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Letter

Metriplectic dynamical systems on contact manifolds

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ABSTRACT

Flows on symplectic, Poisson, contact, and metriplectic manifolds are reviewed in order to describe our main result, which is to associate a natural *metriplectic* dynamical system on the general one-jet bundle $J^1N = T^*N \times \mathbb{R}$, which is at once a (trivial) Poisson manifold and a contact manifold. Unlike the standard contact Hamiltonian system, our metriplectic system is thermodynamically consistent in that

$$\dot{H} = 0 \quad \text{and} \quad \dot{S} \geq 0$$

under the flow. Here H is the Hamiltonian, while S is the *entropy* function which is nothing but the $\mathbb{R}$ coordinate function of J^1N. As an example we derive the Duffing equation (autonomous and nonautonomous versions) either as a contact Hamiltonian system or as a metriplectic system. We show that for both systems the Duffing equation is a subsystem of three dimensional systems that contain a thermodynamic component, a form that facilitates asymptotic stability analysis of the relevant equilibrium state.

1. Introduction

It is widely known that Gibbs [1] understood the contact structure of classical thermodynamics, and that this geometric structure underlies the different potential representations of thermodynamics associated by Legendre transforms. However, classical thermodynamics is not dynamics, so actual temporal evolution is not part of the theory. Thus, in a sense, this paper is about dynamical thermodynamics; in particular, it is about how contact structure appears in this setting and how it relates to another geometric structure, metriplectic dynamics [2], an inclusive dynamical systems theory for thermodynamically consistent systems.

By definition a dynamical system is thermodynamically consistent if it satisfies the first and second laws of thermodynamics, viz., energy conservation and entropy production as dynamical variables. Because thermodynamics is usually applied to continuum systems it is not surprising that early researchers like Eckart [3] developed thermodynamically consistent generalizations of the Navier-Stokes equations (sometimes call the Navier-Stokes-Fourier equations) that entail two dynamical thermodynamic variables, the specific volume and the specific entropy. Although Eckart also considered other systems [4,5], many generalizations have appeared over the years, e.g., the Cahn-Hilliard system combined with the Navier-Stokes system for describing multiphase flows [6–8].

To see how entropy is a dynamical variable in a statistical mechanics setting, we refer readers to [9] for a derivation of nonequilibrium thermodynamics from statistical mechanics as a symplecto-contact reduction on the kinetic theory phase space, which is the (affine) space of probability distributions on the many-body particle phase space in the spirit of the earlier derivations by Jaynes [10,11] and Mrugula [12]. In this derivation, thermodynamic entropy is covariantly derived from Shannon's information entropy which makes it apparent that entropy is indeed *universal* and independent of other observables.

Metriplectic dynamics, being a general framework for thermodynamically consistent dynamical systems, has associated observables (dynamical variables) H, a Hamiltonian corresponding to the energy satisfying $\dot{H} = 0$, and S, an entropy satisfying $\dot{S} \geq 0$. Thus the system has dynamic realizations of the first and second laws of thermodynamics. The axioms of metriplectic dynamics appeared in [13,14] while the name first appeared in [2]. (See [15] for discussion and comparison to some different approaches to nonequilibrium thermodynamics, e.g. [16–18].)

The metriplectic formalism covers a broad range of dynamical systems, both finite- and infinite-dimensional. The multiphase fluid flows of [6,7] and [8] (with minor correction) were shown to be special cases of an encompassing multiphase fluid metriplectic theory given in [19,20]. In the context of kinetic theory, the Vlasov equation with the Laundau collision operator was shown to be metriplectic in [2,13] and a collision operator suitable for gyrokinetics was given in [21]. See [22] for a general discussion of the gradient flow nature of metriplectic dynamics and [15] for many additional examples, finite and infinite.

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The classical *space-time phase space*, which is $T^*\mathbb{R}^n \times \mathbb{R}$ carries the canonical contact form given by $dt - pdq$, the *contactification* of T^*M [23, Appendix 4], provides a natural ground for time-dependent Hamiltonian dynamics. This time-dependent Hamiltonian dynamics combined with the technique of generating functions can be put into a canonical formalism [23], which Arnold [23, Chapter 9] calls an odd-dimensional approach to Hamiltonian phase flows. Given that entropy is an independent observable as mentioned above, it is also natural to say that the space-time phase space is the correct arena for entropy dynamics with entropy playing the role of time; hence, contact manifolds become thermodynamic phase spaces.

