Coupled Oscillations - Four Carts and Five Springs

Theoretical Mechanics - Spring 2010

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Partnered with Joe Wilm

April 19th, 2010

Theory

Consider a system of coupled oscillators with masses defined in *Figure1*, as well as coupling springs defined in said figure.

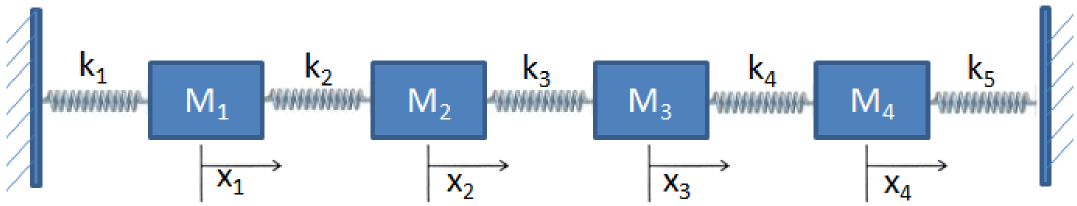

Figure 1: Four carts and five springs

Solving for the normal modes and normal frequencies of this system is best accomplished using matrix methods, which is shown in the following equations. The mass matrix ($\mathbf{M}$) of the system is shown in equation one. This matrix is symmetric and diagonal.

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix} \tag{1}$$

The $\mathbf{K}$ matrix is a mathematical representation of the manner in which the springs of this system are connected, and how they respond to each other. $\mathbf{K}$ for the system in *Figure1* is shown in equation two, which is also symmetric.

$$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 + k_5 \end{pmatrix} \tag{2}$$

The normal modes of the system are found by solving the eigenvalue equation in equation three.

$$(\mathbf{K} - \omega^2 \mathbf{M})\, \mathbf{a} = 0 \tag{3}$$

Where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Correspondingly, the normal frequencies of the system are calculated by solving equation four for ω.

$$\det(\mathbf{K} - \omega^2 \, \mathbf{M}) = 0 \tag{4}$$

The argument of the determinant in equation four is shown (expanded) in equation five.

$$(\mathbf{K} - \omega^2 \, \mathbf{M}) = \begin{pmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 - \omega^2 m_3 & -k_4 \\ 0 & 0 & -k_4 & k_4 + k_5 - \omega^2 m_4 \end{pmatrix} \tag{5}$$

Equation five's determinant is shown in equation six, which is the characteristic polynomial of the system.

$$\begin{aligned} -(k_4)^2(-(k_2)^2 + (k_1 + k_2 - \omega^2 m_1)(k_2 + k_3 - \omega^2 m_2)) + (k_3(-k_1 k_3 - k_2 k_3 + \omega^2 k_3 m_1) \\ +(-(k_2)^2 + k_1 + k_2 - \omega^2 m_1)(k_2 + k_3 - \omega^2 m_2))(k_3 + k_4 - \omega^2 m_3))(k_4 + k_5 - \omega^2 m_4) \quad &= 0 \end{aligned} \tag{6}$$

Solving equation six for ω will reveal the normal frequencies of the system. Solutions to this equation are *extremely* complicated, so some simplifications must be applied to the system for normal modes to be more easily found.

Normal Modes of the Completely Symmetric Case

In the limit where $k_1, k_2, k_3, k_4 \rightarrow k$ and $m_1, m_2, m_3, m_4 \rightarrow m$, equation five reduces to equation seven,

$$(\mathbf{K} - \omega^2 \mathbf{M}) = \begin{pmatrix} 2k - \omega^2 m & -k & 0 & 0 \\ -k & 2k - \omega^2 m & -k & 0 \\ 0 & -k & 2k - \omega^2 m & -k \\ 0 & 0 & -k & 2k - \omega^2 m \end{pmatrix} \tag{7}$$

The characteristic equation, the simplified form of equation six, likewise reduces to the form shown in equation eight.

$$(5k^2 - 5km\omega^2 + m^2\omega^4)(k^2 - 3km\omega^2 + m^2\omega^4) = 0 \tag{8}$$

There are four unique solutions for ω^2, which are shown below.

$$\begin{aligned} (\omega_1)^2 &= \tfrac{(3-\sqrt{5})k}{2m}, \quad (\omega_2)^2 = \frac{\left(3+\sqrt{5}\right)k}{2m} \\ (\omega_3)^2 &= \tfrac{(5-\sqrt{5})k}{2m}, \quad (\omega_4)^2 = \frac{\left(5+\sqrt{5}\right)k}{2m} \end{aligned} \tag{9}$$

Substituting ω_1 into the eigenvalue equation of equation three will give the system of linear equations shown below.

$$\tfrac{1+\sqrt{5}}{2} a_1 - a_2 = 0 \tag{10}$$

$$-a_1 + \tfrac{1+\sqrt{5}}{2} a_2 - a_3 = 0 \tag{11}$$

$$-a_2 + \tfrac{1+\sqrt{5}}{2} a_3 - a_4 = 0 \tag{12}$$

$$-a_3 + \tfrac{1+\sqrt{5}}{2} a_4 = 0 \tag{13}$$

The solutions to these equations show that a_2 and a_3 oscillate at $\frac{1+\sqrt{5}}{2}$ times the amplitude of a_1 and a_4s oscillation at this normal frequency, and that these oscillations are in phase. Mathematically,

$$a_2 = a_3 = \frac{1+\sqrt{5}}{2} a_1$$

$$a_1 = a_4$$

This is illustrated in *Figure2*.

