}{\partial\mathbf{p}}.
$$

Before proceeding to the proof of the Jacobi identity, we present a bracket theorem from Ref. 18 that will be used extensively. This theorem states that the contributions involving second-order functional derivatives cancel each other in the verification of the Jacobi identity.

Theorem 1. *(Bracket theorem[18]) To prove the Jacobi identity of the generic brackets of the form*

$$\{F,G\}(\psi)=\int\frac{\delta F}{\delta\psi}J(\psi)\frac{\delta G}{\delta\psi}\,\mathrm{d}\mathbf{a},$$

where J is an anti-self-adjoint operator depending on ψ and ψ depends on $\mathbf{a}$, *one needs only consider the explicit dependence of J on ψ when taking the functional derivatives of $\{F,G\}$.*

Here, we provide a proof of the Jacobi identity for the Poisson bracket (2).

Theorem 2. *The bracket (2) satisfies the Jacobi identity.*

Proof. By Theorem 1, which was originally established in Ref. 18, to prove the Jacobi identity for bracket (2), we can ignore the second-order functional derivatives when considering the functional derivatives of $\{F,G\}$ and have

$$\frac{\delta\{F,G\}}{\delta f}=\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp},$$
$$\frac{\delta\{F,G\}}{\delta\mathbf{A}}=-\nabla\times\left(\frac{1}{n_e}\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right),$$
$$\frac{\delta\{F,G\}}{\delta n_e}=\frac{\nabla\times\mathbf{A}}{n_e^2}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right).$$

Then for any three smooth functionals F, G, and K depending on f, $\mathbf{A}$, and n_e, we have

$$\begin{aligned}
&\{\{F,G\},K\}(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e)\\
&=\int f\left[\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp},\frac{\delta K}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\
&\quad+\int\frac{1}{n_e}\nabla\times\mathbf{A}\cdot\left(\nabla\times\left(\frac{1}{n_e}\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\times\frac{\delta K}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x}\\
&\quad-\int\nabla\times\left(\frac{1}{n_e}\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\right)\cdot\nabla\frac{\delta K}{\delta n_e}\,\mathrm{d}\mathbf{x}\\
&\quad-\int\frac{\delta K}{\delta\mathbf{A}}\cdot\nabla\left(\frac{1}{n_e^2}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\right)\mathrm{d}\mathbf{x},
\end{aligned}\tag{3}$$

where the third term is 0 via integration by parts. The sum over permutations of the first term vanishes, as it corresponds to a Lie–Poisson bracket.[2,3] The sum over permutations of the second and fourth terms also vanishes, which can be shown using Lemma 4.1 in Ref. 7 by setting $\mathbf{p}=\mathbf{q}=\mathbf{r}=\mathbf{A}$ and $\rho=n_e$. Therefore, we conclude that the bracket (2) satisfies the Jacobi identity, i.e., $\{\{F,G\},K\}+\{\{G,K\},F\}+\{\{K,F\},G\}=0$. □

Based on Theorem 2, we know that $P=C^\infty(\mathbb{R}^6)\times\left[C^\infty(\mathbb{R}^3)\right]^3\times C^\infty(\mathbb{R}^3)$ is a Poisson manifold[1] with the Poisson bracket (2). We define the action Φ of the Lie group $G=C^\infty(\mathbb{R}^3)$ on the Poisson manifold P, i.e., $\Phi: G\times P\to P$, as

$$\Phi_\psi\big(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e\big)=\big(f(\mathbf{x},\mathbf{p}-\nabla\psi),\mathbf{A}+\nabla\psi,n_e\big),\qquad\forall\psi\in G.\tag{4}$$

The Hamiltonian

$$H=\frac{1}{2}\int|\mathbf{p}-\mathbf{A}|^2f\,\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\nabla\times\mathbf{A}|^2\,\mathrm{d}\mathbf{x}+T_e\int n_e\ln n_e\,\mathrm{d}\mathbf{x},$$

is invariant under the group action (4). And the Lie group G acts on the Poisson manifold P canonically,[1] i.e.,

$$\Phi_\psi^*\{F,G\}=\{\Phi_\psi^*F,\Phi_\psi^*G\},$$

where Φ_ψ^* denotes the pullback of the map Φ_ψ. Next, we prove the quasi-neutrality condition is related to a momentum map associated with the Lie group action (4).

Proposition 1. The map $\mathbf{J}: P\to g^*$

$$\mathbf{J}\big(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e\big)=n_e-\int f(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{p},\tag{5}$$

is a momentum map of the action defined in Eq. (4), where $g=C^\infty(\mathbb{R}^3)$ is the Lie algebra of the Lie group $G=C^\infty(\mathbb{R}^3)$, and g^* is the dual space of g in the sense of L^2 integral.

Proof. For $\phi\in g$, the infinitesimal generator ϕ_P is

$$\begin{aligned}\phi_P(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e)&=\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\exp(s\phi)}\big(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e\big)|_{s=0}\\&=\left(-\nabla\phi\cdot\frac{\partial f}{\partial\mathbf{p}},\nabla\phi,0\right).\end{aligned}$$

For the linear map $J: g\to\mathscr{F}(P)$, where $\mathscr{F}(P)$ is the set of all smooth functionals defined on P,

$$J(\phi)(f,\mathbf{A},n_e)=\int n_e\phi\,\mathrm{d}\mathbf{x}-\int f\phi\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p},$$

we can check that the vector field ϕ_P is the Hamiltonian vector field [with the Poisson bracket (2)] of the functional $J(\phi)$. By the definition of momentum map $\mathbf{J}: P\to g^*$ in the Definition 11.2.1 in Ref. 1, we know that

$$\langle\mathbf{J}(f,\mathbf{A},n_e),\phi\rangle=J(\phi)(f,\mathbf{A},n_e),$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. Then, we have $\mathbf{J}(f,\mathbf{A},n_e)=n_e-\int f(\mathbf{x},\mathbf{p},t)\,\mathrm{d}\mathbf{p}$. □

1. Conservation of the momentum map

The Lie algebra g action above, arising from the canonical left Lie group action Φ, acts canonically on the Poisson manifold P and admits the momentum map (5). By Theorem 11.4.1 in Ref. 1, the momentum map is conserved under the Hamiltonian flow; that is, the quasi-neutrality condition holds for all times if it holds initially.

2. Simplified bracket

In Ref. 12, we adopt the formulation with the following antisymmetric bracket for numerical discretization, due to its low numerical noise and favorable conservation properties. This bracket is obtained by removing the last term in Eq. (2) containing the functional derivative of n_e and replacing n_e with $\int f\,\mathrm{d}\mathbf{p}$ in accordance with the quasi-neutrality condition,

$$\begin{aligned}\{F,G\}_s(f(\mathbf{x},\mathbf{p}),\mathbf{A})&=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\&\quad-\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x},\end{aligned}\tag{6}$$

where $n=\int f\,\mathrm{d}\mathbf{p}$ denotes the ion/electron charge density. This bracket does not satisfy the Jacobi identity, as shown below.

Theorem 3. *The bracket (6) does not satisfy the Jacobi identity.*

Proof. To examine the Jacobi identity, we first compute the functional derivatives of $\{F,G\}$, neglecting second-order derivatives according to the bracket Theorem 1, which was originally established in Ref. 18. This yields

$$\frac{\delta\{F,G\}_s}{\delta f}=\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp}+\frac{1}{n^2}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right),$$
$$\frac{\delta\{F,G\}_s}{\delta\mathbf{A}}=-\nabla\times\left(\frac{1}{n}\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right).$$

Based on the functional derivatives computed above, we obtain

$$\begin{aligned}\{\{F,G\}_s,K\}_s&=\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\nabla\times\left(\frac{1}{n}\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\times\frac{\delta K}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x}\\&+\int f\left[\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp},\frac{\delta K}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\&+\int f\left[\frac{1}{n^2}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right),\frac{\delta K}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}.\end{aligned}\tag{7}$$

For functionals F, G depending only on $\mathbf{A}$, we define in particular, a functional depending only on f as $K=\int f\mathbf{p}\cdot\nabla h(\mathbf{x})\,\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}$, where $h(\mathbf{x})=\frac{1}{n^2}\nabla\times\mathbf{A}_0\cdot\left(\frac{\delta F}{\delta\mathbf{A}}|_{\mathbf{A}=\mathbf{A}_0}\times\frac{\delta G}{\delta\mathbf{A}}|_{\mathbf{A}=\mathbf{A}_0}\right)$. Then we have

$$\{\{F,G\}_s,K\}_s|_{(f,\mathbf{A}_0)}+\mathrm{cyc}=\int f|\nabla h|^2\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p},$$

where the "cyc" denotes the results via the permutations of F, G, and K. The integrand is non-negative when f is non-negative, and thus $\{\{F,G\}_s,K\}_s+\mathrm{cyc}$ is not zero at $(f,\mathbf{A}_0)$ for positive $f|\nabla h|^2$. Therefore, we conclude that the antisymmetric bracket (6) does not satisfy the Jacobi identity and is not a Poisson bracket. □

A natural question is under what conditions the Jacobi identity is satisfied for the bracket (6).

Theorem 4. *The bracket (6) satisfies the Jacobi identity when the functionals satisfy the condition,*

$$\nabla\cdot\frac{\delta K}{\delta\mathbf{A}}=-\nabla\cdot\int f\frac{\partial}{\partial\mathbf{p}}\frac{\delta K}{\delta f}\,\mathrm{d}\mathbf{p},\qquad\forall K.\tag{8}$$

Proof. By comparing Eq. (3) with Eq. (7), we know that when the condition (8) is satisfied, we have

$$\{\{F,G\}_s,K\}_s(f(\mathbf{x},\mathbf{p}),\mathbf{A})=\{\{F,G\},K\}(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e)=0,$$

where the "=" holds in the sense that $n_e=n$, and second-order functional derivatives are neglected when computing the functional derivatives of the brackets (2) and (6). This completes the proof of the theorem. □

Remark 1. The condition (8) is satisfied by the Hamiltonian of the hybrid model

$$H=\frac{1}{2}\int|\mathbf{p}-\mathbf{A}|^2f\,\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\nabla\times\mathbf{A}|^2\mathrm{d}\mathbf{x}+T_e\int n\ln n\,\mathrm{d}\mathbf{x}.$$

We here define the following set of functionals satisfying the condition (8):

$$\begin{aligned}\mathscr{A}=\Big\{&F(f(\mathbf{x},\mathbf{p}),\mathbf{A}):=\tilde F(\tilde f(\mathbf{x},\mathbf{v}),\mathbf{B})|\mathbf{B}=\nabla\times\mathbf{A},\\&\mathbf{v}=\mathbf{p}-\mathbf{A},\ \ f(\mathbf{x},\mathbf{p}):=\tilde f(\mathbf{x},\mathbf{v})=\tilde f(\mathbf{x},\mathbf{p}-\mathbf{A})\Big\},\end{aligned}\tag{9}$$

where the time variable is omitted for convenience. This set $\mathscr{A}$ includes functionals determined by all smooth functionals $\tilde F$ about $\tilde f(\mathbf{x},\mathbf{v})$, $\mathbf{B}$ with $\nabla\cdot\mathbf{B}=0$.

Remark 2. The bracket (6) can be obtained from the bracket (2) via the following transformation:

$$(f(\mathbf{x},\mathbf{p}),\,\mathbf{A},\,n_e)\to\left(f(\mathbf{x},\mathbf{p}),\,\mathbf{A},\,C=n_e-\int f\,\mathrm{d}\mathbf{p}\right).$$

Any functional $\tilde F$ of $(f(\mathbf{x},\mathbf{p}),\mathbf{A},C)$ can be regarded as a functional F of $(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e)$ via $F(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e)=\tilde F(f(\mathbf{x},\mathbf{p}),\mathbf{A},C)$. By the chain rules of the functional derivatives, we have

$$\frac{\delta F}{\delta f}=\frac{\delta\tilde F}{\delta f}-\frac{\delta\tilde F}{\delta C},\qquad\frac{\delta F}{\delta\mathbf{A}}=\frac{\delta\tilde F}{\delta\mathbf{A}},\qquad\frac{\delta F}{\delta n_e}=\frac{\delta\tilde F}{\delta C}.$$

Substituting the above relations into bracket (2), we obtain

$$\begin{aligned}\{\tilde F,\tilde G\}&(f(\mathbf{x},\mathbf{p}),\mathbf{A},C)\\&=\int f\left[\frac{\delta\tilde F}{\delta f},\frac{\delta\tilde G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\&-\int\frac{1}{C+\int f\,\mathrm{d}\mathbf{p}}\nabla\times\mathbf{A}\cdot\left(\frac{\delta\tilde F}{\delta\mathbf{A}}\times\frac{\delta\tilde G}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x}\\&+\int\frac{\delta\tilde F}{\delta\mathbf{A}}\cdot\nabla\frac{\delta\tilde G}{\delta C}-\frac{\delta\tilde G}{\delta\mathbf{A}}\cdot\nabla\frac{\delta\tilde F}{\delta C}\,\mathrm{d}\mathbf{x}\\&+\int f\left(-\nabla\frac{\delta\tilde F}{\delta C}\cdot\frac{\partial}{\partial\mathbf{p}}\frac{\delta\tilde G}{\delta f}+\nabla\frac{\delta\tilde G}{\delta C}\cdot\frac{\partial}{\partial\mathbf{p}}\frac{\delta\tilde F}{\delta f}\right)\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}.\end{aligned}\tag{10}$$

The sum of the last two terms is zero when $\tilde F$ and $\tilde G$ are in $\mathscr{A}$ (9). Then for functionals $\tilde F$ and $\tilde G$ in $\mathscr{A}$, bracket (10) becomes

$$\begin{aligned}\{\tilde F,\tilde G\}&(f(\mathbf{x},\mathbf{p}),\mathbf{A},C)\\&=\int f\left[\frac{\delta\tilde F}{\delta f},\frac{\delta\tilde G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\&-\int\frac{1}{C+\int f\,\mathrm{d}\mathbf{v}}\nabla\times\mathbf{A}\cdot\left(\frac{\delta\tilde F}{\delta\mathbf{A}}\times\frac{\delta\tilde G}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x},\end{aligned}$$

which is the bracket (6) when $C=0$. As no functional derivative with respect to C is conducted, C can be regarded as a parameter function.

Before we can conclude that the bracket (6) defines a Poisson bracket for functionals in $\mathscr{A}$, we must verify that $\{F,G\}_s\in\mathscr{A}$ for $\forall F,G\in\mathscr{A}$. The answer is affirmative, since the proof relies on the relation with the xvB formulation; we defer the argument to Theorem 7, where the result is established in a unified manner.