On the other hand, general *contact Hamiltonian* flows on a contact manifold are born dissipative, in that they do not preserve the natural Liouville measure and hence Poincaré recurrence fails. (See [24], however, for some discussion on the construction of a Liouville-type invariant measure of contact Hamiltonian flows when the Hamiltonian H is nowhere vanishing.) While there are many similarities in their Hamiltonian calculi, this dynamical perspective makes a stark difference from the behavior of the classical (symplectic) Hamiltonian dynamics. This being said, a fundamental question in contact dynamics is the question of how one can systematically study this dissipative aspect of contact dynamics in terms of the underlying contact Hamiltonian formalism. (See [9,24–28] in which the relevant questions are also discussed.) This paper attempts to make the first step towards this question in the context of the metriplectic formalism on manifolds that have both Poisson and contact structure. In previous work [15], the Kulkarni-Nomizu (K-N) product [29,30] was used to construct metriplectic systems, with the involved factors physically motivated in [20]. By contrast, the present work provides a natural geometric motivation for these choices on vector bundles (see Definition 1 in Section 5 below).

Preliminary material on contact, symplectic, Poisson, and metriplectic manifolds is given in Section 2, while Section 3 describes flows on these geometrical structures. Section 4 compares contact Hamiltonian systems and metriplectic systems. Section 5 discusses a coordinate-free construction of metriplectic geometry on a one-jet bundle of a Riemannian manifold. Section 6 shows how the Duffing equation can be 'derived' in either the contact Hamiltonian formalism or the metriplectic formalism, and some ramifications of the formalisms are explored. Finally, in Section 7 we offer some concluding remarks.

2. Preliminaries

We consider structures on a smooth real manifold $\mathcal{Z}$ of dimension n, with a point of $\mathcal{Z}$ denoted by P. The set of vector fields on a manifold $\mathcal{Z}$, which is the space $\Gamma(T\mathcal{Z})$ of sections of the tangent bundle $T\mathcal{Z}$, is denoted by $\mathfrak{X}(\mathcal{Z})$), and differential forms on $\mathcal{Z}$ by $\Omega^k(\mathcal{Z}) = \Gamma(\Lambda^k(\mathcal{Z}))$, where $\Omega^0(\mathcal{Z})$ is the set of smooth functions $\mathcal{Z} \mapsto \mathbb{R}$, $\Omega^1(\mathcal{Z}) = \Gamma(T^*\mathcal{Z})$ the set of 1-forms, etc. We denote the exterior derivative by $\mathbf{d}$ and wedge product by $\wedge$. We also use Einstein's summation convention throughout the paper.

(i) The manifold $\mathcal{Z}$ is a *symplectic manifold* if it is equipped with a closed nondegenerate 2-form $\omega \in \Omega^2(\mathcal{Z})$. For vectors $X, Y \in T_P\mathcal{Z}$ we have $\omega(X,Y) \in \mathbb{R}$. Because ω is nondegenerate we have the *Poisson bivector* defined by $\{f,g\} = J(\mathbf{d}f, \mathbf{d}g)$ for $f, g \in \Omega^0(\mathcal{Z})$, and $\mathbf{d}f, \mathbf{d}g \in \Omega^1(\mathcal{Z})$. In coordinates J is determined by the *Poisson tensor*. As usual, symplectic manifolds are denoted by the pair $(\mathcal{Z}, \omega)$ and the interior product (contraction) by $\lrcorner$.

In *Darboux coordinates* the Poisson tensor has the form

$$J_c = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix}, \tag{1}$$

where $n = 2N$, I_N is the $N \times N$ identity and O_N is an $N \times N$ block of zeros. In these coordinates, say $z = (z^1, z^2, \dots, z^{2N})$, there is a split into the canonical coordinates $z = (q, p)$ and the Poisson bracket is given by

$$\{f,g\} = J_c^{ij}\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^i} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q^a}\frac{\partial f}{\partial p_a}, \tag{2}$$

where $i, j = 1, 2, \dots, 2N$, $a = 1, 2, \dots, N$, and repeated sum notation is assumed.