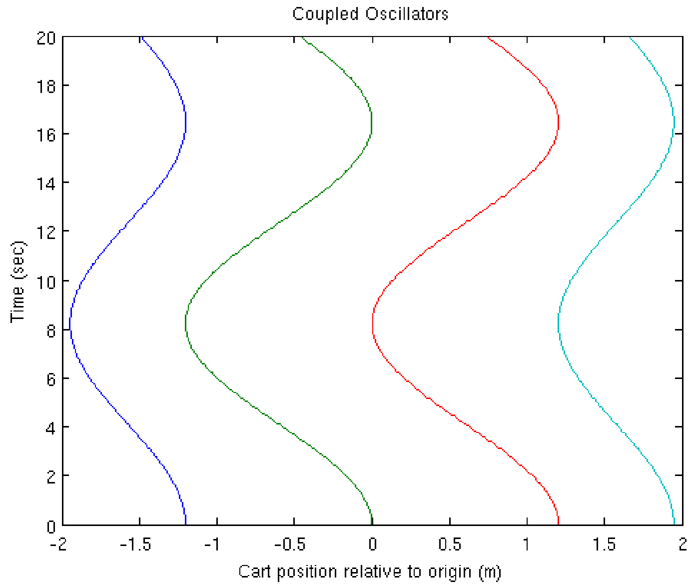

Figure 2: *Behavior of the system in ω_1.*

The system of linear equations resulting from ω_2 are shown in the following equations.

$$\tfrac{1-\sqrt{5}}{2}a_1 - a_2 = 0 \tag{14}$$

$$-a_1 + \tfrac{1-\sqrt{5}}{2}a_2 - a_3 = 0 \tag{15}$$

$$-a_2 + \tfrac{1-\sqrt{5}}{2}a_3 - a_4 = 0 \tag{16}$$

$$-a_3 + \tfrac{1-\sqrt{5}}{2}a_4 = 0 \tag{17}$$

Solving the equations for the individual amplitudes shows

$$a_2 = -a_3 = \frac{-1+\sqrt{5}}{2}a_1$$

$$a_1 = -a_4$$

These relations indicate that m_2 and m_3 are oscillating with an amplitude $\frac{-1+\sqrt{5}}{2}$ of the other two masses, completely out of phase of a_1. This is also shown in *Figure3*.

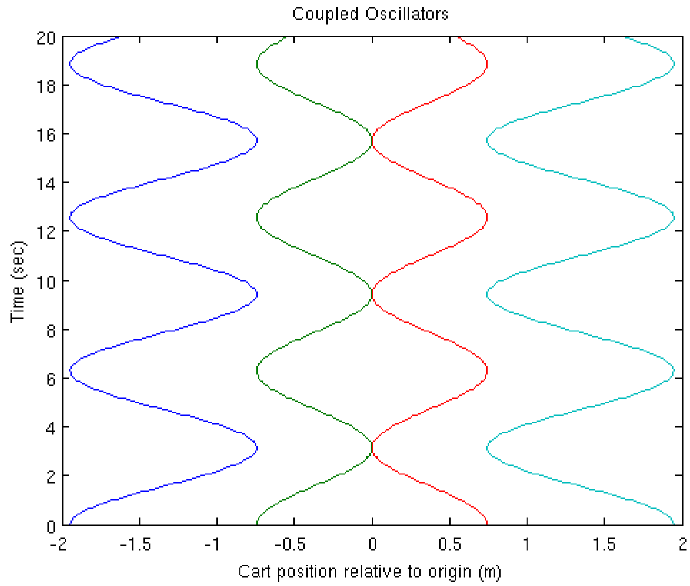

Figure 3: *Behavior of the system in ω_2.*

The corresponding system of linear equations for ω_3 is shown below.

$$\tfrac{-1+\sqrt{5}}{2}a_1 - a_2 = 0 \tag{18}$$

$$-a_1 + \tfrac{-1+\sqrt{5}}{2}a_2 - a_3 = 0 \tag{19}$$

$$-a_2 + \tfrac{-1+\sqrt{5}}{2}a_3 - a_4 = 0 \tag{20}$$

$$-a_3 + \tfrac{-1+\sqrt{5}}{2}a_4 = 0 \tag{21}$$

The amplitudes of the masses in this mode are

$$a_1 = -a_4$$

and

$$a_3 = a_2 = \frac{-1+\sqrt{5}}{2}a_1$$

These relations indicate that m_1 and m_4 are exactly out of phase and that m_2 and m_3 are in phase with amplitudes of $\frac{-1+\sqrt{5}}{2}$ times the amplitude of a_1. This is also shown in *Figure4*.