3. Coulomb gauge

When Coulomb gauge is used, i.e., $\nabla \cdot \mathbf{A} = 0$, we let $\psi_A = -\Delta^{-1}(\nabla \cdot \mathbf{A})$, then the map $(f(\mathbf{x}, \mathbf{p}), \mathbf{A}) \rightarrow (f(\mathbf{x}, \mathbf{p} - \nabla\psi_A), \mathbf{A} + \nabla\psi_A)$[3] transforms the bracket (6) as

$$
\begin{aligned}
&\{F, G\}_A^{\text{Coulomb}}(f(\mathbf{x}, \mathbf{p}), \mathbf{A}) \\
&\quad = \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xp} \mathbf{dxdp} - \int \frac{1}{n} \nabla \times \mathbf{A} \cdot (\mathbf{M} \times \mathbf{N}) \mathbf{dx},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{M} &= \frac{\delta F}{\delta \mathbf{A}} + \nabla\Delta^{-1}\nabla \cdot \frac{\delta F}{\delta \mathbf{A}} + \nabla\Delta^{-1}\nabla \cdot \int f \frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \mathbf{dp}, \\
\mathbf{N} &= \frac{\delta G}{\delta \mathbf{A}} + \nabla\Delta^{-1}\nabla \cdot \frac{\delta G}{\delta \mathbf{A}} + \nabla\Delta^{-1}\nabla \cdot \int f \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f} \mathbf{dp}.
\end{aligned}
$$

When the magnetic field $\mathbf{B}$ is adopted, we have the following bracket by $\mathbf{B} = \nabla \times \mathbf{A}$ and $\frac{\delta F}{\delta \mathbf{A}} = \nabla \times \frac{\delta F}{\delta \mathbf{B}}$,

$$
\begin{aligned}
&\{F, G\}_B^{\text{Coulomb}}(f(\mathbf{x}, \mathbf{p}), \mathbf{B}) \\
&\quad = \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xp} \mathbf{dxdp} - \int \frac{1}{n} \mathbf{B} \cdot (\tilde{\mathbf{M}} \times \tilde{\mathbf{N}}) \mathbf{dx},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathbf{M}} &= \nabla \times \frac{\delta F}{\delta \mathbf{B}} + \nabla\Delta^{-1}\nabla \cdot \int f \frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \mathbf{dp}, \\
\tilde{\mathbf{N}} &= \nabla \times \frac{\delta G}{\delta \mathbf{B}} + \nabla\Delta^{-1}\nabla \cdot \int f \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f} \mathbf{dp}.
\end{aligned}
$$

B. xvA formulation

Next, we consider the xvA formulation,[8,12,19] for which the Poisson bracket is given by formula (76) in Ref. 8,

$$
\begin{aligned}
&\{F, G\}(f(\mathbf{x}, \mathbf{v}), \mathbf{A}, n_e) \\
&= \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xv} \mathbf{dxdv} \\
&\quad + \int f\nabla \times \mathbf{A} \cdot \left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right) \mathbf{dxdv} \\
&\quad - \int \frac{1}{n_e} \nabla \times \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{A}} \times \frac{\delta G}{\delta \mathbf{A}}\right) \mathbf{dx} \\
&\quad - \int \frac{f}{n_e} \nabla \times \mathbf{A} \cdot \left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f} \times \frac{\delta F}{\delta \mathbf{A}} - \frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f} \times \frac{\delta G}{\delta \mathbf{A}}\right) \mathbf{dxdv} \\
&\quad - \int \frac{1}{n_e} \nabla \times \mathbf{A} \cdot \left(\int f \frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f} \mathbf{dv} \times \int f \frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f} \mathbf{dv}\right) \mathbf{dx} \\
&\quad + \int \frac{\delta F}{\delta \mathbf{A}} \cdot \nabla \frac{\delta G}{\delta n_e} - \frac{\delta G}{\delta \mathbf{A}} \cdot \nabla \frac{\delta F}{\delta n_e} \mathbf{dx} \\
&\quad + \int f\left(-\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f} \cdot \nabla \frac{\delta G}{\delta n_e} + \frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f} \cdot \nabla \frac{\delta F}{\delta n_e}\right) \mathbf{dxdv} \\
&= \{F, G\}_0 + \{F, G\}_{n_e},
\end{aligned}
\tag{11}
$$

where we denote the sum of the first six terms in the above bracket by $\{F, G\}_0$, and the last term by $\{F, G\}_{n_e}$.

Here, we explicitly write out the terms needed to verify the Jacobi identity. As before, we first compute the functional derivatives of $\{F, G\}$, neglecting second-order derivatives according to the bracket Theorem 1, which was originally established in Ref. 18. Then, we have

$$
\begin{aligned}
&\{\{F, G\}, K\} \\
&= -\int \frac{1}{n_e} \nabla \times \mathbf{A} \cdot \left(\frac{\delta\{F, G\}}{\delta \mathbf{A}} \times \frac{\delta K}{\delta \mathbf{A}}\right) \mathbf{dx} \\
&\quad + \int f\left[\frac{\delta\{F, G\}_0}{\delta f}, \frac{\delta K}{\delta f}\right]_{xv} \mathbf{dxdv} \\
&\quad + \int f\nabla \times \mathbf{A} \cdot \left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta\{F, G\}_0}{\delta f} \times \frac{\partial}{\partial \mathbf{v}}\frac{\delta K}{\delta f}\right) \mathbf{dxdv} \\
&\quad - \int \frac{f}{n_e} \nabla \times \mathbf{A} \cdot \left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta K}{\delta f} \times \frac{\delta\{F, G\}}{\delta \mathbf{A}} - \frac{\partial}{\partial \mathbf{v}}\frac{\delta\{F, G\}_0}{\delta f} \times \frac{\delta K}{\delta \mathbf{A}}\right) \mathbf{dxdv} \\
&\quad - \int \frac{1}{n_e} \nabla \times \mathbf{A} \cdot \left(\int f \frac{\partial}{\partial \mathbf{v}}\frac{\delta\{F, G\}_0}{\delta f} \mathbf{dv} \times \int f \frac{\partial}{\partial \mathbf{v}}\frac{\delta K}{\delta f} \mathbf{dv}\right) \mathbf{dx} \\
&\quad + \int f\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta K}{\delta f} \cdot \nabla \frac{\delta\{F, G\}}{\delta n_e}\right) \mathbf{dxdv} - T_{n_e}^1 + T_{n_e}^2,
\end{aligned}
\tag{12}
$$

where $T_{n_e}^1 = \int \frac{\delta K}{\delta \mathbf{A}} \cdot \nabla \frac{\delta\{F,G\}}{\delta n_e} \mathbf{dx}$, and all other terms involving the functional derivative with respect to n_e are included in $T_{n_e}^2$, i.e.,

$$
\begin{aligned}
T_{n_e}^2 &= \int \frac{\delta\{F, G\}}{\delta \mathbf{A}} \cdot \nabla \frac{\delta K}{\delta n_e} \mathbf{dx} + \int f\left(-\frac{\partial}{\partial \mathbf{v}}\frac{\delta\{F, G\}}{\delta f} \cdot \nabla \frac{\delta K}{\delta n_e}\right) \mathbf{dxdv} \\
&\quad + \int f\left[\frac{\delta\{F, G\}_{n_e}}{\delta f}, \frac{\delta K}{\delta f}\right]_{xv} \mathbf{dxdv} \\
&\quad + \int f\nabla \times \mathbf{A} \cdot \ \frac{\partial}{\partial \mathbf{v}}\frac{\delta\{F, G\}_{n_e}}{\delta f}\Bigg) \times \frac{\partial}{\partial \mathbf{v}}\frac{\delta K}{\delta f} \mathbf{dxdv} \\
&\quad + \int \frac{f}{n_e} \nabla \times \mathbf{A} \cdot \ \frac{\partial}{\partial \mathbf{v}} \ \frac{\delta\{F, G\}_{n_e}}{\delta f}\Bigg) \times \frac{\delta K}{\delta \mathbf{A}}\Bigg) \mathbf{dxdv} \\
&\quad - \int \frac{1}{n_e} \nabla \times \mathbf{A} \cdot \left(\int f \frac{\partial}{\partial \mathbf{v}}\frac{\delta\{F, G\}_{n_e}}{\delta f} \mathbf{dv} \times \int f \frac{\partial}{\partial \mathbf{v}}\frac{\delta K}{\delta f} \mathbf{dv}\right) \mathbf{dx},
\end{aligned}
$$

where the first term vanishes by integration by parts. Since the Poisson bracket (11) satisfies the Jacobi identity, the sum over permutations of $T_{n_e}^2$, which contains all terms involving the functional derivatives of n_e, also vanishes.

Remark 3. The quasi-neutrality condition, $n_e = \int f \, \mathbf{dv}$, corresponds to a class of Casimir functionals of the bracket (11), i.e., for any given $\mathbf{x}_0$,

$$
C(f, n_e, \mathbf{A}) = \int \delta(\mathbf{x} - \mathbf{x}_0)\left(n_e - \int f \, \mathbf{dv}\right) \mathbf{dx}.
$$

4. Simplified bracket

The simplified bracket presented in Ref. 12 is obtained by removing the terms involving the functional derivatives with respect to n_e and replacing n_e with $\int f \, \mathbf{dv}$,

$$\begin{aligned}
&\{F,G\}_s(f(\mathbf{x},\mathbf{v}),\mathbf{A})\\
&\quad=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv}\mathbf{dxdv}\\
&\qquad+\int f\nabla\times\mathbf{A}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathbf{dx}\\
&\qquad-\int\frac{f}{n}\nabla\times\mathbf{A}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\times\frac{\delta F}{\delta\mathbf{A}}-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\mathbf{dv}\times\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\mathbf{dv}\right)\mathbf{dx}. \qquad (13)
\end{aligned}$$

In the following theorem, we state the condition under which the bracket (13) satisfies the Jacobi identity.

Theorem 5. *The bracket (13) satisfies the Jacobi identity when the functionals satisfy the condition,* $\nabla\cdot\frac{\delta K}{\delta\mathbf{A}}=0,\quad\forall K$.

Proof. In order to prove the Jacobi identity, we need to calculate the functional derivatives of the bracket (13),

$$\frac{\delta\{F,G\}_s}{\delta f}=\frac{\delta\{F,G\}_0}{\delta f}+\frac{\delta\{F,G\}}{\delta n_e},\qquad\frac{\delta\{F,G\}_s}{\delta\mathbf{A}}=\frac{\delta\{F,G\}}{\delta\mathbf{A}},$$

where $\{\cdot,\cdot\}$ is the bracket (11), the "=" holds in the sense that we replace n_e with $\int f\,\mathbf{dv}$ on the right-hand side and ignore the second-order functional derivatives when calculating the functional derivatives of $\{F,G\}$ and $\{F,G\}_s$. Then, we have

$$\begin{aligned}
&\{\{F,G\}_s,K\}_s\\
&\quad=\int f\left[\frac{\delta\{F,G\}_0}{\delta f}+\frac{\delta\{F,G\}}{\delta n_e},\frac{\delta K}{\delta f}\right]_{xv}\\
&\qquad+\int f\nabla\times\mathbf{A}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta\{F,G\}_0}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta K}{\delta f}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\frac{\delta\{F,G\}}{\delta\mathbf{A}}\times\frac{\delta K}{\delta\mathbf{A}}\right)\mathbf{dx}\\
&\qquad-\int\frac{f}{n}\nabla\times\mathbf{A}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta K}{\delta f}\times\frac{\delta\{F,G\}}{\delta\mathbf{A}}-\frac{\partial}{\partial\mathbf{v}}\frac{\delta\{F,G\}_0}{\delta f}\times\frac{\delta K}{\delta\mathbf{A}}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta\{F,G\}_0}{\delta f}\mathbf{dv}\times\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta K}{\delta f}\mathbf{dv}\right)\mathbf{dx}.
\end{aligned} \qquad (14)$$

Since the Poisson bracket (11) satisfies the Jacobi identity, the sum over permutations of $T^2_{n_e}$ in Eq. (12) vanishes. The remaining terms in Eq. (12), except for $T^1_{n_e}$ and $T^2_{n_e}$, coincide with the terms in Eq. (14). When $\nabla\cdot\frac{\delta K}{\delta\mathbf{A}}=0,\forall K$, we have $T^1_{n_e}=0$, and hence the bracket (13) satisfies the Jacobi identity. □

We can define the following set of functionals satisfying the condition $\nabla\cdot\frac{\delta K}{\delta\mathbf{A}}=0$,

$$\mathscr{B}=\{F(f(\mathbf{x},\mathbf{v}),\mathbf{A}):=\tilde{F}(f(\mathbf{x},\mathbf{v}),\mathbf{B})|\mathbf{B}=\nabla\times\mathbf{A}\}, \qquad (15)$$

which includes the functionals determined by all smooth functionals $\tilde{F}$ about $f(\mathbf{x},\mathbf{v})$, $\mathbf{B}$ with $\nabla\cdot\mathbf{B}=0$.

As with the bracket (6), we must verify that $\{F,G\}_s\in\mathscr{B}$ for $\forall F,G\in\mathscr{B}$, before we can conclude that the bracket (13) defines a Poisson bracket for functionals in $\mathscr{B}$. The answer is affirmative, since the proof relies on the relation with the xvB formulation; as in the case of the xpA formulation, we defer the argument to Theorem 7, where the result is established in a unified manner.

C. xvB formulation

Finally, we consider the magnetic-field-based formulation. When the electron charge density n_e is treated as an independent variable, a Poisson bracket can be derived from Ref. 8 as follows:

$$\begin{aligned}
&\{F,G\}(f(\mathbf{x},\mathbf{v}),\mathbf{B},n_e)\\
&\quad=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv}\mathbf{dxdv}\\
&\qquad+\int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n_e}\mathbf{B}\cdot\left(\nabla\times\frac{\delta F}{\delta\mathbf{B}}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathbf{dx}\\
&\qquad-\int\frac{1}{n_e}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\mathbf{dv}\times\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\mathbf{dv}\right)\mathbf{dx}\\
&\qquad-\int\frac{f}{n_e}\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta\mathbf{B}}-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathbf{dxdv}\\
&\qquad+\int f\left(-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\cdot\nabla\frac{\delta G}{\delta n_e}+\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\cdot\nabla\frac{\delta F}{\delta n_e}\right)\mathbf{dxdv}. \qquad (16)
\end{aligned}$$

Remark 4. The quasi-neutrality condition $n_e=\int f\,\mathbf{dv}$ corresponds to a class of Casimir functional of the bracket (16), i.e., for any given $\mathbf{x}_0$,

$$C(f,n_e,\mathbf{B})=\int\delta(\mathbf{x}-\mathbf{x}_0)\left(n_e-\int f\,\mathbf{dv}\right)\mathbf{dx}.$$

5. Simplified bracket

In Ref. 10, we propose the following bracket for the xvB formulation by removing the terms involving the functional derivatives with respect to n_e and replacing n_e with $n=\int f\,\mathbf{dv}$,

$$\begin{aligned}
&\{F,G\}_s(f(\mathbf{x},\mathbf{v}),\mathbf{B})\\
&\quad=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv}\mathbf{dxdv}\\
&\qquad+\int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\mathbf{B}\cdot\left(\nabla\times\frac{\delta F}{\delta\mathbf{B}}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathbf{dx}\\
&\qquad-\int\frac{f}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta\mathbf{B}}-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\mathbf{dv}\times\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\mathbf{dv}\right)\mathbf{dx}. \qquad (17)
\end{aligned}$$

Remark 5. Since $n_e = \int f\,\mathrm{d}\mathbf{v}$ corresponds to a Casimir functional of the Poisson bracket (16), we choose the Casimir functional as an independent variable to simplify the Poisson bracket, following the spirit of Darboux's theorem. Specifically, we perform the following transformation of variables:

$$\left(f(\mathbf{x},\mathbf{v}),\ \mathbf{B},\ n_e\right) \to \left(f(\mathbf{x},\mathbf{v}),\ \mathbf{B},\ C = n_e - \int f\,\mathrm{d}\mathbf{v}\right).$$

Then any smooth functional $\tilde{F}$ depending on $f, \mathbf{B}, C$ can be considered as a functional F depending on $f, \mathbf{B}, n_e$, with the following relations between their functional derivatives:

$$\frac{\delta F}{\delta f} = \frac{\delta \tilde{F}}{\delta f} - \frac{\delta \tilde{F}}{\delta C}, \qquad \frac{\delta F}{\delta \mathbf{A}} = \frac{\delta \tilde{F}}{\delta \mathbf{A}}, \qquad \frac{\delta F}{\delta n_e} = \frac{\delta \tilde{F}}{\delta C}.$$

Substituting the above relations for the functional derivatives into Eq. (16), we obtain (the tilde is omitted for convenience)

$$\begin{aligned}
&\{F,G\}(f(\mathbf{x},\mathbf{v}),\mathbf{B},C)\\
&= \int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad - \int f\left(\nabla\frac{\delta F}{\delta C}\cdot\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f} - \nabla\frac{\delta G}{\delta C}\cdot\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad + \int f\mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad - \int \frac{1}{\tilde{n}}\mathbf{B}\cdot\left(\nabla\times\frac{\delta F}{\delta \mathbf{B}}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\mathrm{d}\mathbf{x}\\
&\quad - \int \frac{1}{\tilde{n}}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\mathrm{d}\mathbf{v}\times\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\mathrm{d}\mathbf{v}\right)\mathrm{d}\mathbf{x}\\
&\quad - \int \frac{f}{\tilde{n}}\mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta \mathbf{B}} - \frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad + \int f\left(-\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\cdot\nabla\frac{\delta G}{\delta C} + \frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\cdot\nabla\frac{\delta F}{\delta C}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v},
\end{aligned}$$

where $\tilde{n} = C + \int f\mathrm{d}\mathbf{v}$, the second term cancels with the last term. In the final expression, there is no functional derivative with respect to C, so C can be treated as a parameter. Note that when $C = 0$, the bracket reduces to Eq. (17).