(ii) The manifold $\mathcal{Z}$ is a *Poisson manifold* if it is equipped with a bracket $\{\,,\,\}: \Omega^0(\mathcal{Z}) \times \Omega^0(\mathcal{Z}) \to \Omega^0(\mathcal{Z})$ producing a Lie algebra realization with the addition of a Leibniz rule (a Poisson algebra). Being a Lie algebra realization means it is antisymmetric bilinear (here over the field $\mathbb{R}$), and satisfies the Jacobi identity. When degeneracy exists, there exist Casimir invariants C that satisfy $\{f, C\} \equiv 0$ for all $f \in \Omega^0(\mathcal{Z})$. Poisson manifolds are denoted by the pair $(\mathcal{Z}, \{\,,\,\})$.

In *Darboux coordinates* the Poisson tensor has the form

$$J = \begin{bmatrix} 0_N & I_N & 0_{N\times r} \\ -I_N & 0_N & 0_{N\times r} \\ 0_{r\times N} & 0_{r\times N} & 0_r \end{bmatrix}, \tag{3}$$

where J is an $n \times n$ matrix with $n = 2N + r$ and, as before, I_N is the $N \times N$ identity and O_N is an $N \times N$ block of zeros, while $0_{r\times N}$ is a rectangular matrix of zeros. A global Poisson tensor of this form is only possible if J has constant rank. In these coordinates it is obvious that there are r independent (nonparallel gradients) Casimir invariants.

(iii) The manifold $\mathcal{Z}$ is a *contact manifold* if it has odd dimension, say $n = 2N + 1$, and is equipped with a smooth maximally nonintegrable hyperplane field (does not define a distribution) $\xi \subset T\mathcal{Z}$. This means that locally $\xi = \ker \alpha$, where α is a 1-form that satisfies $\alpha \wedge (\mathbf{d}\alpha)^n \neq 0$. The quantity ξ is called a contact structure and the quantity α a contact 1-form that locally defines ξ. Contact manifolds are denoted by the pair $(\mathcal{Z}, \xi)$.

In *Darboux coordinates* the 1-form α takes the form

$$\alpha = dz - pdq \qquad \rightarrow \qquad \mathbf{d}\alpha = dq \wedge dp\,. \tag{4}$$

The Reeb vector field, say R, which serves as a sort of analog in contact geometry to a Hamiltonian vector field, is given by

$$R \lrcorner\, \alpha = 1 \qquad \text{and} \qquad R \lrcorner\, \mathbf{d}\alpha = 0\,. \tag{5}$$

(iv) The manifold $\mathcal{Z}$ is a *metriplectic manifold* [13,14] if it is a Poisson manifold and possess an additional multilinear 4-bracket

$$(\,,\,;\,,\,): \Omega^0(\mathcal{Z}) \times \Omega^0(\mathcal{Z}) \times \Omega^0(\mathcal{Z}) \times \Omega^0(\mathcal{Z}) \to \Omega^0(\mathcal{Z}),$$

where for any $f, k, g, n \in \Omega^0(\mathcal{Z})$ the following properties hold:
(a) the algebraic identities/symmetries

$$\begin{aligned} (f,k;g,n) &= -(k,f;g,n) \\ (f,k;g,n) &= -(f,k;n,g) \\ (f,k;g,n) &= (g,n;f,k) \end{aligned} \tag{6}$$

and

$$(f,k;g,n) + (f,g;n,k) + (f,n;k,g) = 0\,, \tag{7}$$

which are not axiomatically minimal, and
(b) derivation in all arguments, e.g.,

$$(fh,k;g,n) = f(h,k;g,n) + (f,k;g,n)h\,, \tag{8}$$

where fh denotes pointwise multiplication.

One way to obtain a metriplectic 4-bracket is via a covariant version of the K-N product. (See [29,30] where the contravariant version was given.) Given two symmetric bivector fields, say σ and μ, operating on 1-forms $\mathbf{d}f, \mathbf{d}k$ and $\mathbf{d}g, \mathbf{d}n$, the K-N product is defined by

$$\begin{aligned} \sigma \owedge \mu\,(\mathbf{d}f, \mathbf{d}k, \mathbf{d}g, \mathbf{d}n) = {}& \sigma(\mathbf{d}f, \mathbf{d}g)\,\mu(\mathbf{d}k, \mathbf{d}n) \\ & - \sigma(\mathbf{d}f, \mathbf{d}n)\,\mu(\mathbf{d}k, \mathbf{d}g) \\ & + \mu(\mathbf{d}f, \mathbf{d}g)\,\sigma(\mathbf{d}k, \mathbf{d}n) \\ & - \mu(\mathbf{d}f, \mathbf{d}n)\,\sigma(\mathbf{d}k, \mathbf{d}g)\,. \end{aligned} \tag{9}$$