Next, we present a direct proof of the Jacobi identity for the bracket (17) under Gauss's law for magnetism $\nabla\cdot\mathbf{B} = 0$.

Theorem 6. *The bracket (17) satisfies the Jacobi identity under Gauss's law for magnetism* $\nabla\cdot\mathbf{B} = 0$.

Proof. By the bracket Theorem 1, which was originally established in Ref. 18, to prove the Jacobi identity for a bracket of the form $\{F,G\} = \int \frac{\delta F}{\delta \psi}J(\psi)\frac{\delta G}{\delta \psi}\,\mathrm{d}\mathbf{a}$, where J is an operator depending on ψ and ψ depends on $\mathbf{a}$, it is sufficient to consider only the explicit dependence of J on ψ when computing the functional derivative $\frac{\delta\{F,G\}}{\delta\psi}$. Applying this to the bracket in Eq. (17), we obtain the following functional derivatives, where the index s is omitted for simplicity:

$$\begin{aligned}
\frac{\delta\{F,G\}}{\delta f} &= \left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv} + \mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right)\\
&\quad + \frac{1}{n^2}\mathbf{B}\cdot\left(\nabla\times\frac{\delta F}{\delta \mathbf{B}}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\\
&\quad + \int\frac{f\mathbf{B}}{n^2}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta \mathbf{B}} - \frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\mathrm{d}\mathbf{v}\\
&\quad - \frac{1}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta \mathbf{B}} - \frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\\
&\quad + \frac{1}{n^2}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\mathrm{d}\mathbf{v}\times\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\mathrm{d}\mathbf{v}\right)\\
&\quad - \frac{1}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\mathrm{d}\mathbf{v}\right)\\
&\quad - \frac{1}{n}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\mathrm{d}\mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right),\\
\frac{\delta\{F,G\}}{\delta \mathbf{B}} &= \int f\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right)\mathrm{d}\mathbf{v}\\
&\quad - \left(\frac{1}{n}\nabla\times\frac{\delta F}{\delta \mathbf{B}}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\\
&\quad - \int\frac{f}{n}\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta \mathbf{B}} - \frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right)\mathrm{d}\mathbf{v}\\
&\quad - \frac{1}{n}\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\mathrm{d}\mathbf{v}\times\int f\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\mathrm{d}\mathbf{v}.
\end{aligned}$$

Here we write out $\{\{F,G\},K\}$ as

$$\begin{aligned}
\{\{F,G\},K\} &= \{\{F,G\},K\}_{fff} + \{\{F,G\},K\}_{ffB}\\
&\quad + \{\{F,G\},K\}_{fBB} + \{\{F,G\},K\}_{BBB},
\end{aligned}$$

where $\{\{F,G\},K\}_{abc}$ denotes the terms involving the functional derivatives with respect to a, b, and c. Next, we explicitly write out the expressions for each group of terms,

$$\begin{aligned}
\{\{F,G\},K\}_{fff} &= \{\{F,G\},K\}_{fff}^{f} + \{\{F,G\},K\}_{fff}^{fB}\\
&\quad + \{\{F,G\},K\}_{fff}^{ffB/n} + \{\{F,G\},K\}_{fff}^{fffB/n^2}\\
&\quad + \{\{F,G\},K\}_{fff}^{fB^2} + \{\{F,G\},K\}_{fff}^{ffB^2/n}\\
&\quad + \{\{F,G\},K\}_{fff}^{fffB^2/n^2},\\
\{\{F,G\},K\}_{ffB} &= \{\{F,G\},K\}_{ffB}^{fB/n} + \{\{F,G\},H\}_{ffB}^{ffB/n^2}\\
&\quad + \{\{F,G\},K\}_{ffB}^{fB^2/n} + \{\{F,G\},K\}_{ffB}^{ffB^2/n^2},\\
\{\{F,G\},K\}_{fBB} &= \{\{F,G\},K\}_{fBB}^{fB/n^2} + \{\{F,G\},K\}_{fBB}^{fB^2/n^2},\\
\{\{F,G\},K\}_{BBB} &= \{\{F,G\},K\}_{BBB}^{B/n^2},
\end{aligned} \tag{18}$$

where the superscripts indicate the functions involved, excluding the functional derivatives.

The proof that the sum over permutations of each of the above terms vanishes is given in Appendix, with some results holding under

Gauss's law for magnetism $\nabla \cdot \mathbf{B} = 0$. It then follows that the Jacobi identity is satisfied, i.e.,

$$\{\{F, G\}, K\} + \{\{G, K\}, F\} + \{\{K, F\}, G\} = 0,$$

under Gauss's law for magnetism $\nabla \cdot \mathbf{B} = 0$. □

D. The relations between the brackets

Here, we show the relationships among the three simplified brackets. A similar approach was used to derive the magnetic-field-based Poisson bracket from the canonical momentum-based Poisson bracket in Refs. 3, 8, and 17.

(1) The bracket (17) can be obtained from the bracket (6) via the following transformation and corresponding relations between the functional derivatives:

$$f(\mathbf{x},\mathbf{p}) = \tilde{f}(\mathbf{x},\mathbf{v}), \qquad \mathbf{p} = \mathbf{v} + \mathbf{A}, \qquad \nabla\times\mathbf{A} = \mathbf{B},$$
$$[f(\mathbf{x},\mathbf{p}), g(\mathbf{x},\mathbf{p})]_{xp} = \left[\tilde{f}(\mathbf{x},\mathbf{v}), \tilde{g}(\mathbf{x},\mathbf{v})\right]_{xv} + \nabla\times\mathbf{A}\cdot\frac{\partial}{\partial\mathbf{v}}\tilde{f}(\mathbf{x},\mathbf{v}) \times\frac{\partial}{\partial\mathbf{v}}\tilde{g}(\mathbf{x},\mathbf{v}),$$
$$\frac{\delta F}{\delta f} = \frac{\delta\tilde{F}}{\delta\tilde{f}}, \qquad \frac{\delta F}{\delta\mathbf{A}} = \nabla\times\frac{\delta\tilde{F}}{\delta\mathbf{B}} - \int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta\tilde{F}}{\delta\tilde{f}}\,\mathrm{d}\mathbf{v}. \tag{19}$$

(2) The bracket (13) can be obtained from the bracket (6) via the following transformation and the corresponding relations between the functional derivatives:

$$f(\mathbf{x},\mathbf{p}) = \tilde{f}(\mathbf{x},\mathbf{v}), \quad \mathbf{p} = \mathbf{v} + \mathbf{A},$$
$$[f(\mathbf{x},\mathbf{p}), g(\mathbf{x},\mathbf{p})]_{xp} = \left[\tilde{f}(\mathbf{x},\mathbf{v}), \tilde{g}(\mathbf{x},\mathbf{v})\right]_{xv} + \nabla\times\mathbf{A}\cdot\frac{\partial}{\partial\mathbf{v}}\tilde{f}(\mathbf{x},\mathbf{v}) \times\frac{\partial}{\partial\mathbf{v}}\tilde{g}(\mathbf{x},\mathbf{v}),$$
$$\frac{\delta F}{\delta f} = \frac{\delta\tilde{F}}{\delta\tilde{f}}, \quad \frac{\delta F}{\delta\mathbf{A}} = \frac{\delta\tilde{F}}{\delta\mathbf{A}} - \int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta\tilde{F}}{\delta\tilde{f}}\,\mathrm{d}\mathbf{v}.$$

(3) The bracket (17) can be obtained from the bracket (13) via the following transformation and the corresponding relations between the functional derivatives:

$$\nabla\times\mathbf{A} = \mathbf{B}, \qquad \frac{\delta F}{\delta\mathbf{A}} = \nabla\times\frac{\delta\tilde{F}}{\delta\mathbf{B}}.$$

Theorem 7. The bracket (6) of two arbitrary smooth functionals in $\mathscr{A}$ (9) belongs also to $\mathscr{A}$ (9). The bracket (13) of arbitrary smooth functionals in $\mathscr{B}$ (15) belongs also to $\mathscr{B}$ (15).

Proof. For two arbitrary smooth functionals F and G in Eq. (9) depending on $f(\mathbf{x},\mathbf{p})$ and $\mathbf{A}$, we have

$$F(f(\mathbf{x},\mathbf{p}),\mathbf{A}) = \tilde{F}(\tilde{f}(\mathbf{x},\mathbf{v}),\mathbf{B}), \qquad G(f(\mathbf{x},\mathbf{p}),\mathbf{A}) = \tilde{G}(\tilde{f}(\mathbf{x},\mathbf{v}),\mathbf{B}),$$

where $\tilde{F}$ and $\tilde{G}$ are two functionals determining F and G through the definition in Eq. (9). Then, we have

$$\{F, G\}(f(\mathbf{x},\mathbf{p}),\mathbf{A}) = \left\{\tilde{F},\tilde{G}\right\}(\tilde{f}(\mathbf{x},\mathbf{v}),\mathbf{B}), \tag{20}$$

where the bracket on the left-hand side is the bracket (6) of the xpA formulation, while the bracket on the right-hand side is the bracket (17) of the xvB formulation. In Eq. (20), the equality holds because the bracket (17) can be obtained from the bracket (6) via the transformation with the chain rules for functional derivatives (19). Note that equality (20) implies that the functional $\{F, G\}(f(\mathbf{x},\mathbf{p}),\mathbf{A})$ is determined by the functional $\left\{\tilde{F},\tilde{G}\right\}(\tilde{f}(\mathbf{x},\mathbf{v}),\mathbf{B})$, and is therefore an element of the functional set $\mathscr{A}$ (9), satisfying the condition (8).

By similar arguments, the bracket (13) of arbitrary smooth functionals in $\mathscr{B}$ (15) also belongs to $\mathscr{B}$ (15). □

Here we summarize the above xpA, xvA, and xvB formulations in Table I.

III. GENERALIZATION

In this section, we generalize the above results to broader classes of plasma models, including the HKM model with electron entropy, the kinetic Hall MHD hybrid model,[17] Hall MHD,[6] and the kinetic multi-fluid model.[8] The two-fluid model has been considered previously in Ref. 20.

A. The HKM model with electron entropy

We now relax the isothermal assumption and consider a general electron fluid endowed with an advected entropy field s.[16] This extension allows for spatial and temporal variations of the electron thermodynamic state and provides a more general description of electron pressure than the isothermal closure. Accordingly, the electron pressure is taken to satisfy the equation of state $p_e = p_e(n_e, s)$, which extends the Hamiltonian structure of the HKM model to non-isothermal electron dynamics.

We include a new term associated with the entropy per unit mass, s, in the bracket proposed in Ref. 8, yielding

$$\begin{aligned}\{F, G\}&(f(\mathbf{x},\mathbf{p}),\mathbf{A},n_e,s)\\ &= \int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} - \int\frac{1}{n_e}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x}\\ &\quad + \int\frac{\delta F}{\delta\mathbf{A}}\cdot\nabla\frac{\delta G}{\delta n_e} - \frac{\delta G}{\delta\mathbf{A}}\cdot\nabla\frac{\delta F}{\delta n_e}\,\mathrm{d}\mathbf{x} + \int\frac{\nabla s}{n_e}\cdot\left(\frac{\delta F}{\delta s}\frac{\delta G}{\delta\mathbf{A}} - \frac{\delta G}{\delta s}\frac{\delta F}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x},\end{aligned} \tag{21}$$

TABLE I. Comparison of the xpA, xvA, and xvB formulations in terms of the choice of variables, the role of quasi-neutrality, and the conditions required for the Jacobi identity. Here, K denotes an arbitrary smooth functional.

Formulations	xpA	xvA	xvB
Variables	$f(\mathbf{x},\mathbf{p})$, $\mathbf{A}$	$f(\mathbf{x},\mathbf{v})$, $\mathbf{A}$	$f(\mathbf{x},\mathbf{v})$, $\mathbf{B}$
Quasi neutrality	Momentum map	Casimir	Casimir
Jacobi identity	$\nabla\cdot\frac{\delta K}{\delta\mathbf{A}} = -\nabla\cdot\int f\frac{\partial}{\partial\mathbf{p}}\frac{\delta K}{\delta f}\,\mathrm{d}\mathbf{p}$	$\nabla\cdot\frac{\delta K}{\delta\mathbf{A}} = 0$	$\nabla\cdot\mathbf{B} = 0$

with the following Hamiltonian

$$H=\frac{1}{2}\int|\mathbf{p}-\mathbf{A}|^2 f\,\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\nabla\times\mathbf{A}|^2\,\mathrm{d}\mathbf{x}+\int n_e\,\mathscr{U}(n_e,s)\,\mathrm{d}\mathbf{x}.$$

The Jacobi identity for this bracket can be verified via a direct proof. By removing the terms involving functional derivatives with respect to n_e and replacing n_e with n in the remaining terms, we obtain the following simplified bracket, which satisfies the Jacobi identity under condition (8), along with the corresponding Hamiltonian,

$$\begin{aligned}
&\{F,G\}_s(f(\mathbf{x},\mathbf{p}),\mathbf{A},s)\\
&=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}-\int\frac{1}{n}\nabla\times\mathbf{A}\cdot\left(\frac{\delta F}{\delta\mathbf{A}}\times\frac{\delta G}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x}\\
&\quad+\int\frac{\nabla s}{n}\cdot\left(\frac{\delta F}{\delta s}\frac{\delta G}{\delta\mathbf{A}}-\frac{\delta G}{\delta s}\frac{\delta F}{\delta\mathbf{A}}\right)\mathrm{d}\mathbf{x},\qquad n=\int f\,\mathrm{d}\mathbf{p},
\end{aligned}$$

$$H=\frac{1}{2}\int|\mathbf{p}-\mathbf{A}|^2 f\,\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\nabla\times\mathbf{A}|^2\,\mathrm{d}\mathbf{x}+\int n\,\mathscr{U}(n,s)\,\mathrm{d}\mathbf{x}. \tag{22}$$

Next, we present the results for the magnetic-field-based formulation. By adding two entropy-related terms to the bracket (16), we obtain

$$\begin{aligned}
&\{F,G\}(f(\mathbf{x},\mathbf{v}),\mathbf{B},n_e,s)\\
&=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}+\int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad-\int\frac{1}{n_e}\mathbf{B}\cdot\left(\nabla\times\frac{\delta F}{\delta\mathbf{B}}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\\
&\quad-\int\frac{1}{n_e}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\mathrm{d}\mathbf{v}\times\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\mathrm{d}\mathbf{v}\right)\mathrm{d}\mathbf{x}\\
&\quad-\int f\frac{\mathbf{B}}{n_e}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta\mathbf{B}}-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad+\int f\left(-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\cdot\nabla\frac{\delta G}{\delta n_e}+\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\cdot\nabla\frac{\delta F}{\delta n_e}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad+\int\frac{\nabla s}{n_e}\cdot\left(\frac{\delta F}{\delta s}\nabla\times\frac{\delta G}{\delta\mathbf{B}}-\frac{\delta G}{\delta s}\nabla\times\frac{\delta F}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\\
&\quad-\int f\,\frac{\nabla s}{n_e}\cdot\left(\frac{\delta F}{\delta s}\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}-\frac{\delta G}{\delta s}\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v},
\end{aligned}$$

$$H=\frac{1}{2}\int|\mathbf{v}|^2 f\,\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\mathbf{B}|^2\,\mathrm{d}\mathbf{x}+\int n_e\,\mathscr{U}(n_e,s)\,\mathrm{d}\mathbf{x}. \tag{23}$$

Similar to Sec. II C, we get the following simplified Poisson bracket and corresponding Hamiltonian:

$$\begin{aligned}
&\{F,G\}_s(f(\mathbf{x},\mathbf{v}),\mathbf{B},s)\\
&=\int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xv}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}+\int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad-\int\frac{1}{n}\mathbf{B}\cdot\left(\nabla\times\frac{\delta F}{\delta\mathbf{B}}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\\
&\quad-\int\frac{f}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\times\nabla\times\frac{\delta F}{\delta\mathbf{B}}-\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\times\nabla\times\frac{\delta G}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\\
&\quad-\int\frac{1}{n}\mathbf{B}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\mathrm{d}\mathbf{v}\times\int f\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}\mathrm{d}\mathbf{v}\right)\mathrm{d}\mathbf{x}\\
&\quad+\int\frac{\nabla s}{n}\cdot\left(\frac{\delta F}{\delta s}\nabla\times\frac{\delta G}{\delta\mathbf{B}}-\frac{\delta G}{\delta s}\nabla\times\frac{\delta F}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\\
&\quad-\int f\,\frac{\nabla s}{n}\cdot\left(\frac{\delta F}{\delta s}\frac{\partial}{\partial\mathbf{v}}\frac{\delta G}{\delta f}-\frac{\delta G}{\delta s}\frac{\partial}{\partial\mathbf{v}}\frac{\delta F}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v},\qquad n=\int f\,\mathrm{d}\mathbf{v},
\end{aligned}$$

$$H=\frac{1}{2}\int|\mathbf{v}|^2 f\,\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}+\frac{1}{2}\int|\mathbf{B}|^2\,\mathrm{d}\mathbf{x}+\int n\,\mathscr{U}(n,s)\,\mathrm{d}\mathbf{x}. \tag{24}$$

The Poisson brackets expressed in terms of the entropy density, $\sigma=n_e s$, or the electron pressure, $p_e=n_e^2\frac{\partial\mathscr{U}}{\partial n_e}$, can be obtained from the above brackets using the chain rule for functional derivatives. Similar simplifications can be applied using the quasi-neutrality condition. Moreover, analogous numerical discretizations can be implemented based on these brackets, as in Refs. 10 and 12.