Thus, the following produces a 4-bracket:

$$(f,k;g,n) = \sigma \owedge \mu(\mathbf{d}f, \mathbf{d}k, \mathbf{d}g, \mathbf{d}n)\,. \tag{10}$$

In a coordinate patch the metriplectic 4-bracket can be expressed in index form as

$$(f,k;g,n) = R^{ijkl}(z)\frac{\partial f}{\partial z^i}\frac{\partial k}{\partial z^j}\frac{\partial g}{\partial z^k}\frac{\partial n}{\partial z^l}\,. \tag{11}$$

Here and henceforth $i,j,k,l = 1,2,\dots,n$

Any manifold that is at once both a Poisson manifold and a Riemannian manifold or a Poisson manifold with a connection, has metriplectic structure. However, metriplectic manifolds need not be Riemannian.

3. Hamiltonian, contact Hamiltonian, and metriplectic Systems

3.1. Hamilton's equation on Poisson manifolds

Given a Hamiltonian $H \in \Omega^0(\mathcal{Z})$, canonical (symplectic) Hamiltonian vector fields X_H are given by

$$X_H \rfloor \omega = dH\,. \tag{12}$$

In Darboux coordinates this leads to the *canonical Hamiltonian system* of differential equations:

$$\dot q^a = \frac{\partial H}{\partial p_a} \quad \text{and} \quad \dot p_a = -\frac{\partial H}{\partial q^a}, \quad a = 1,2\dots N\,, \tag{13}$$

which follows because ω is nondegenerate.

A Hamiltonian vector field on general Poisson manifolds is determined in terms of a Hamiltonian $H \in \Omega^0(\mathcal{Z})$ via the bracket derivation

$$\mathcal{V}_{NCH} = \{\,\cdot\,,H\} = J(\,\cdot\,,\mathbf{d}H)\,. \tag{14}$$

(See, e.g., [31] and [32].)

Thus, in a coordinate patch a *Hamiltonian system* on a Poisson manifold is governed by the differential equations

$$\dot{z^i} = \{z^i,H\} = J^{ij}\frac{\partial H}{\partial z^j}\,, \qquad i,j = 1,2,\dots,n \tag{15}$$

in terms of the Poisson bivector field

$$J = J^{ij}\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j}\,.$$

In (Poisson) Darboux coordinates [31], say (q,p,C), this gives the *noncanonical Hamiltonian system* of differential equations:

$$\dot q^a = \frac{\partial H}{\partial p_a}\,, \quad \dot p_a = -\frac{\partial H}{\partial q^a}, \quad a = 1,2,\dots,N$$
$$\dot C^r = 0\,, \qquad r = 1,2,\dots,n-2N. \tag{16}$$

3.2. Hamilton's equation on contact manifolds

Let (M,α) be a contact manifold of dimension $2n+1$. A contact Hamiltonian vector field is defined by

$$\begin{cases} X_H \rfloor \alpha = -H \\ X_H \rfloor d\alpha = dH - R_\alpha[H]\alpha\,, \end{cases} \tag{17}$$

where R_α denotes the Reeb vector field and $R_\alpha[H]$ is the directional derivative action on the 0-form H. (See [33,34] for discussion – we follow their sign conventions.) In Darboux coordinates (q,p,z) *contact Hamiltonian systems* are given by

$$\dot q^i = \frac{\partial H}{\partial p_i}\,, \qquad \dot p_i = -\frac{\partial H}{\partial q^i} - p_i\frac{\partial H}{\partial z}\,, \tag{18}$$
$$\dot z = -H + p_i\frac{\partial H}{\partial p_i}\,, \qquad i = 1,2,\dots,n\,, \tag{19}$$

which displays both Hamiltonian and dissipative parts. Consequently, the energy evolution is

$$\dot H = \frac{\partial H}{\partial q^i}\dot q^i + \frac{\partial H}{\partial p_i}\dot p_i + \frac{\partial H}{\partial z}\dot z = -H\frac{\partial H}{\partial z}\,, \tag{20}$$

which does not vanish in general, but will if $H = 0$ or $\partial H/\partial z = 0$.

3.3. Metriplectic dynamical systems and thermodynamic consistency

As already noted, thermodynamic consistency means $\dot H = 0$ and $\dot S \geq 0$, which are dynamic realizations of the first and second laws of thermodynamics. (See [13–15] for relevant discussions.) Neither of Hamilton's equations is suitable to accommodate this thermodynamic consistency, which motivated the first-named author to introduce *metriplectic dynamical systems*.

A metriplectic vector field has both Hamiltonian and dissipative parts, generated as follows:

$$\mathcal{V}_{MP} = \{\,\cdot\,,H\} + (\,\cdot\,,H;S,H)\,, \tag{21}$$

where $S, H \in \Omega^0(\mathcal{Z})$ are an entropy function and Hamiltonian, respectively.

In a coordinate patch, a *metriplectic dynamical system* is given in terms of a metriplectic vector field according to

$$\dot z^i = \{z^i,H\} + (z^i\cdot,H;S,H)$$
$$= J^{ij}\frac{\partial H}{\partial z^j} + R^{ijkl}\frac{\partial H}{\partial z^j}\frac{\partial S}{\partial z^k}\frac{\partial H}{\partial z^l}\,, \tag{22}$$

where $i,j,k,l = 1,2,\dots,n$. Thermodynamic consistency follows if S is a Casimir invariant, i.e., $\{f,S\} \equiv 0$ for all $f \in \Omega^0(\mathcal{Z})$ and $(f,k;g,n)$ is a 4-bracket, e.g., generated by a K-N product. In the next section we will display a particular form for the 4-bracket that facilitates comparison with contact Hamiltonian systems.

4. Contact Hamiltonian systems versus metriplectic dynamical systems

To motivate our introduction of metriplectic dynamical systems on the one-jet bundle $J^1M = T^*M\times\mathbb{R}$, which we may regard as the *spacetime phase space*. The upshot is that this bundle carries the canonical contact form given by $dt - p dq$. For the simplicity of presentation, which however contains the essence of the comparison, we start with the simplest contact Hamiltonian and metriplectic systems on $\mathbb{R}^2\times\mathbb{R} = \mathbb{R}^3$.

First consider a *3D contact Hamiltonian system*. For the coordinates $(q,p,z)\in\mathbb{R}^3$ Eqs. (18) and (19) reduce to the following three ODEs:

$$\dot q = \frac{\partial H}{\partial p}\,, \quad \dot p = -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial z}\,,$$
$$\dot z = -H + p\frac{\partial H}{\partial p}\,, \tag{23}$$

and as above,

$$\dot H = -H\frac{\partial H}{\partial z} \neq 0\,. \tag{24}$$

Thus energy is not conserved unless H is independent of z, $\partial H/\partial z = 0$ or one begins on the $H = 0$ level set. So, in general contact Hamiltonian systems are **not** thermodynamically consistent.

We build our *3D metriplectic system* atop a trivial Poisson manifold, which is a stack of symplectic planes $T^*\mathcal{Z}\times\mathbb{R}$ (the first jet bundle). In 3D we have $\mathcal{Z} = T^*\mathbb{R}\times\mathbb{R}$, which has the following Poisson bivector J in Darboux coordinates $(q,p,z)\in\mathbb{R}^3$:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

That J satisfies the Jacobi identity is trivial. For this construction, these coordinates are in fact global. Thus the Hamiltonian part of our 3D metriplectic system is generated by this Poisson bracket as

$$\dot q = \{q,H\}\,, \quad \dot p = \{p,H\}\,, \quad \text{and} \quad \dot z = \{z,H\} = 0$$

or

$$\begin{bmatrix} \dot q \\ \dot p \\ \dot z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \partial H/\partial q \\ \partial H/\partial p \\ \partial H/\partial z \end{bmatrix}. \tag{26}$$

Here z is the Casimir invariant that plays the role of entropy, i.e., $\{f, z\} = 0$ for all f. Thus we are on the road to thermodynamic consistency since evidently $\dot{H} \equiv 0$.