B. Kinetic Hall MHD hybrid model

Next, we consider a more general hybrid model,[17] in which only a subset of the ions is described by kinetic equations. For simplicity, we focus on the case of a single ion species with unit charge,

$$\begin{aligned}
\frac{\partial f}{\partial t}&=-\mathbf{v}\cdot\nabla f-(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f,\\
\frac{\partial\mathbf{B}}{\partial t}&=-\nabla\times\mathbf{E},\\
\frac{\partial n_e}{\partial t}&=-\nabla\cdot(n_i\mathbf{V})-\nabla\cdot\mathbf{J}_k,\\
\frac{\partial n_i}{\partial t}&=-\nabla\cdot(n_i\mathbf{V}),\\
n_i\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)&=\left[\frac{n_i}{n_e}n_k\mathbf{V}+\frac{n_i}{n_e}(\nabla\times\mathbf{B}-\mathbf{J}_k)\right]\times\mathbf{B}\\
&\quad-\frac{n_i}{n_e}\nabla\cdot\mathbf{P}_e-\nabla P_i,
\end{aligned}\tag{25}$$

where $\mathbf{J}=\nabla\times\mathbf{B}$, $n_k=\int f\,\mathrm{d}\mathbf{v}$, and $\mathbf{J}_k=\int\mathbf{v}f\,\mathrm{d}\mathbf{v}$,

$$\begin{aligned}
\mathbf{E}&=-\frac{n_i}{n_e}\mathbf{V}\times\mathbf{B}+\frac{1}{n_e}(\nabla\times\mathbf{B}-\mathbf{J}_k)\times\mathbf{B}-\frac{\nabla\cdot\mathbf{P}_e}{n_e},\\
\mathbf{P}_e&=\frac{P_{e\parallel}-P_\perp}{B^2}\mathbf{B}\mathbf{B}+P_{e\perp}\mathbf{I},\qquad P_{e\parallel}=\mathbf{P}_e:\mathbf{b}\mathbf{b},\\
P_{e\perp}&=\frac{1}{2}\mathbf{P}_e:(\mathbf{I}-\mathbf{b}\mathbf{b}),\qquad \mathbf{b}:=\mathbf{B}/|\mathbf{B}|.
\end{aligned}\tag{26}$$

Here, $f(\mathbf{x},\mathbf{v},t)$ represents the distribution function of the ions depending on time t, space $\mathbf{x}$, and velocity $\mathbf{v}$. $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields, respectively. P_i is isotropic ion fluid pressure. The kinetic ion number density is n_k, the fluid ion number density is n_i, the electron number density is n_e, which satisfy $n_e = n_i + n_k$ because of the quasi-neutrality condition. $\mathbf{V}$ denotes the ion fluid velocity field. The electron internal energy is taken as $\mathscr{U}_e = \mathscr{U}_e(n_e, |\mathbf{B}|)$, which gives

$$\frac{\partial \mathscr{U}_e}{\partial n_e} = \frac{P_{e\parallel}}{n_e^2}, \qquad \frac{\partial \mathscr{U}_e}{\partial |\mathbf{B}|} = \frac{P_{e\perp} - P_{e\parallel}}{n_e|\mathbf{B}|},$$
$$\frac{\partial \mathscr{U}_e}{\partial \mathbf{B}} = -\gamma \mathbf{B}, \qquad \gamma = \frac{P_{e\parallel} - P_{e\perp}}{n_e|\mathbf{B}|^2}.$$

In Ref. 17, various Poisson brackets are proposed for different, but equivalent, formulations of the kinetic Hall MHD hybrid model (25). Similar to the HKM model, simplified anti-symmetric brackets for the kinetic Hall MHD hybrid model (25) can be obtained by neglecting the terms involving the functional derivative with respect to n_e, while still preserving the equivalence of the resulting equations with the corresponding Hamiltonians. In the following, we determine the condition under which the simplified "canonical bracket" satisfies the Jacobi identity and provide remarks on the magnetic-field-based brackets in Remark 6.

1. Canonical bracket

In Ref. 17, the following canonical bracket, consisting of only five terms, is proposed for the kinetic Hall MHD hybrid model (25). Here, "canonical" refers to the use of canonical ion fluid momentum and canonical ion kinetic momentum,

$$\begin{aligned}
&\{F, G\}(n_i, \bar{\mathbf{M}}, \mathbf{A}, f(\mathbf{x}, \mathbf{p}), n_e) \\
&= \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xp} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int \bar{\mathbf{M}} \cdot \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta \bar{\mathbf{M}}} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta \bar{\mathbf{M}}}\right) \mathrm{d}\mathbf{x} \\
&+ \int n_i \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta n_i} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta n_i}\right) \mathrm{d}\mathbf{x} \\
&- \int \left(\frac{\delta G}{\delta \mathbf{A}} \cdot \nabla \frac{\delta F}{\delta n_e} - \frac{\delta F}{\delta \mathbf{A}} \cdot \nabla \frac{\delta G}{\delta n_e}\right) \mathrm{d}\mathbf{x} \\
&- \int \frac{1}{n_e} \nabla \times \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{A}} \times \frac{\delta G}{\delta \mathbf{A}}\right) \mathrm{d}\mathbf{x}.
\end{aligned} \tag{27}$$

We define the following transformation of the unknowns:

$$(f, n_i, \bar{\mathbf{M}}, \mathbf{A}, n_e) \rightarrow \left(f, n_i, \bar{\mathbf{M}}, \mathbf{A}, C = n_e - n_i - \int f \, \mathrm{d}\mathbf{p}\right).$$

Then any functional $\tilde{F}$ about $f, n_i, \bar{\mathbf{M}}, \mathbf{A}$, and C can be regarded as a functional F about $f, n_i, \bar{\mathbf{M}}, \mathbf{A}$, and n_e. We have the following relations of the functional derivatives:

$$\frac{\delta F}{\delta f} = \frac{\delta \tilde{F}}{\delta f} - \frac{\delta \tilde{F}}{\delta C}, \qquad \frac{\delta F}{\delta n_i} = \frac{\delta \tilde{F}}{\delta n_i} - \frac{\delta \tilde{F}}{\delta C},$$
$$\frac{\delta F}{\delta \bar{\mathbf{M}}} = \frac{\delta \tilde{F}}{\delta \bar{\mathbf{M}}}, \qquad \frac{\delta F}{\delta \mathbf{A}} = \frac{\delta \tilde{F}}{\delta \mathbf{A}}, \qquad \frac{\delta F}{\delta n_e} = \frac{\delta \tilde{F}}{\delta C}.$$

By substituting the above relations into the Poisson bracket (27), we obtain (the tilde is omitted for convenience)

$$\begin{aligned}
&\{F, G\}(n_i, \bar{\mathbf{M}}, \mathbf{A}, f(\mathbf{x}, \mathbf{p}), C) \\
&= \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xp} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} - \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta C}\right]_{xp} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} \\
&- \int f\left[\frac{\delta F}{\delta C}, \frac{\delta G}{\delta f}\right]_{xp} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int \bar{\mathbf{M}} \cdot \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta \bar{\mathbf{M}}} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta \bar{\mathbf{M}}}\right) \mathrm{d}\mathbf{x} \\
&+ \int n_i \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta n_i} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta n_i}\right) \mathrm{d}\mathbf{x} \\
&- \int n_i \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta C} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta C}\right) \mathrm{d}\mathbf{x} \\
&- \int \left(\frac{\delta G}{\delta \mathbf{A}} \cdot \nabla \frac{\delta F}{\delta C} - \frac{\delta F}{\delta \mathbf{A}} \cdot \nabla \frac{\delta G}{\delta C}\right) \mathrm{d}\mathbf{x} \\
&- \int \frac{1}{C + n_i + \int f \, \mathrm{d}\mathbf{p}} \nabla \times \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{A}} \times \frac{\delta G}{\delta \mathbf{A}}\right) \mathrm{d}\mathbf{x}.
\end{aligned} \tag{28}$$

The sum of the second, third, sixth, and seventh terms vanishes when the following condition is satisfied by the functionals,

$$-\nabla \cdot \frac{\delta K}{\delta \mathbf{A}} = \nabla \cdot \left(\int f \frac{\partial}{\partial \mathbf{p}} \frac{\delta K}{\delta f} \mathrm{d}\mathbf{p} + n_i \frac{\delta K}{\delta \bar{\mathbf{M}}}\right), \qquad \forall K. \tag{29}$$

We can verify that this condition (29) is satisfied by the Hamiltonian,

$$\int \frac{1}{2} n_i |n_i^{-1}\bar{\mathbf{M}} - \mathbf{A}|^2 \, \mathrm{d}\mathbf{x} + \int n_i \mathscr{U}(n_i) \, \mathrm{d}\mathbf{x} + \int \frac{1}{2} |\nabla \times \mathbf{A}|^2 \, \mathrm{d}\mathbf{x}$$
$$+ \frac{1}{2} \int |\mathbf{p} - \mathbf{A}|^2 f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int n_e \mathscr{U}_e(n_e, |\nabla \times \mathbf{A}|) \, \mathrm{d}\mathbf{x}.$$

We can define the following set of functionals satisfying the above condition (29):

$$\begin{aligned}
\mathscr{D} = \Big\{ &F(f(\mathbf{x}, \mathbf{p}), \mathbf{A}, n_i, \bar{\mathbf{M}}) := F(\tilde{f}(\mathbf{x}, \mathbf{v}), \mathbf{B}, n_i, \mathbf{V}) | \mathbf{B} = \nabla \times \mathbf{A}, \\
&\mathbf{v} = \mathbf{p} - \mathbf{A}, \mathbf{V} = n_i^{-1}\bar{\mathbf{M}} - \mathbf{A}, f(\mathbf{x}, \mathbf{p}, t) := \tilde{f}(\mathbf{x}, \mathbf{v}) = \tilde{f}(\mathbf{x}, \mathbf{p} - \mathbf{A}) \Big\},
\end{aligned}$$

which is a set of functionals determined by all smooth functionals $\tilde{F}$ about $\tilde{f}(\mathbf{x}, \mathbf{v})$, $\mathbf{B}$, n_i, $\mathbf{V}$ with $\nabla \cdot \mathbf{B} = 0$.

In the bracket (28), the functional derivative with respect to C does not appear. When $C = 0$ and condition (29) is satisfied, the resulting simplified bracket is a Poisson bracket for the functionals in $\mathscr{D}$, which can be proved similarly to Theorem 7 by considering the magnetic-field formulation,

$$\begin{aligned}
&\{F, G\}(n_i, \bar{\mathbf{M}}, \mathbf{A}, f(\mathbf{x}, \mathbf{p})) \\
&= \int f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xp} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int \bar{\mathbf{M}} \cdot \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta \bar{\mathbf{M}}} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta \bar{\mathbf{M}}}\right) \mathrm{d}\mathbf{x} \\
&+ \int n_i \left(\frac{\delta G}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta F}{\delta n_i} - \frac{\delta F}{\delta \bar{\mathbf{M}}} \cdot \nabla \frac{\delta G}{\delta n_i}\right) \mathrm{d}\mathbf{x} \\
&- \int \frac{1}{n_i + \int f \, \mathrm{d}\mathbf{p}} \nabla \times \mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{A}} \times \frac{\delta G}{\delta \mathbf{A}}\right) \mathrm{d}\mathbf{x}.
\end{aligned} \tag{30}$$

Remark 6. For the magnetic-field-based formulations of the kinetic Hall MHD hybrid model (25), the Poisson brackets given in Ref. 17 can be simplified by removing the terms involving the functional derivative of n_e and replacing n_e with $n_i + \int f\, d\mathbf{v}$. Alternatively, the simplified bracket can be obtained via a transformation using the Casimir function $C = n_e - n_i - \int f\, d\mathbf{v}$, as noted in Remark 5. The resulting simplified bracket is a Poisson bracket under Gauss's law for magnetism $\nabla \cdot \mathbf{B} = 0$.

Remark 7. Since Hall MHD is a subset of the hybrid model (25), the terms involving the functional derivative of n_e in the Poisson bracket proposed in Ref. 6 can likewise be removed. For the resulting simplified bracket, the Jacobi identity is satisfied for functionals that fulfill the condition

$$-\nabla \cdot \frac{\delta K}{\delta \mathbf{A}} = \nabla \cdot \left(n_i \frac{\delta K}{\delta \bar{\mathbf{M}}} \right), \qquad \forall K.$$

The simplified magnetic field-based Poisson bracket of Hall MHD has been proposed in Ref. 7.

2. Equivalence of the equations derived from the canonical brackets

Here, we prove that the equations derived from the two brackets above are equivalent. The Hamiltonian proposed in Ref. 17 is given by

$$\begin{aligned} &H_1(f(\mathbf{x},\mathbf{p}), n_i, \bar{\mathbf{M}}, \mathbf{A}, n_e) \\ &= \int \frac{1}{2} n_i |n_i^{-1}\bar{\mathbf{M}} - \mathbf{A}|^2\, d\mathbf{x} + \int n_i \mathscr{U}(n_i)\, d\mathbf{x} + \int n_e \mathscr{U}_e(n_e, |\nabla \times \mathbf{A}|)\, d\mathbf{x} \\ &\quad + \int \frac{1}{2} |\nabla \times \mathbf{A}|^2\, d\mathbf{x} + \frac{1}{2} \int |\mathbf{p} - \mathbf{A}|^2 f\, d\mathbf{x} d\mathbf{p}. \end{aligned} \tag{31}$$

In the Hamiltonian (31), we represent n_e by $n_i + \int f\, d\mathbf{p}$ via the quasi-neutrality condition, and get

$$\begin{aligned} &H_2(f(\mathbf{x},\mathbf{p}), n_i, \bar{\mathbf{M}}, \mathbf{A}) \\ &= \int \frac{1}{2} n_i |n_i^{-1}\bar{\mathbf{M}} - \mathbf{A}|^2\, d\mathbf{x} + \int n_i \mathscr{U}(n_i)\, d\mathbf{x} + \int \frac{1}{2} |\nabla \times \mathbf{A}|^2\, d\mathbf{x} \\ &\quad + \frac{1}{2} \int |\mathbf{p} - \mathbf{A}|^2 f\, d\mathbf{x} d\mathbf{p} \\ &\quad + \int \left(n_i + \int f\, d\mathbf{p} \right) \mathscr{U}_e\left(n_i + \int f\, d\mathbf{p}, |\nabla \times \mathbf{A}| \right) d\mathbf{x}. \end{aligned} \tag{32}$$

Theorem 8. The equations derived from bracket (27) are equivalent to the equations derived from bracket (30) with energy H_1 (31) and H_2 (32), respectively.