The full metriplectic dynamics in the standard Darboux coordinates of $\mathbb{R}^2 \times \mathbb{R}$ will be given by

$$\dot{z}^i = \{z^i, H\} + (z^i, H; S, H), \qquad i = 1, 2, 3, \tag{27}$$

yielding

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial z},$$
$$\dot{z} = \left(\frac{\partial H}{\partial p}\right)^2. \tag{28}$$

Here we have equated the entropy S with the coordinate z and particular choices were made for μ and σ, which will be addressed in Section 5. (See (30).)

It remains to compare (28) with (23). The first observation to make is that for so-called *natural* Hamiltonians where $H = p^2/2 + V(q)$, $\partial H/\partial p = p$. So, (28) gives $\dot{z} = p^2$, while (23) yields the Legendre transform of H, i.e. the Lagrangian

$$L = p\frac{\partial H}{\partial p} - H = p^2/2 - V(q), \tag{29}$$

which is not sign definite. However, if H is the kinetic energy which is Euler homogeneous of degree 2 in p, then $\dot{z} = -H + p\partial H/\partial p = H = (\partial H/\partial p)^2/2$ and the above two systems can be seen to coincide within a factor of 2. If in addition the Hamiltonian is linear in z so that $\partial H/\partial z = 1$ then the equations for $\dot{p}$ of the two systems coincides. This demonstrates that for some Hamiltonians and entropies, contact Hamiltonian systems can be thermodynamically consistent. This can generalizes to the kinetic energy on the cotangent bundle T^*N equipped with a Riemannian metric of arbitrary dimension.

By altering μ, σ, S, and H one can obtain other corresponding systems with various properties.

5. Metriplectic systems on one-jet bundles

Let (N, g) be any Riemannian manifold and consider its one-jet bundle $J^1N = T^*N \times \mathbb{R}$ equipped with the canonical contact form

$$\alpha = dz - p\,dq = dz - p_i dq^i, \qquad i = 1, 2, \dots, n.$$

We denote by ω_0 the standard symplectic form ω_0 on T^*N and pull it back to $\pi^*\omega_0$ on J^1N which is a closed two-form. Thus, it defines a Poisson bracket on $J^1N = T^*N \times \mathbb{R}$ with coordinate function z as a Casimir. We denote this bracket by $\{\cdot, \cdot\}$, which governs the Hamiltonian part of the dynamical system.

To naturally incorporate the dissipative aspect of certain dynamical systems, we introduce some natural metriplectic 4-brackets associated to the contact structure.

For this purpose, we use a simple K-N product to construct the metriplectic 4-bracket, viz.

$$\mu(df, dg) = \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \quad \text{and} \quad \sigma = \left\langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial p} \right\rangle, \tag{30}$$

where $\left\langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial p} \right\rangle$ denotes the dot product of the *fiber derivatives* $d^{\text{fiber}} f$ of f and g for the projection $T^*N \to N$. This choice for μ singles out the entropy variable, while the choice for σ emphasizes the fiber. By altering μ, σ, S, and H one can obtain other corresponding systems. However, the choices of (30) are suited to compare with contact Hamiltonian flows, where the $\dot{q}$ equation contains no entropy term as seen in (23).

Here we recall the precise definition of the fiber derivative that applies to any vector bundle $E \to B$, which we apply to $T^*N \to N$ (or $J^1N = T^*N \times \mathbb{R}$) in the present paper.

Definition 1. Let $\pi : E \to B$ be a (finite dimensional) vector bundle and consider its associated dual bundle $\pi^* : E^* \to B$. Let $f : E \to \mathbb{R}$ be a smooth function on E. Then the *fiber derivative* of f at $b \in B$, denoted by $d_b^{\text{fiber}} f$, is defined as follows: For each $e_b \in E_b$, we put

$$d^{\text{fiber}} f(e_b)(\eta_b) := \frac{d}{ds}\Big|_{s=0} f(e_b + s\eta_b) \tag{31}$$

for $\eta_b \in E_b$. The map $b \mapsto d_b^{\text{fiber}} f$ defines a section of the associated (infinite dimensional) fiber bundle $C^\infty_{\text{fiber}}(E, E^*) \to B$ consisting of fiberwise smooth maps, i.e., we have

$$C^\infty_{\text{fiber}}(E, E^*) := \bigcup_{b \in B} \{b\} \times C^\infty(E_b, E_b^*).$$

In our current context where $E = J^1B$ for B equipped with a Riemannian metric g, the precise meaning of our heuristic notation $\left\langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial p} \right\rangle$ above is given by

$$\left\langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial p} \right\rangle(e_b) := g^\#\left(d^{\text{fiber}} f(e_b), d^{\text{fiber}} g(e_b)\right)$$

on $E_b = J^1_b B$, where $g^\#$ is the dual metric of g. Then the function $g^\#(d^{\text{fiber}} f, d^{\text{fiber}} g) : J^1B \to \mathbb{R}$, which we define by

$$g^\#(d^{\text{fiber}} f, d^{\text{fiber}} g)(e) := g^\#\left(d^{\text{fiber}} f(e), d^{\text{fiber}} g(e)\right)$$

for $e \in J^1B$, is well-defined as a function on J^1B.

In the contact canonical coordinates

$$(q^1, \dots, q^n, p_1, \dots, p_n, z)$$

of $J^1B = T^*B \times \mathbb{R}$ associated to canonical coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ of T^*N, we have

$$g^\#(d^{\text{fiber}} f, d^{\text{fiber}} g)(e_b) = \left(g^{ij}\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_j}\right)(q, p, z)$$

for $e_b = (q, p, z)$. (Clearly the right hand side does not depend on the choice of such canonical coordinates.)

Remark 2.

1. The fiber derivative can be also defined on any fiber bundle $E \to B$, not just for a vector bundle. It can be also defined for infinite-dimensional bundles in the sense of variational calculus modulo some technical functional analytical issues. (See [35] for an excellent exposition on such function theoretic matters.)
2. Having given the above covariant definition, we will keep the ‘heuristic’ coordinate expressions below, so as not to make formulas too abstract.

With (30) the K-N product $\sigma \owedge \mu$ yields the following 4-bracket:

$$\begin{aligned}(f, k; g, n) = {} & \\ & \left\langle \frac{\partial f}{\partial p}, \frac{\partial g}{\partial p} \right\rangle \frac{\partial k}{\partial z}\frac{\partial n}{\partial z} - \left\langle \frac{\partial f}{\partial p}, \frac{\partial n}{\partial p} \right\rangle \frac{\partial k}{\partial z}\frac{\partial g}{\partial z} \\ & + \left\langle \frac{\partial k}{\partial p}, \frac{\partial n}{\partial p} \right\rangle \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} - \left\langle \frac{\partial k}{\partial p}, \frac{\partial g}{\partial p} \right\rangle \frac{\partial f}{\partial z}\frac{\partial n}{\partial z}.\end{aligned} \tag{32}$$

Recall that $S = z$ is a Casimir. Then a straightforward calculation leads to the equations of motion.

The full metriplectic dynamics in the standard Darboux coordinates of J^1N $(q^1, \dots, q^n, p_1, \dots p_n, z)$ is given by

$$\dot{z}^i = \{z^i, H\} + (z^i, H; S, H), \qquad i = 1, 2, \dots, n,, \tag{33}$$

yielding

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k} - g_{kj}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial z},$$
$$\dot{z} = \left\| \frac{\partial H}{\partial p} \right\|_g^2. \tag{34}$$

Theorem 3. *For any choice of Hamiltonian $H = H(q, p, z)$, we have*

$$\dot{H} = (H, H; S, H) = 0 \tag{35}$$

while

$$\dot{S} = (S, H; S, H) = \left\| \frac{\partial H}{\partial p} \right\|_g^2 \geq 0 . \tag{36}$$

In particular, the system of equations (28) are indeed thermodynamically consistent.

Proof. Since both equations are coordinate independent, we consider the standard Darboux coordinates of $J^1 N$ $(q^1, \ldots, q^n, p_1, \ldots p_n, z)$ noting that z is a global function indepependent of this choice.

Clearly $\dot{z} \geq 0$ and by the metriplectic symmetries, we derive

$$\dot{H} = \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial p}\frac{\partial H}{\partial z}\right) + \frac{\partial H}{\partial z}\left(\frac{\partial H}{\partial p}\right)^2 = 0 . \tag{37}$$

□

6. An example: Duffing's equation

The well-known Duffing equation [36,37], being an equation with both damping (friction) and drive, provides an interesting example that shows how nonautonomous systems can be at once contact Hamiltonian and metriplectic. We will derive it in this section as a contact Hamiltonian system and as a metriplectic system. We will see in both cases that with the appropriate Hamiltonian the usual Duffing equation can be a two-dimensional subsystem of three-dimensional systems that contain a thermodynamic component.