Proof. From bracket (27) we have

$$\begin{aligned} \frac{\partial n_i}{\partial t} &= -\nabla \cdot \left(n_i \frac{\delta H_1}{\delta \bar{\mathbf{M}}} \right) = -\nabla \cdot (\bar{\mathbf{M}} - n_i \mathbf{A}) = -\nabla \cdot (n_i \mathbf{V}), \\ \frac{\partial \bar{M}_i}{\partial t} &= \int \bar{\mathbf{M}} \cdot \left(\frac{\delta H_1}{\delta \bar{\mathbf{M}}} \cdot \nabla \delta_i \right) d\mathbf{x} - \int \bar{\mathbf{M}} \cdot \left(\delta_i \cdot \nabla \frac{\delta H_1}{\delta \bar{\mathbf{M}}} \right) d\mathbf{x} - n_i \frac{\partial}{\partial x_i} \frac{\delta H_1}{\delta n_i}, \\ \frac{\partial \mathbf{A}}{\partial t} &= \nabla \frac{\delta H_1}{\delta n_e} + \frac{1}{n_e} \nabla \times \mathbf{A} \times \frac{\delta H_1}{\delta \mathbf{A}}, \\ \frac{\partial n_e}{\partial t} &= \nabla \cdot \frac{\delta H_1}{\delta \mathbf{A}}, \quad \frac{\partial f}{\partial t} = \int f \left[\delta, \frac{\delta H_1}{\delta f} \right]_{xp} d\mathbf{x} d\mathbf{p}, \end{aligned} \tag{33}$$

where $\delta_i = \delta(\mathbf{x} - \mathbf{x}')\mathbf{e}_i$, $\mathbf{e}_i$ is the ith unit vector.

From bracket (30) we have

$$\begin{aligned} \frac{\partial n_i}{\partial t} &= -\nabla \cdot \left(n_i \frac{\delta H_2}{\delta \bar{\mathbf{M}}} \right) = -\nabla \cdot (\bar{\mathbf{M}} - n_i \mathbf{A}) = -\nabla \cdot (n_i \mathbf{V}), \\ \frac{\partial \bar{M}_i}{\partial t} &= \int \bar{\mathbf{M}} \cdot \left(\frac{\delta H_2}{\delta \bar{\mathbf{M}}} \cdot \nabla \delta_i \right) d\mathbf{x} - \int \bar{\mathbf{M}} \cdot \left(\delta_i \cdot \nabla \frac{\delta H_2}{\delta \bar{\mathbf{M}}} \right) d\mathbf{x} - n_i \frac{\partial}{\partial x_i} \frac{\delta H_2}{\delta n_i}, \\ \frac{\partial \mathbf{A}}{\partial t} &= \frac{1}{n_e} \nabla \times \mathbf{A} \times \frac{\delta H_2}{\delta \mathbf{A}}, \quad \frac{\partial f}{\partial t} = \int f \left[\delta, \frac{\delta H_2}{\delta f} \right]_{xp} d\mathbf{x} d\mathbf{p}, \end{aligned} \tag{34}$$

where again $\delta_i = \delta(\mathbf{x} - \mathbf{x}')\mathbf{e}_i$, $\mathbf{e}_i$ is the ith unit vector.

The only difference in the equations for the vector potential $\mathbf{A}$ is the presence of a gradient term, $\nabla \frac{\delta H_1}{\delta n_e}$; the equations for the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ remain unchanged. It is also straightforward to see that the equation for n_i is identical.

As for the distribution function, the equation satisfied by f in Eq. (33) is

$$\frac{\partial f}{\partial t} + (\mathbf{p} - \mathbf{A}) \cdot \nabla f - \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}} \right)^{\top} (\mathbf{A} - \mathbf{p}) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0. \tag{35}$$

We define a new distribution function depending on velocity $\mathbf{v}$ via the relation $\mathbf{p} = \mathbf{v} - \mathbf{A}$ as

$$F(\mathbf{x}, \mathbf{v}, t) = f(\mathbf{x}, \mathbf{p}, t). \tag{36}$$

Then, by plugging the following relations into Eq. (35):

$$\frac{\partial f}{\partial t} = \frac{\partial F}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\partial F}{\partial \mathbf{v}}, \qquad \nabla f = \nabla F - \frac{\partial \mathbf{A}}{\partial \mathbf{x}} \frac{\partial F}{\partial \mathbf{v}}, \qquad \frac{\partial f}{\partial \mathbf{p}} = \frac{\partial F}{\partial \mathbf{v}},$$

we have

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F - \frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\partial F}{\partial \mathbf{v}} + (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial F}{\partial \mathbf{v}} = 0. \tag{37}$$

By similar calculations for the distribution function of f in Eq. (34), we have

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F - \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \frac{\delta H_1}{\delta n_e} \right) \cdot \frac{\partial F}{\partial \mathbf{v}} + (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial F}{\partial \mathbf{v}} = 0. \tag{38}$$

Equation (37) is the same as Eq. (38), due to the difference between the $\frac{\partial \mathbf{A}}{\partial t}$ in Eqs. (37) and (38) is $\nabla \frac{\delta H_1}{\delta n_e}$, which can be seen from the equations satisfied by the vector potential.

Next, we show that the second equations in Eqs. (33) and (34) yield the same equation for $n_i \mathbf{V} := \bar{\mathbf{M}} - n_i \mathbf{A}$. Note that the unknown $\bar{\mathbf{M}}$ in these two equations is defined differently, as the gauge of the vector potential involved in its definition differs.

For the Eq. (33) derived from the bracket (27), with the definition of $\bar{\mathbf{M}}$, we have

$$\begin{aligned} \frac{\partial (n_i \mathbf{V})}{\partial t} &= \frac{\partial \bar{\mathbf{M}}}{\partial t} - \frac{\partial (n_i \mathbf{A})}{\partial t}, \\ \frac{\partial \bar{\mathbf{M}}}{\partial t} &= -\nabla \cdot (n_i \mathbf{V}) \mathbf{V} - n_i \mathbf{V} \cdot \nabla \mathbf{V} - n_i \nabla \big(\mathscr{U}(n_i) + n_i \mathscr{U}'(n_i) \big) \\ &\quad + n_i \mathbf{V} \times \nabla \times \mathbf{A} - \nabla \cdot (n_i \mathbf{V}) \mathbf{A}, \\ \frac{\partial (n_i \mathbf{A})}{\partial t} &= \frac{\partial n_i}{\partial t} \mathbf{A} + n_i \frac{\partial \mathbf{A}}{\partial t} = -\nabla \cdot (n_i \mathbf{V}) \mathbf{A} + n_i \nabla \frac{\delta H_1}{\delta n_e} \\ &\quad + \frac{n_i}{n_e} \nabla \times \mathbf{A} \times \left(\nabla \times \nabla \times \mathbf{A} - n_i \mathbf{V} - \int (\mathbf{p} - \mathbf{A}) f\, d\mathbf{p} \right) \\ &\quad - n_i \nabla \times (\gamma \nabla \times \mathbf{A}). \end{aligned}$$

Then, we obtain the equation satisfied by $n_i\mathbf{V}$,

$$\frac{\partial(n_i\mathbf{V})}{\partial t} = -\nabla\cdot(n_i\mathbf{V})\mathbf{V} - n_i\mathbf{V}\cdot\nabla\mathbf{V} - n_i\nabla\big(\mathcal{U}(n_i) + n_i\mathcal{U}'(n_i)\big) + n_i\mathbf{V}\times\nabla\times\mathbf{A} - n_i\nabla\frac{\delta H_1}{\delta n_e} + n_i\nabla\times(\gamma\mathbf{B}) - \frac{n_i}{n_e}\mathbf{B}\times\left(\nabla\times\mathbf{B} - n_i\mathbf{V} - \int(\mathbf{p}-\mathbf{A})f\,\mathrm{d}\mathbf{p}\right). \tag{39}$$

Next, for the Eq. (34) derived from the bracket (30), we show that the equation satisfied by $n_i\mathbf{V}$ is identical to Eq. (39). By similar calculations, we obtain

$$\frac{\partial(n_i\mathbf{V})}{\partial t} = \frac{\partial\bar{\mathbf{M}}}{\partial t} - \frac{\partial(n_i\mathbf{A})}{\partial t},$$

$$\frac{\partial\bar{\mathbf{M}}}{\partial t} = -\nabla\cdot(n_i\mathbf{V})\mathbf{V} - n_i\mathbf{V}\cdot\nabla\mathbf{V} + n_i\mathbf{V}\times\nabla\times\mathbf{A} - \nabla\cdot(n_i\mathbf{V})\mathbf{A} - n_i\nabla\left(\mathcal{U}(n_i) + n_i\mathcal{U}'(n_i) + \frac{\delta H_1}{\delta n_e}\right),$$

$$\frac{\partial(n_i\mathbf{A})}{\partial t} = -\nabla\cdot(n_i\mathbf{V})\mathbf{A} - n_i\nabla\times(\gamma\nabla\times\mathbf{A}) + \frac{n_i}{n_e}\nabla\times\mathbf{A}\times\left(\nabla\times\nabla\times\mathbf{A} - n_i\mathbf{V} - \int(\mathbf{p}-\mathbf{A})f\,\mathrm{d}\mathbf{p}\right).$$

Then, we have

$$\frac{\partial(n_i\mathbf{V})}{\partial t} = -\nabla\cdot(n_i\mathbf{V})\mathbf{V} - n_i\mathbf{V}\cdot\nabla\mathbf{V} - n_i\nabla\big(\mathcal{U}(n_i) + n_i\mathcal{U}'(n_i)\big) + n_i\mathbf{V}\times\nabla\times\mathbf{A} - n_i\nabla\frac{\delta H_1}{\delta n_e} + n_i\nabla\times(\gamma\mathbf{B}) - \frac{n_i}{n_e}\mathbf{B}\times\left(\nabla\times\mathbf{B} - n_i\mathbf{V} - \int(\mathbf{p}-\mathbf{A})f\,\mathrm{d}\mathbf{p}\right), \tag{40}$$

which is the same as (39). Note that the kinetic currents in Eqs. (39) and (40) are equal, as $\int(\mathbf{p}-\mathbf{A})f\,\mathrm{d}\mathbf{p} = \int\mathbf{v}F(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}$, where F is defined in Eq. (36). □

C. Kinetic-multifluid model

For the two-fluid model, the electron charge density is eliminated using Gauss's law for the electric field in Ref. 20. As a direct generalization, we here consider the kinetic multi-fluid model,

$$\rho_i\frac{\partial\mathbf{u}_i}{\partial t} + \rho_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i = a_i\rho_i(\mathbf{E} + \mathbf{u}_i\times\mathbf{B}) - \nabla p_i,$$

$$\rho_e\frac{\partial\mathbf{u}_e}{\partial t} + \rho_e(\mathbf{u}_e\cdot\nabla)\mathbf{u}_e = a_e\rho_i(\mathbf{E} + \mathbf{u}_e\times\mathbf{B}) - \nabla p_e,$$

$$\frac{\partial\rho_i}{\partial t} + \nabla\cdot(\rho_i\mathbf{u}_i) = 0, \qquad \frac{\partial\rho_e}{\partial t} + \nabla\cdot(\rho_e\mathbf{u}_e) = 0,$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_h}\cdot\nabla f + q_h\left(\mathbf{E} + \frac{\mathbf{p}}{m_h}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial\mathbf{p}} = 0,$$

$$\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} = \nabla\times\mathbf{B} - \mu_0(a_i\rho_i\mathbf{u}_i + a_e\rho_e\mathbf{u}_e) - \mu_0 a_h\int\mathbf{p}f\,\mathrm{d}\mathbf{p},$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \qquad \nabla\cdot\mathbf{B} = 0,$$

$$\varepsilon_0\nabla\cdot\mathbf{E} = a_i\rho_i + a_e\rho_e + q_h\int f\,\mathrm{d}\mathbf{p}, \tag{41}$$

where we use the notations: the charge-to-mass ratio of the fluid s-th species $a_s = q_s/m_s$, mass density ρ_s, velocity $\mathbf{u}_s$, the distribution function $f(\mathbf{x},\mathbf{p},t)$ of the ions depending on time t, position $\mathbf{x}$, and momentum $\mathbf{p}$, the kinetic ion mass m_h, kinetic ion charge q_h, electric field $\mathbf{E}$, and magnetic field $\mathbf{B}$.

In Ref. 8, the following Poisson bracket is proposed:

$$\begin{aligned}\{F,G\} &= \sum_{s=i,e}\int\mathbf{m}_s\cdot\left[\frac{\delta F}{\delta\mathbf{m}_s},\frac{\delta G}{\delta\mathbf{m}_s}\right]\mathrm{d}\mathbf{x} + \int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]_{xp}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\ &- \sum_{s=i,e}\int\rho_s\left(\frac{\delta F}{\delta\mathbf{m}_s}\cdot\nabla\frac{\delta G}{\delta\rho_s} - \frac{\delta G}{\delta\mathbf{m}_s}\cdot\nabla\frac{\delta F}{\delta\rho_s}\right)\mathrm{d}\mathbf{x}\\ &+ \sum_{s=i,e}\int a_s\rho_s\left(\frac{1}{\varepsilon_0}\frac{\delta F}{\delta\mathbf{m}_s}\cdot\frac{\delta G}{\delta\mathbf{E}} - \frac{1}{\varepsilon_0}\frac{\delta G}{\delta\mathbf{m}_s}\cdot\frac{\delta F}{\delta\mathbf{E}}\right)\mathrm{d}\mathbf{x}\\ &+ q_h\int f\left(\frac{1}{\varepsilon_0}\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\cdot\frac{\delta G}{\delta\mathbf{E}} - \frac{1}{\varepsilon_0}\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}\cdot\frac{\delta F}{\delta\mathbf{E}}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}\\ &+ \frac{1}{\varepsilon_0}\int\left(\frac{\delta F}{\delta\mathbf{E}}\cdot\nabla\times\frac{\delta G}{\delta\mathbf{B}} - \frac{\delta G}{\delta\mathbf{E}}\cdot\nabla\times\frac{\delta F}{\delta\mathbf{B}}\right)\mathrm{d}\mathbf{x}\\ &+ \sum_{s=i,e}\int a_s\rho_s\left(\mathbf{B}\cdot\frac{\delta F}{\delta\mathbf{m}_s}\times\frac{\delta G}{\delta\mathbf{m}_s}\right)\mathrm{d}\mathbf{x}\\ &+ q_h\int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{p}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial\mathbf{p}}\frac{\delta G}{\delta f}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p},\end{aligned} \tag{42}$$

where $\mathbf{m}_s = \rho_s\mathbf{u}_s$. Hamiltonian is

$$\frac{1}{2}\sum_{s=i,e}\int\frac{|\mathbf{m}_s|^2}{\rho_s}\mathrm{d}\mathbf{x} + \frac{1}{2m_h}\int f|\mathbf{p}|^2\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \sum_{s=i,e}\int\rho_s\mathcal{U}(\rho_s)\,\mathrm{d}\mathbf{x} + \frac{\varepsilon_0}{2}\int|\mathbf{E}|^2\mathrm{d}\mathbf{x} + \frac{1}{2\mu_0}\int|\mathbf{B}|^2\mathrm{d}\mathbf{x}. \tag{43}$$

Gauss's law for the electric field $\varepsilon_0\nabla\cdot\mathbf{E} = a_i\rho_i + a_e\rho_e + q_h\int f\,\mathrm{d}\mathbf{p}$, is treated here as a constraint and yields

$$\rho_e = a_e^{-1}\left(\varepsilon_0\nabla\cdot\mathbf{E} - a_i\rho_i - q_h\int f\,\mathrm{d}\mathbf{p}\right),$$

based on which we can define the following transformation and get a new bracket:

$$(\rho_i,\rho_e,\mathbf{m}_i,\mathbf{m}_e,f,\mathbf{E},\mathbf{B}) \to \left(\rho_i, C = \rho_e - a_e^{-1}\left(\varepsilon_0\nabla\cdot\mathbf{E} - a_i\rho_i - q_h\int f\,\mathrm{d}\mathbf{p}\right),\mathbf{m}_i,\mathbf{m}_e,f,\mathbf{E},\mathbf{B}\right).$$