The most general form of the Duffing equation is given by the following nonlinear ODE with external periodic forcing:

$$\ddot{q} + \delta\dot{q} + \alpha q + \beta q^3 = \gamma \sin(\omega t + \phi), \tag{38}$$

with parameters δ, α, β, γ, ω, ϕ.

6.1. Derivation of Duffing's equation and energy flow

All that is needed to obtain the contact Hamiltonian form is to specify a contact Hamiltonian. We assume

$$H = \frac{p^2}{2} + \frac{\alpha q^2}{2} + \frac{\beta q^4}{4} - \gamma q \sin(\omega t + \phi) + \delta z. \tag{39}$$

Observe, if we drop the last term of (39), δz, we have an ordinary Hamiltonian for the Duffing equation without the damping term, $\delta\dot{q}$, of (38). However, if we substitute (39) into (23) we obtain the contact Hamiltonian system given by

$$\dot{q} = p \tag{40}$$

$$\dot{p} = -\alpha q - \beta q^3 + \gamma \sin(\omega t + \phi) - \delta p \tag{41}$$

$$\dot{z} = \frac{p^2}{2} - \delta z - \frac{\alpha q^2}{2} - \frac{\beta q^4}{4} + \gamma q \sin(\omega t + \phi). \tag{42}$$

Upon differentiating (40) with respect to time and using (41), we obtain precisely (38). Thus, given a solution of Duffing's equation, the right hand side of (42) becomes a function of z and t, and it can be solved after the fact.

Because this system is a nonautonomous contact Hamiltonian system, (24) becomes

$$\dot{H} = -H\frac{\partial H}{\partial z} + \frac{\partial H}{\partial t} = -\delta H - \gamma\omega q \cos(\omega t + \phi)\,; \tag{43}$$

upon dropping the driving term we see that the Hamiltonian decays according to

$$H = H_0\, e^{-\delta t}\,, \tag{44}$$

and the entropy Eq. (42) depends on all the variables (q, p, z, t).

On the other hand, the associated metriplectic system becomes the following upon substitution of the H of (39) into (28):

$$\dot{q} = p \tag{45}$$

$$\dot{p} = -\alpha q - \beta q^3 + \gamma \sin(\omega t + \phi) - \delta\, p \tag{46}$$

$$\dot{z} = p^2 \tag{47}$$

Noting that the first two equations of this system coincide with (40) and (41); we again obtain precisely Duffing's equation. And, again (47) can be solved after the fact, but now it only has dependence on p; it does not have explicit time dependence arising from the nonautonomous term of H nor any dependence on q.

Upon dropping the driving term, the system satisfies $\dot{H} = 0$, as it must since it is autonomous metriplectic. On the other hand for the case with driving when it is nonautonomous metriplectic,

$$\dot{H} = \frac{\partial H}{\partial t} = -\gamma\omega q \cos(\omega t + \phi)\,. \tag{48}$$

However, it is interesting that we still have

$$\dot{z} \geq 0\,,$$

even when the system is driven!

In this metriplectic case the addition of the δz term to the Hamiltonian of (39) makes physical sense as an internal energy term. This parallels the case of the metriplectic Navier-Stokes-Fourier system (see e.g. [13,15]) where viscous dissipation of energy is compensated for by the production of heat in an entropy equation. This is an example of a kind of metriplectic thermodynamic completion, where a thermodynamically inconsistent system is made consistent via an entropy equation that is consistent with the second law.

7. Conclusion

We have shown that both contact Hamiltonian and metriplectic dynamical systems naturally occur on the one-jet bundle. We have compared and contrasted the resulting flows, and have provided the Duffing equation as an example, which can be derived either as a contact Hamiltonian system or as a metriplectic dynamical system. This example shows how damping/friction can appear in both settings.

Both contact Hamiltonian and metriplectic dynamical systems have analogous constructions for infinite-dimensional systems, field theories (PDEs). Lifting the results of this setting to such systems is a natural next step. As noted in Section 1, understanding how both systems emerge from underlying large dimensional Hamiltonian systems is a goal.

CRediT authorship contribution statement

Philip J. Morrison: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization; **Yong-Geun Oh:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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