Any smooth functional $\tilde{F}$ about $(\rho_i, C, \mathbf{m}_i, \mathbf{m}_e, f, \mathbf{E}, \mathbf{B})$ can be considered as a functional F about $(\rho_i, \rho_e, \mathbf{m}_i, \mathbf{m}_e, f, \mathbf{E}, \mathbf{B})$, and we have the following relations of the functional derivatives:

$$\frac{\delta F}{\delta f} = \frac{\delta \tilde{F}}{\delta f} + a_e^{-1} q_h \frac{\delta \tilde{F}}{\delta C}, \qquad \frac{\delta F}{\delta \rho_i} = \frac{\delta \tilde{F}}{\delta \rho_i} + a_e^{-1} a_i \frac{\delta \tilde{F}}{\delta C},$$
$$\frac{\delta F}{\delta \rho_e} = \frac{\delta \tilde{F}}{\delta C}, \qquad \frac{\delta F}{\delta \mathbf{m}_i} = \frac{\delta \tilde{F}}{\delta \mathbf{m}_i}, \tag{44}$$
$$\frac{\delta F}{\delta \mathbf{m}_e} = \frac{\delta \tilde{F}}{\delta \mathbf{m}_e}, \qquad \frac{\delta F}{\delta \mathbf{B}} = \frac{\delta \tilde{F}}{\delta \mathbf{B}}, \qquad \frac{\delta F}{\delta \mathbf{E}} = \frac{\delta \tilde{F}}{\delta \mathbf{E}} + a_e^{-1} \varepsilon_0 \nabla \frac{\delta \tilde{F}}{\delta C}.$$

By substituting the above relations into the bracket (42), we obtain the simplified Poisson bracket (the tilde is omitted here for convenience),

$$\begin{aligned}
\{F, G\}_s = & \sum_{s=i,e} \int \mathbf{m}_s \cdot \left[\frac{\delta F}{\delta \mathbf{m}_s}, \frac{\delta G}{\delta \mathbf{m}_s}\right] \mathbf{dx} + \int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xp} \mathbf{dxdp} \\
& - \int \rho_i \left(\frac{\delta F}{\delta \mathbf{m}_i} \cdot \nabla \frac{\delta G}{\delta \rho_i} - \frac{\delta G}{\delta \mathbf{m}_i} \cdot \nabla \frac{\delta F}{\delta \rho_i}\right) \mathbf{dx} \\
& + \sum_{s=i,e} \int a_s \rho_s \left(\frac{1}{\varepsilon_0} \frac{\delta F}{\delta \mathbf{m}_s} \cdot \frac{\delta G}{\delta \mathbf{E}} - \frac{1}{\varepsilon_0} \frac{\delta G}{\delta \mathbf{m}_s} \cdot \frac{\delta F}{\delta \mathbf{E}}\right) \mathbf{dx} \\
& + q_h \int f \left(\frac{1}{\varepsilon_0} \frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \cdot \frac{\delta G}{\delta \mathbf{E}} - \frac{1}{\varepsilon_0} \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f} \cdot \frac{\delta F}{\delta \mathbf{E}}\right) \mathbf{dxdp} \\
& + \frac{1}{\varepsilon_0} \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \nabla \times \frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{E}} \cdot \nabla \times \frac{\delta F}{\delta \mathbf{B}}\right) \mathbf{dx} \\
& + \sum_{s=i,e} \int a_s \rho_s \left(\mathbf{B} \cdot \frac{\delta F}{\delta \mathbf{m}_s} \times \frac{\delta G}{\delta \mathbf{m}_s}\right) \mathbf{dx} \\
& + q_h \int f \mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{p}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{p}} \frac{\delta G}{\delta f}\right) \mathbf{dxdp}.
\end{aligned} \tag{45}$$

As seen in Eq. (45), there is no term involving C or the functional derivative with respect to C; therefore, C can be treated as a parameter.

We substitute the relation

$$\rho_e = a_e^{-1}\left(\varepsilon_0 \nabla \cdot \mathbf{E} - a_i \rho_i - q_h \int f \, \mathbf{dp}\right),$$

into the Hamiltonian (43), obtaining a new Hamiltonian that no longer depends on ρ_e. Using this new Hamiltonian together with the bracket (45), one can derive the hybrid model (41) without the equation for ρ_e and without Gauss's law for the electric field $\varepsilon_0 \nabla \cdot \mathbf{E} = a_i \rho_i + a_e \rho_e + q_h \int f \, \mathbf{dp}$. From a numerical perspective, there is no need to explicitly enforce this constraint, as it is automatically satisfied by the formulation.

IV. CONCLUSION

In this work, we investigate the quasi-neutrality condition for the HKM model (hybrid model with kinetic ions and massless electrons). In the canonical-momentum formulation, this condition is associated with a momentum map, whereas in the velocity-based formulations it appears as a Casimir invariant.

We then determine the conditions under which the three simplified Poisson brackets proposed in Refs. 10 and 12 based on the use of the quasi-neutrality condition satisfy the Jacobi identity. The Jacobi identity is proved either by comparison with the terms in the Jacobi identity for the Poisson brackets of Ref. 8, or by deriving the simplified brackets from the brackets of Ref. 8 through variable transformations associated with the quasi-neutrality condition. In particular, for the magnetic-field-based bracket,[10] we provide a direct and self-contained proof of the Jacobi identity.

These results are further extended to the HKM model with electron entropy and to the kinetic Hall MHD hybrid model with anisotropic-pressure electrons. In both cases, the Poisson brackets can be simplified under the quasi-neutrality condition, and the corresponding Jacobi identity conditions are obtained. For the kinetic-multifluid model, an analogous simplification of the Poisson bracket is achieved using Gauss's law for the electric field.[20]

The simplified Poisson brackets and their associated formulations possess several numerical advantages. In particular, because the quasi-neutrality constraint or Gauss's law for the electric field is eliminated, it does not need to be enforced numerically, being already built into the continuous system. In addition, the antisymmetry of the simplified Poisson brackets provides a foundation for constructing energy-conserving numerical schemes with good long-time properties, following the approach of Ref. 21.

These properties are particularly important for hybrid kinetic simulations of collisionless plasmas, where long-time integration is often required to capture multiscale processes such as magnetic reconnection, collisionless shocks, plasma turbulence, and solar wind magnetosphere interactions. In such applications, numerical violations of fundamental invariants may lead to artificial dissipation or spurious energy growth that obscures the underlying plasma dynamics. By incorporating quasi-neutrality or Gauss's law for the electric field directly into the Hamiltonian structure, the simplified Poisson brackets developed here provide a rigorous theoretical foundation for structure-preserving numerical methods. We expect that these formulations will facilitate the development of robust and efficient simulation tools for large-scale studies of space-weather phenomena and other astrophysical plasma processes.

ACKNOWLEDGMENTS

Philip J. Morrison received support from the DOE Office of Fusion Energy Sciences under No. DE-FG02-04ER-54742. Yingzhe Li would like to acknowledge his late grandfather, Kaojin Li (1944–2025), who planted in him the seeds of a lifelong pursuit of truth and knowledge.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Yingzhe Li: Conceptualization (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal). **Philip J. Morrison:** Conceptualization (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal). **Stefan Possanner:** Conceptualization (supporting); Investigation (supporting); Methodology (supporting); Supervision (equal); Writing – original draft (supporting). **Eric Sonnendrüecker:** Conceptualization (supporting); Investigation (supporting); Methodology (supporting); Supervision (equal); Writing – original draft (supporting).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: DETAILS OF THE PROOF OF THEOREM 6

In this appendix, we provide further details of Theorem 6. In particular, we present a detailed proof that the sum over all permutations of each term in Eq. (18) vanishes. In the following, "cyc" denotes the terms by permutation of F, G, and K. For convenience, we use the notations for $\forall F$,

$$\mathbf{F} = \int f \frac{\partial}{\partial \mathbf{v}} \frac{\delta F}{\delta f} \mathrm{d}\mathbf{v}, \qquad \mathbf{F}^A = \nabla \times \frac{\delta F}{\delta \mathbf{B}}, \qquad F_f = \frac{\delta F}{\delta f},$$

and the ith components of $\mathbf{F}$ and $\mathbf{F}^A$ are denoted as F_i and F_i^A, respectively. Also, we use $\partial_{x_i} = \frac{\partial}{\partial x_i}, \partial_{v_i} = \frac{\partial}{\partial v_i}, i = 1, 2, 3$. The following lemma given in Ref. 18 and the following three-vector identities are used in many places.

Lemma 1. Morrison[18] For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and vector field $\mathbf{d}$ in $\mathbb{R}^3$,

$$\mathbf{c} \cdot \nabla \mathbf{d} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot \nabla \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \cdot \nabla \mathbf{d} \cdot (\mathbf{c} \times \mathbf{a}) = \nabla \cdot \mathbf{d}(\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})),$$

where the notation $\cdot \nabla \mathbf{d} \cdot$ is defined as $\mathbf{m} \cdot \nabla \mathbf{d} \cdot \mathbf{n} = m_i n_j \partial_{x_i} d_j$, here $\mathbf{m}$ and $\mathbf{n}$ are two any vectors in $\mathbb{R}^3$, and x_i is the coordinate variable corresponding to the ith direction in $\mathbb{R}^3$.

Vector identities

$$\nabla \times (\mathbf{X} \times \mathbf{Y}) = \mathbf{X}(\nabla \cdot \mathbf{Y}) - \mathbf{Y}(\nabla \cdot \mathbf{X}) + (\mathbf{Y} \cdot \nabla)\mathbf{X} - (\mathbf{X} \cdot \nabla)\mathbf{Y}, \tag{A1}$$

$$\mathbf{Z} \times (\mathbf{X} \times \mathbf{Y}) = (\mathbf{Z} \cdot \mathbf{Y})\mathbf{X} - (\mathbf{Z} \cdot \mathbf{X})\mathbf{Y}, \tag{A2}$$

$$\nabla \cdot (\mathbf{X} \times \mathbf{Y}) = (\nabla \times \mathbf{X}) \cdot \mathbf{Y} - (\nabla \times \mathbf{Y}) \cdot \mathbf{X}, \tag{A3}$$

$$\nabla \times (f\mathbf{X}) = \nabla f \times \mathbf{X} + f(\nabla \times \mathbf{X}), \tag{A4}$$

where $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are three-vector fields and f is a scalar function defined on $\mathbb{R}^3$.

The first term

$\{\{F, G\}, K\}_{fff}^{f} + \text{cyc} = \int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xv} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} + \text{cyc} = 0,$ because $\int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]_{xv} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$ is a Lie–Poisson bracket.[2,3]

The second term

$$\begin{aligned}
&\{\{F, G\}, K\}_{fff}^{fB} + \text{cyc} \\
&= \int f \left[\mathbf{B} \cdot \left(\frac{\partial F_f}{\partial \mathbf{v}} \times \frac{\partial G_f}{\partial \mathbf{v}}\right), K_f\right]_{xv} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad + \int f\mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{v}}[F_f, G_f]_{xv} \times \frac{\partial K_f}{\partial \mathbf{v}}\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} + \text{cyc} \\
&= \underbrace{\int f \frac{\partial K_f}{\partial \mathbf{v}} \cdot \nabla \mathbf{B} \cdot \left(\frac{\partial F_f}{\partial \mathbf{v}} \times \frac{\partial G_f}{\partial \mathbf{v}}\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{\text{Term 1}} \\
&\quad + \underbrace{\int f \frac{\partial K_f}{\partial \mathbf{v}} \cdot \nabla \left(\frac{\partial F_f}{\partial \mathbf{v}} \times \frac{\partial G_f}{\partial \mathbf{v}}\right) \cdot \mathbf{B}\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{\text{Term 2}} \\
&\quad \underbrace{- \int f \nabla K_f \cdot \frac{\partial}{\partial \mathbf{v}} \left(\frac{\partial F_f}{\partial \mathbf{v}} \times \frac{\partial G_f}{\partial \mathbf{v}}\right) \cdot \mathbf{B}\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{\text{Term 3}} \\
&\quad + \underbrace{\int f\mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{v}}[F_f, G_f]_{xv} \times \frac{\partial K_f}{\partial \mathbf{v}}\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{\text{Term 4}} + \text{cyc},
\end{aligned} \tag{A5}$$

where $M \cdot \mathscr{D}N \cdot K = M_j \mathscr{D}_j N_i K_i$, M, N, K are three-vector fields in $\mathbb{R}^3$, and $\mathscr{D}$ is ∇ or $\frac{\partial}{\partial \mathbf{v}}$.

For the Term 4 +cyc, we have

$$\begin{aligned}
&\int f\mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{v}}[F_f, G_f]_{xv} \times \frac{\partial K_f}{\partial \mathbf{v}}\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} + \text{cyc} \\
&= \underbrace{\int f B_n \varepsilon_{nlm} \partial_{v_l} \partial_{x_i} F_f \partial_{v_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{c} \\
&\quad + \underbrace{\int f B_n \varepsilon_{nlm} \partial_{x_i} F_f \partial_{v_l} \partial_{v_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{a} \\
&\quad \underbrace{- \int f B_n \varepsilon_{nlm} \partial_{v_l} \partial_{v_i} F_f \partial_{x_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{b} \\
&\quad \underbrace{- \int f B_n \varepsilon_{nlm} \partial_{v_i} F_f \partial_{v_l} \partial_{x_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{d} + \text{cyc},
\end{aligned}$$

where we use the product rule to differentiate the product of two functions, and express the cross product using the Levi–Civita symbol ε.

For the Term 2 +cyc, by the permutation of F, G, and K, we have

$$\begin{aligned}
&\underbrace{\int f B_n \varepsilon_{nlm} \partial_{v_i} F_f \partial_{v_l} \partial_{x_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{d} \\
&\quad + \underbrace{\int f B_n \varepsilon_{nlm} \partial_{v_m} \partial_{x_i} F_f \partial_{v_i} G_f \partial_{v_l} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{c} + \text{cyc}.
\end{aligned}$$

For Term 3 +cyc, by the permutation of F, G, and K, we have

$$\begin{aligned}
&\underbrace{- \int f B_n \varepsilon_{nlm} \partial_{x_i} F_f \partial_{v_l} \partial_{v_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{a} \\
&\quad + \underbrace{\int f B_n \varepsilon_{nlm} \partial_{v_l} \partial_{v_i} F_f \partial_{x_i} G_f \partial_{v_m} K_f \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}}_{b} + \text{cyc}.
\end{aligned}$$

The terms with the same label cancel with each other.

By Lemma 1, we have the sum of the permutations of Term 1 as

$$\int f(\nabla \cdot \mathbf{B}) \frac{\partial K_f}{\partial \mathbf{v}} \cdot \left(\frac{\partial F_f}{\partial \mathbf{v}} \times \frac{\partial G_f}{\partial \mathbf{v}}\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v},$$

which gives that $\{\{F, G\}, K\}_{fff}^{fB} + \text{cyc} = 0$ under Gauss's law for magnetism $\nabla \cdot \mathbf{B} = 0$.

The third term

$$
\begin{aligned}
&\{\{F,G\},K\}^{ffB/n}_{fff}+\mathrm{cyc}\\
&\quad=-\int f\left[\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}\right),K_f\right]_{xv}\mathbf{dxdv}\\
&\qquad-\int f\left[\frac{\mathbf{B}}{n}\cdot\left(\mathbf{F}\times\frac{\partial G_f}{\partial\mathbf{v}}\right),K_f\right]_{xv}\mathbf{dxdv}\\
&\qquad-\int\frac{1}{n}\mathbf{B}\cdot\left(\mathbf{K}\times\nabla\times\int f\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\mathbf{dv}\right)\mathbf{dx}\\
&\qquad-\int\frac{\mathbf{B}}{n}\left(\int f\frac{\partial}{\partial\mathbf{v}}[F_f,G_f]_{xv}\mathbf{dv}\times\mathbf{K}\right)\mathbf{dx}+\mathrm{cyc}.
\end{aligned}
$$

By integrating by parts, the derivatives with respect to $\mathbf{x}$ in the first part of the Lie–Poisson brackets in the first and second terms, and using formula (A4) for the third term, we obtain

$$
\begin{aligned}
&\{\{F,G\},K\}^{ffB/n}_{fff}+\mathrm{cyc}\\
&=\int\left(\nabla f\cdot\frac{\partial K_f}{\partial\mathbf{v}}+f\nabla\cdot\frac{\partial K_f}{\partial\mathbf{v}}\right)\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}\right)\mathbf{dvdx}\\
&\quad+\int\frac{f}{n}\nabla K_f\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}\right)\cdot\mathbf{Bdvdx}\\
&\quad+\int\left(\nabla f\cdot\frac{\partial K_f}{\partial\mathbf{v}}+f\nabla\cdot\frac{\partial K_f}{\partial\mathbf{v}}\right)\frac{\mathbf{B}}{n}\cdot\left(\mathbf{F}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\mathbf{dvdx}\\
&\quad+\int\frac{f}{n}\nabla K_f\cdot\frac{\partial}{\partial\mathbf{v}}\left(\mathbf{F}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\cdot\mathbf{Bdvdx}\\
&\quad-\int\frac{\mathbf{B}}{n}\cdot\left(\mathbf{K}\times\int\nabla f\times\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\mathbf{dv}\right)\mathbf{dx}\\
&\quad-\int\frac{\mathbf{B}}{n}\cdot\left(\mathbf{K}\times\int f\nabla\times\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\mathbf{dv}\right)\mathbf{dx}\\
&\quad-\int\frac{\mathbf{B}}{n}\cdot\left(\int f\frac{\partial}{\partial\mathbf{v}}[F_f,G_f]_{xv}\mathbf{dv}\times\mathbf{K}\right)\mathbf{dx}+\mathrm{cyc}\\
&=\int f\mathbf{K}\times\frac{\mathbf{B}}{n}\cdot\text{Term 2 }\mathbf{dxdv}+\int\mathbf{K}\times\frac{\mathbf{B}}{n}\cdot\text{Term 3 }\mathbf{dxdv}+\mathrm{cyc},
\end{aligned}
$$

where Term 3 is

$$
\frac{\partial G_f}{\partial\mathbf{v}}\nabla f\cdot\frac{\partial F_f}{\partial\mathbf{v}}-\frac{\partial F_f}{\partial\mathbf{v}}\nabla f\cdot\frac{\partial G_f}{\partial\mathbf{v}}+\nabla f\times\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)=0,
$$

with the help of formula (A2). As for Term 2, we have

$$
\begin{aligned}
\text{Term 2}&=\frac{\partial G_f}{\partial\mathbf{v}}\nabla\cdot\frac{\partial F_f}{\partial\mathbf{v}}+\nabla F_f\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\partial G_f}{\partial\mathbf{v}}\right)-\frac{\partial F_f}{\partial\mathbf{v}}\nabla\cdot\frac{\partial G_f}{\partial\mathbf{v}}\\
&\quad-\nabla G_f\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\partial F_f}{\partial\mathbf{v}}\right)+\nabla\times\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\\
&\quad-\frac{\partial}{\partial\mathbf{v}}\left(\nabla F_f\cdot\frac{\partial G_f}{\partial\mathbf{v}}-\frac{\partial F_f}{\partial\mathbf{v}}\cdot\nabla G_f\right)=0,
\end{aligned}
$$

with the help of formula (A1). Then, we have shown that $\{\{F,G\},K\}^{ffB/n}_{fff}+\mathrm{cyc}=0.$

The fourth term

$$
\begin{aligned}
&\{\{F,G\},K\}^{fffB/n^2}_{fff}+\mathrm{cyc}\\
&=\int f\left[\frac{\mathbf{B}}{n^2}\cdot(\mathbf{F}\times\mathbf{G}),K_f\right]_{xv}\mathbf{dxdv}\\
&\quad+\int\frac{1}{n}\mathbf{B}\cdot\left(\mathbf{K}\times\nabla\times\left(\frac{1}{n}\mathbf{F}\times\mathbf{G}\right)\right)\mathbf{dx}\\
&=\int\nabla\left(\frac{\mathbf{B}}{n^2}\cdot(\mathbf{F}\times\mathbf{G})\right)\cdot\mathbf{Kdx}+\int\frac{1}{n^2}\mathbf{B}\cdot(\mathbf{K}\times\nabla\times(\mathbf{F}\times\mathbf{G}))\mathbf{dx}\\
&\quad+\frac{1}{2}\int\mathbf{B}\cdot\left(\mathbf{K}\times\left(\nabla\left(\frac{1}{n^2}\right)\times(\mathbf{F}\times\mathbf{G})\right)\right)\mathbf{dx}+\mathrm{cyc}\\
&=\int\frac{1}{n^2}\left(\underbrace{-\mathbf{B}\cdot(\mathbf{F}\times\mathbf{G})\nabla\cdot\mathbf{K}+\mathrm{cyc}}_{\text{Term 1}}\right)\mathbf{dx},\\
&\quad+\int\frac{1}{n^2}\left(\underbrace{\frac{\mathbf{B}}{2}\cdot(\mathbf{K}\times\nabla\times(\mathbf{F}\times\mathbf{G}))+\mathrm{cyc}}_{\text{Term 2}}\right)\mathbf{dx},\\
&\quad+\int\frac{1}{n^2}\underbrace{\frac{1}{2}(\mathbf{F}\times\mathbf{G})\cdot\nabla\times(\mathbf{B}\times\mathbf{K})+\mathrm{cyc}\ \mathbf{dx}}_{\text{Term 3}},
\end{aligned}
$$

where we have used formula (A3).

By using formula (A1), the Term 2 can be expanded as

$$
\begin{aligned}
&\frac{\mathbf{B}}{2}\cdot(\mathbf{K}\times\nabla\times(\mathbf{F}\times\mathbf{G}))+\mathrm{cyc}\\
&\quad=\underbrace{(\nabla\cdot\mathbf{G})\frac{\mathbf{B}}{2}\cdot(\mathbf{K}\times\mathbf{F})-(\nabla\cdot\mathbf{F})\frac{\mathbf{B}}{2}\cdot(\mathbf{K}\times\mathbf{G})+\mathrm{cyc}}_{=\mathbf{B}\cdot(\mathbf{F}\times\mathbf{G})\nabla\cdot\mathbf{K}+\mathrm{cyc}=-\text{Term 1}}\\
&\qquad+\underbrace{\frac{\mathbf{B}}{2}\cdot(\mathbf{K}\times(\mathbf{G}\cdot\nabla)\mathbf{F})-\frac{\mathbf{B}}{2}\cdot(\mathbf{K}\times(\mathbf{F}\cdot\nabla)\mathbf{G})}_{\text{Term 4}}+\mathrm{cyc}.
\end{aligned}
\tag{A6}
$$

By using formula (A1), Term 3 can be expanded as

$$
\begin{aligned}
&\underbrace{\frac{1}{2}(\nabla\cdot\mathbf{K})(\mathbf{F}\times\mathbf{G})\cdot\mathbf{B}+\mathrm{cyc}}_{\text{Term 6}}\underbrace{-\frac{1}{2}(\nabla\cdot\mathbf{B})(\mathbf{F}\times\mathbf{G})\cdot\mathbf{K}+\mathrm{cyc}}_{\text{Term 7}}\\
&\quad+\underbrace{\frac{1}{2}(\mathbf{F}\times\mathbf{G})\cdot(\mathbf{K}\cdot\nabla)\mathbf{B}+\mathrm{cyc}}_{\text{Term 8}}\underbrace{-\frac{1}{2}(\mathbf{F}\times\mathbf{G})\cdot(\mathbf{B}\cdot\nabla)\mathbf{K}+\mathrm{cyc}.}_{\text{Term 5}}
\end{aligned}
\tag{A7}
$$

By permutation of the F, G, and K in Term 4 and summing up with Term 5, we obtain

$$
\begin{aligned}
&\frac{\mathbf{B}}{2}\cdot(\mathbf{G}\times(\mathbf{F}\cdot\nabla)\mathbf{K})-\frac{\mathbf{B}}{2}\cdot(\mathbf{F}\times(\mathbf{G}\cdot\nabla)\mathbf{K})-\frac{1}{2}(\mathbf{F}\times\mathbf{G})\cdot(\mathbf{B}\cdot\nabla)\mathbf{K}\\
&\quad=\frac{1}{2}(\nabla\cdot\mathbf{K})(\mathbf{F}\cdot(\mathbf{B}\times\mathbf{G}))=-\text{Term 6},
\end{aligned}
\tag{A8}
$$

where we have used Lemma 1. Then, we know that

$$
\text{Term 4}+\mathrm{cyc}+\text{Term 5}+\mathrm{cyc}+\text{Term 6}+\mathrm{cyc}=0.
$$

By Lemma 1, we have

$$
\text{Term 8}+\mathrm{cyc}=\frac{1}{2}\nabla\cdot\mathbf{B}(\mathbf{K}\cdot(\mathbf{F}\times\mathbf{G})).
$$

As we know that Term 7 + cyc $= -\frac{3}{2}\nabla\cdot\mathbf{B}(\mathbf{K}\cdot(\mathbf{F}\times\mathbf{G}))$, we have

$$\text{Term } 7 + \text{cyc} + \text{Term } 8 + \text{cyc} = -\nabla\cdot\mathbf{B}(\mathbf{K}\cdot(\mathbf{F}\times\mathbf{G})).$$

In the end, we know

$$\{\{F,G\},K\}_{fff}^{fffB/n^2} + \text{cyc} = \int \frac{1}{n^2}(-\nabla\cdot\mathbf{B}(\mathbf{K}\cdot(\mathbf{F}\times\mathbf{G})))\mathbf{dx} = 0,$$

under Gauss's law for magnetism $\nabla\cdot\mathbf{B}=0$.

The fifth term

$$\begin{aligned}&\{\{F,G\},K\}_{fff}^{fB^2} + \text{cyc}\\ &= \int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\left(\mathbf{B}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\right)\times\frac{\partial K_f}{\partial\mathbf{v}}\right)\mathbf{dxdv} + \text{cyc}.\end{aligned}$$

We use the product rule to differentiate the product of two functions, and express the cross product using the Levi–Civita symbol ε, and have

$$\begin{aligned}&\{\{F,G\},K\}_{fff}^{fB^2} + \text{cyc}\\ &= \int fB_iB_l\varepsilon_{ijk}\varepsilon_{lmn}\partial_{v_m}\partial_{v_j}F_f\partial_{v_k}K_f\partial_{v_n}G_f\mathbf{dxdv}\\ &\quad - \int fB_iB_l\varepsilon_{ijk}\varepsilon_{lmn}\partial_{v_m}\partial_{v_j}G_f\partial_{v_k}F_f\partial_{v_n}K_f\mathbf{dxdv} + \text{cyc}\\ &= \int fB_iB_l\varepsilon_{ijk}\varepsilon_{lmn}\partial_{v_m}\partial_{v_j}F_f\partial_{v_k}K_f\partial_{v_n}G_f\mathbf{dxdv}\\ &\quad - \int fB_iB_l\varepsilon_{ijk}\varepsilon_{lmn}\partial_{v_m}\partial_{v_j}F_f\partial_{v_k}K_f\partial_{v_n}G_f\mathbf{dxdv} + \text{cyc} = 0,\end{aligned}$$

where in the second last step, the permutation of F, G, K is used.

The sixth term

$$\begin{aligned}&\{\{F,G\},K\}_{fff}^{ffB^2/n} + \text{cyc}\\ &= -\int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}\right)\right)\times\frac{\partial K_f}{\partial\mathbf{v}}\right)\mathbf{dxdv}\\ &\quad - \int f\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\mathbf{F}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\right)\times\frac{\partial K_f}{\partial\mathbf{v}}\right)\mathbf{dxdv}\\ &\quad - \int\frac{\mathbf{B}}{n}\cdot\left(\left(\int f\frac{\partial}{\partial\mathbf{v}}\left(\mathbf{B}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\mathbf{dv}\right)\times\mathbf{K}\right)\mathbf{dx} + \text{cyc}\\ &= -\int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_f\partial_{v_k}K_fG_n\,\mathbf{dxdv}\\ &\quad + \int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_f\partial_{v_k}K_fF_n\,\mathbf{dxdv}\\ &\quad + \int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_f\partial_{v_k}F_fK_n\,\mathbf{dxdv}\\ &\quad - \int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_f\partial_{v_k}G_fK_n\,\mathbf{dxdv} + \text{cyc},\end{aligned}$$

where we use the product rule to differentiate the product of two functions, and express the cross product using the Levi–Civita symbol ε. By the permutation of F, G, and K, we know that $\{\{F,G\},K\}_{fff}^{ffB^2/n}$+cyc $= 0$.

The seventh term

$$\begin{aligned}&\{\{F,G\},K\}_{fff}^{fffB^2/n^2}\text{+cyc}\\ &= \int\frac{\mathbf{B}}{n}\cdot\left(\left(\int f\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}\right)\mathbf{dv}\right)\times\mathbf{K}\right)\mathbf{dx}\\ &\quad + \int\frac{\mathbf{B}}{n}\cdot\left(\left(\int f\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\mathbf{F}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\mathbf{dv}\right)\times\mathbf{K}\right)\mathbf{dx} + \text{cyc}\\ &= \int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_fG_kK_n\,\mathbf{dxdv}\\ &\quad - \int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_fF_kK_n\,\mathbf{dxdv} + \text{cyc},\end{aligned}$$

where we express the cross product using the Levi–Civita symbol ε. By the permutation of F, G, and K, we know that $\{\{F,G\},K\}_{fff}^{fffB^2/n^2} + \text{cyc} = 0$.

The eighth term

$$\begin{aligned}&\{\{F,G\},K\}_{ffB}^{fB/n}\text{+cyc}\\ &= -\int f\left[\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{F}^A - \frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}^A\right),K_f\right]_{xv}\mathbf{dxdv}\\ &\quad - \int\frac{\mathbf{B}}{n}\cdot\left(\left(\nabla\times\int f\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\mathbf{dv}\right)\times\mathbf{K}^A\right)\mathbf{dx}\\ &\quad + \int\frac{f}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}[F_f,G_f]_{xv}\times\mathbf{K}^A\right)\mathbf{dxdv} + \text{cyc}.\end{aligned}$$

By integrating by parts, the derivatives with respect to $\mathbf{x}$ in the first part of the Lie–Poisson bracket in the first term, and using formula (A4) for the second term, we obtain

$$\begin{aligned}&\{\{F,G\},K\}_{ffB}^{fB/n}\text{+cyc}\\ &= \int\nabla f\cdot\frac{\partial G_f}{\partial\mathbf{v}}\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\mathbf{dxdv}\\ &\quad - \int\nabla f\cdot\frac{\partial F_f}{\partial\mathbf{v}}\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\mathbf{dxdv}\\ &\quad - \int\frac{\mathbf{B}}{n}\cdot\left(\nabla f\times\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\right)\times\mathbf{K}^A\mathbf{dxdv}\\ &\quad + \int f\left(\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\right)\right)\cdot\frac{\partial G_f}{\partial\mathbf{x}}\mathbf{dxdv}\\ &\quad - \int f\left(\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\right)\right)\cdot\frac{\partial F_f}{\partial\mathbf{x}}\mathbf{dxdv}\\ &\quad - \int f\frac{\mathbf{B}}{n}\cdot\left(\nabla\times\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\times\mathbf{K}^A\right)\mathbf{dxdv}\\ &\quad + \int\frac{f}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}[F_f,G_f]_{xv}\times\mathbf{K}^A\right)\mathbf{dxdv}\\ &\quad + \int f\nabla\cdot\frac{\partial G_f}{\partial\mathbf{v}}\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\mathbf{dxdv}\\ &\quad - \int f\nabla\cdot\frac{\partial F_f}{\partial\mathbf{v}}\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\mathbf{dxdv} + \text{cyc},\end{aligned}$$

where we have used the permutation of F, G, and K. The first three terms cancel with each other by using formula (A2) in the third line. The last six terms cancel to each other by using formula (A1) to the fourth last term.

The ninth term

$$\{\{F,G\},K\}_{ffB}^{ffB/n^2}+\text{cyc}$$
$$=\int f\left[\int\frac{f}{n^2}\mathbf{B}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{F}^A-\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}^A\right)\mathrm{d}\mathbf{v},K_f\right]_{xv}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{\mathbf{B}}{n}\cdot\left(\left(\nabla\times\left(\frac{1}{n}\mathbf{F}\times\mathbf{G}\right)\right)\times\mathbf{K}^A\right)\mathrm{d}\mathbf{x}$$
$$+\int\frac{1}{n}\mathbf{B}\cdot\left(\mathbf{K}\times\nabla\times\frac{1}{n}(\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A)\right)\mathrm{d}\mathbf{x}+\text{cyc}.$$

By integrating by parts, the derivatives with respect to $\mathbf{x}$ in the first term, and using formula (A4) for the second and third terms, we obtain

$$\{\{F,G\},K\}_{ffB}^{ffB/n^2}+\text{cyc}$$
$$=-\int\nabla\cdot\mathbf{K}\frac{1}{n^2}\mathbf{B}\cdot(\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A)\mathrm{d}\mathbf{x}$$
$$+\int\frac{\mathbf{B}}{n^2}\cdot((\nabla\times(\mathbf{F}\times\mathbf{G}))\times\mathbf{K}^A)\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int\mathbf{B}\cdot\left(\left(\nabla\left(\frac{1}{n^2}\right)\times(\mathbf{F}\times\mathbf{G})\right)\times\mathbf{K}^A\right)\mathrm{d}\mathbf{x}$$
$$+\int\frac{1}{n^2}\mathbf{B}\cdot(\mathbf{K}\times\nabla\times(\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A))\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int\mathbf{B}\cdot\left(\mathbf{K}\times\left(\nabla\left(\frac{1}{n^2}\right)\times(\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A)\right)\right)\mathrm{d}\mathbf{x}+\text{cyc}.$$

By integrating by parts, the derivatives with respect to $\mathbf{x}$ in the terms containing $\nabla\left(\frac{1}{n^2}\right)$, we obtain

$$\{\{F,G\},K\}_{ffB}^{ffB/n^2}+\text{cyc}$$
$$=-\int\nabla\cdot\mathbf{K}\left(\frac{1}{n^2}\mathbf{B}\cdot(\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A)\right)\mathrm{d}\mathbf{x}$$
$$+\int\frac{1}{n^2}\mathbf{B}\cdot((\nabla\times(\mathbf{F}\times\mathbf{G}))\times\mathbf{K}^A)\mathrm{d}\mathbf{x}-\int\frac{1}{2n^2}\nabla\cdot((\mathbf{F}\times\mathbf{G})$$
$$\times(\mathbf{K}^A\times\mathbf{B}))\mathrm{d}\mathbf{x}+\int\frac{1}{n^2}\mathbf{B}\cdot(\mathbf{K}\times\nabla\times(\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A))\mathrm{d}\mathbf{x}$$
$$-\frac{1}{2}\int\frac{1}{n^2}\nabla\cdot((\mathbf{G}\times\mathbf{F}^A-\mathbf{F}\times\mathbf{G}^A)\times(\mathbf{B}\times\mathbf{K}))\mathrm{d}\mathbf{x}+\text{cyc}$$
$$=-\int\frac{1}{n^2}\nabla\cdot\mathbf{G}(\mathbf{B}\cdot(\mathbf{F}\times\mathbf{K}^A))-\frac{1}{n^2}\nabla\cdot\mathbf{F}(\mathbf{B}\cdot(\mathbf{G}\times\mathbf{K}^A))\mathrm{d}\mathbf{x}$$
$$+\int\frac{1}{n^2}\mathbf{B}\cdot((\nabla\times(\mathbf{F}\times\mathbf{G}))\times\mathbf{K}^A)\mathrm{d}\mathbf{x}$$
$$-\int\frac{1}{2n^2}\nabla\cdot((\mathbf{F}\times\mathbf{G})\times(\mathbf{K}^A\times\mathbf{B}))\mathrm{d}\mathbf{x}$$
$$+\int\frac{1}{n^2}\mathbf{B}\cdot(\mathbf{G}\times\nabla\times(\mathbf{F}\times\mathbf{K}^A)-\mathbf{F}\times\nabla\times(\mathbf{G}\times\mathbf{K}^A))\mathrm{d}\mathbf{x}$$
$$-\frac{1}{2}\int\frac{1}{n^2}\nabla\cdot\left((\mathbf{F}\times\mathbf{K}^A)\times(\mathbf{B}\times\mathbf{G})-\frac{1}{n^2}(\mathbf{G}\times\mathbf{K}^A)\times(\mathbf{B}\times\mathbf{F})\right)\mathrm{d}\mathbf{x}$$
$$+\text{cyc}$$
$$=\int\frac{1}{n^2}\nabla\cdot\mathbf{B}(\mathbf{K}^A\cdot(\mathbf{F}\times\mathbf{G}))\mathrm{d}\mathbf{x}-\int\frac{1}{n^2}\nabla\cdot\mathbf{K}^A(\mathbf{B}\cdot(\mathbf{F}\times\mathbf{G}))\mathrm{d}\mathbf{x}+\text{cyc},$$

where we used the permutations of F, G, and K, formula (A1), formula (A3), and Lemma 1. As $\nabla\cdot\mathbf{K}^A=\nabla\cdot\nabla\times\frac{\delta K}{\delta\mathbf{B}}=0$, we know that $\{\{F,G\},K\}_{ffB}^{ffB/n^2}+\text{cyc}$ is zero under Gauss's law for magnetism $\nabla\cdot\mathbf{B}=0$.

The tenth term

$$\{\{F,G\},K\}_{ffB}^{fB^2/n}+\text{cyc}$$
$$=-\int f\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{F}^A-\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}^A\right)\right)\times\frac{\partial K_f}{\partial\mathbf{v}}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{f}{n}\mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\left(\mathbf{B}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\right)\right)\times\mathbf{K}^A\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}+\text{cyc}$$
$$=-\int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_fF_k^A\partial_{v_n}K_f\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_fG_k^A\partial_{v_n}K_f\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_fK_n^A\partial_{v_k}G_f\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$-\int\frac{f}{n}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_fK_k^A\partial_{v_n}F_f\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}+\text{cyc},$$

where we use the product rule to differentiate the product of two functions, and express the cross product using the Levi–Civita symbol ε. By the permutations of F, G, and K, the first term cancels with the third term, and the second term cancels with the fourth term. So $\{\{F,G\},K\}_{ffB}^{fB^2/n}+\text{cyc}$ is zero.

The eleventh term

$$\{\{F,G\},K\}_{ffB}^{ffB^2/n^2}+\text{cyc}$$
$$=-\int\frac{f}{n}\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}\right)\right)\times\mathbf{K}^A\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$-\int\frac{f}{n}\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\mathbf{F}\times\frac{\partial G_f}{\partial\mathbf{v}}\right)\right)\times\mathbf{K}^A\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{\mathbf{B}}{n}\cdot\left(\left(\int f\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{F}^A-\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}^A\right)\mathrm{d}\mathbf{v}\right)\times\mathbf{K}\right)\mathrm{d}\mathbf{x}$$
$$+\text{cyc}$$
$$=-\int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_fG_kK_n^A\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_fF_kK_n^A\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$+\int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_fK_kF_n^A\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$$
$$-\int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_fK_kG_n^A\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}+\text{cyc},$$

where we use the product rule to differentiate the product of two functions, and express the cross product using the Levi–Civita symbol ε. By the permutations of F,G, and K, the first term cancels with the third term, and the second term cancels with the fourth term, so $\{\{F,G\},K\}_{ffB}^{ffB^2/n^2}+\text{cyc}$ is zero.

The twelfth term

$$
\begin{aligned}
&\{\{F,G\},K\}_{fBB}^{fB/n^2}+\mathrm{cyc}\\
&=\int f\left[\frac{\mathbf{B}}{n^2}\cdot(\mathbf{F}^A\times\mathbf{G}^A),\frac{\delta K}{\delta f}\right]_{xv}\mathbf{dxdv}\\
&\quad+\int\frac{\mathbf{B}}{n}\cdot\left(\nabla\times\int\frac{f}{n}\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{F}^A-\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{K}^A\right)\mathbf{dv}\times\mathbf{K}^A\right)\mathbf{dx}\\
&\quad+\int\frac{f}{n}\mathbf{B}\cdot\left(\frac{\partial K_f}{\partial\mathbf{v}}\times\nabla\times\left(\frac{1}{n}\mathbf{F}^A\times\mathbf{G}^A\right)\right)\mathbf{dvdx}+\mathrm{cyc},\\
&=\int-\frac{\mathbf{B}}{n^2}\cdot(\mathbf{F}^A\times\mathbf{G}^A)\nabla\cdot\mathbf{K}\,\mathbf{dx}\\
&\quad+\int\frac{\mathbf{B}}{n}\cdot\left(\left(\nabla\times\left(\frac{1}{n}\mathbf{G}\times\mathbf{F}^A-\frac{1}{n}\mathbf{F}\times\mathbf{G}^A\right)\right)\times\mathbf{K}^A\right)\mathbf{dx}\\
&\quad+\int\frac{\mathbf{B}}{n}\cdot\left(\mathbf{K}\times\nabla\times\left(\frac{1}{n}\mathbf{F}^A\times\mathbf{G}^A\right)\right)\mathbf{dx}+\mathrm{cyc}.
\end{aligned}\tag{A9}
$$

The last three terms of Eq. (A9) are in the same form as formula (45) in Ref. 22, and the result of Eq. (A9) is thus zero, for which Gauss's law for magnetism $\nabla\cdot\mathbf{B}=0$ is used, as mentioned in Ref. 22.

The thirteenth term

$$
\begin{aligned}
&\{\{F,G\},K\}_{fBB}^{fB^2/n^2}+\mathrm{cyc}\\
&\quad=-\int\frac{f}{n}\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{B}}{n}\cdot\left(\frac{\partial G_f}{\partial\mathbf{v}}\times\mathbf{F}^A-\frac{\partial F_f}{\partial\mathbf{v}}\times\mathbf{G}^A\right)\right)\\
&\qquad\times\mathbf{K}^A\mathbf{dxdv}+\mathrm{cyc}\\
&\quad=-\int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}G_fF_k^AK_n^A\,\mathbf{dxdv}\\
&\qquad+\int\frac{f}{n^2}\varepsilon_{ijk}\varepsilon_{lmn}B_iB_l\partial_{v_m}\partial_{v_j}F_fG_k^AK_n^A\,\mathbf{dxdv}+\mathrm{cyc},
\end{aligned}
$$

where we express the cross product using the Levi–Civita symbol ε, and the above two terms cancel by the permutations of F,G, and K. So $\{\{F,G\},K\}_{fBB}^{fB^2/n^2}+\mathrm{cyc}=0$.

The fourteenth term

$$
\{\{F,G\},K\}_{BBB}^{B/n^2}+\mathrm{cyc}=\int\frac{\mathbf{B}}{n}\cdot\left(\nabla\times\left(\frac{\mathbf{F}^A}{n}\times\mathbf{G}^A\right)\times\mathbf{K}^A\right)\mathbf{dx}+\mathrm{cyc},
$$

which is equal to zero as shown in Ref. 22 under Gauss's law for magnetism $\nabla\cdot\mathbf{B}=0$.

As the above 14 groups of terms are zero under Gauss's law for magnetism, that $\nabla\cdot\mathbf{B}=0$, we have proven the bracket (17) satisfies the Jacobi identity under Gauss's law for magnetism $\nabla\cdot\mathbf{B}=0$.

REFERENCES

[1] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems* (Springer Science & Business Media, 2013), Vol. 17.
[2] P. J. Morrison, "The Maxwell-Vlasov equations as a continuous Hamiltonian system," Phys. Lett. A **80**, 383–386 (1980).
[3] J. E. Marsden and A. Weinstein, "The Hamiltonian structure of the Maxwell-Vlasov equations," Phys. D **4**, 394–406 (1982).
[4] P. J. Morrison and J. M. Greene, "Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics," Phys. Rev. Lett. **45**, 790 (1980).
[5] R. G. Spencer and A. N. Kaufman, "Hamiltonian structure of two-fluid plasma dynamics," Phys. Rev. A **25**, 2437 (1982).
[6] D. D. Holm, "Hall magnetohydrodynamics: Conservation laws and Lyapunov stability," Phys. Fluids **30**, 1310–1322 (1987).
[7] H. M. Abdelhamid, Y. Kawazura, and Z. Yoshida, "Hamiltonian formalism of extended magnetohydrodynamics," J. Phys. A: Math. Theor. **48**, 235502 (2015).
[8] C. Tronci, "Hamiltonian approach to hybrid plasma models," J. Phys. A: Math. Theor. **43**, 375501 (2010).
[9] D. Winske, H. Karimabadi, A. Y. Le, N. N. Omidi, V. Roytershteyn, and A. J. Stanier, "Hybrid-kinetic approach: Massless electrons," in *Space and Astrophysical Plasma Simulation: Methods, Algorithms, and Applications* (Springer, 2023), pp. 63–91.
[10] Y. Li, M. C. Pinto, F. Holderied, S. Possanner, and E. Sonnendrücker, "Geometric particle-in-cell discretizations of a plasma hybrid model with kinetic ions and mass-less fluid electrons," J. Comput. Phys. **498**, 112671 (2024).
[11] S. Von Alfthan, D. Pokhotelov, Y. Kempf, S. Hoilijoki, I. Honkonen, A. Sandroos, and M. Palmroth, "Vlasiator: First global hybrid-Vlasov simulations of Earth's foreshock and magnetosheath," J. Atmos. Sol.-Terr. Phys. **120**, 24–35 (2014).
[12] Y. Li, F. Holderied, S. Possanner, and E. Sonnendrücker, "Canonical variables based numerical schemes for hybrid plasma models with kinetic ions and massless electrons," J. Comput. Phys. **505**, 112916 (2024).
[13] F. Pinto and J. Burby, "Poisson–Dirac submanifolds as a paradigm for imposing constraints in non-dissipative plasma models," Phys. Plasmas **32**, 072305 (2025).
[14] D. Kaltsas, J. Burby, P. Morrison, E. Tassi, and G. Throumoulopoulos, "Imposing quasineutrality on electrostatic plasmas via the Dirac theory of constraints," arXiv:2508.02389 (2025).
[15] J. Burby, D. Kaltsas, P. Morrison, E. Tassi, and G. Throumoulopoulos, "Hamiltonian formulation of the quasineutral Vlasov-Poisson system," arXiv:2506.21415 (2025).
[16] K. Kimura and P. Morrison, "On energy conservation in extended magnetohydrodynamics," Phys. Plasmas **21**, 082101 (2014).
[17] D. Kaltsas, G. Throumoulopoulos, and P. Morrison, "Hamiltonian kinetic-Hall magnetohydrodynamics with fluid and kinetic ions in the current and pressure coupling schemes," J. Plasma Phys. **87**, 835870502 (2021).
[18] P. J. Morrison, "A general theory for gauge-free lifting," Phys. Plasmas **20**, 012104 (2013).
[19] A. Stanier, L. Chacón, and G. Chen, "A fully implicit, conservative, non-linear, electromagnetic hybrid particle-ion/fluid-electron algorithm," J. Comput. Phys. **376**, 597–616 (2019).
[20] J. W. Burby, "Magnetohydrodynamic motion of a two-fluid plasma," Phys. Plasmas **24**, 082104 (2017).
[21] K. Kormann and E. Sonnendrücker, "Energy-conserving time propagation for a structure-preserving particle-in-cell Vlasov–Maxwell solver," J. Comput. Phys. **425**, 109890 (2021).
[22] M. Lingam, P. Morrison, and G. Miloshevich, "Remarkable connections between extended magnetohydrodynamics models," Phys. Plasmas **22**, 072111 (2015).