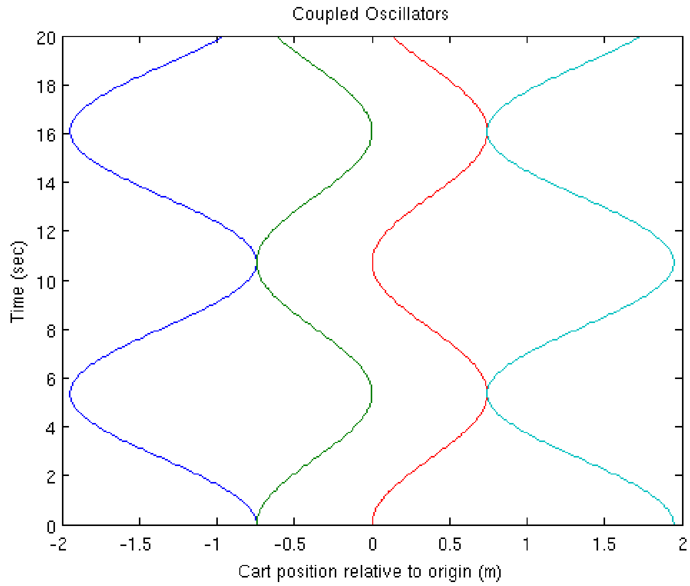

Figure 4: *Behavior of the system in* ω_3.

The equations from ω_4 are shown below.

$$\frac{-1-\sqrt{5}}{2}a_1 - a_2 = 0 \tag{22}$$

$$-a_1 + \frac{-1-\sqrt{5}}{2}a_2 - a_3 = 0 \tag{23}$$

$$-a_2 + \frac{-1-\sqrt{5}}{2}a_3 - a_4 = 0 \tag{24}$$

$$-a_3 + \frac{-1-\sqrt{5}}{2}a_4 = 0 \tag{25}$$

The amplitudes of the individual masses are

$$a_1 = -a_4$$

and

$$a_3 = -a_2 = \frac{1+\sqrt{5}}{2}a_1$$

Which is shown in *Figure5*.

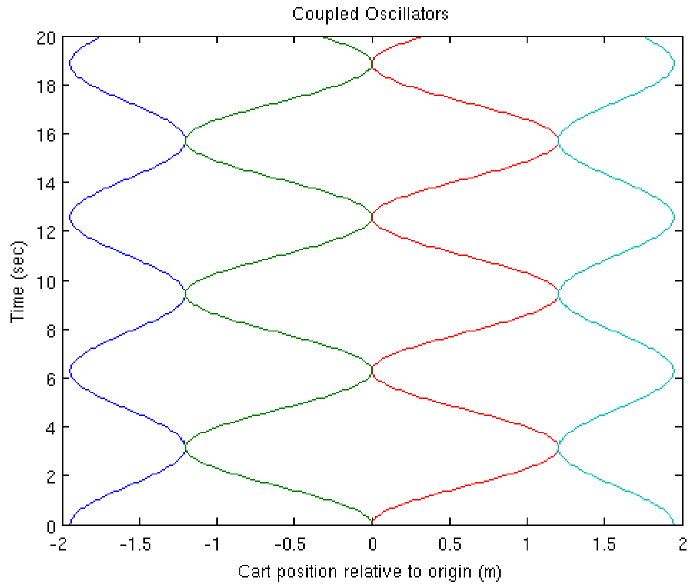

Figure 5: *Behavior of the system in* ω_4.

A comparison of all four normal modes is shown in *Figure6*.

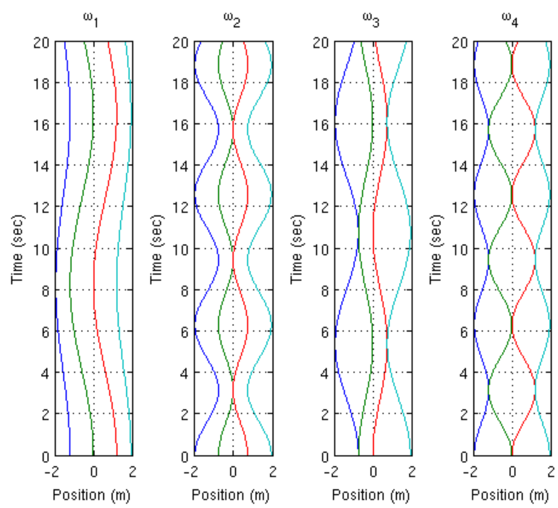

Figure 6: *Normal modes corresponding to different normal frequencies of the symmetric case. The blue line represents* m_1*, green* m_2*, red* m_3*, and teal* m_4*. The first mode has all carts in phase, with the amplitudes of oscillation of the middle masses* $\frac{1+\sqrt{5}}{2}$ *times that of the outside masses. The second mode shows* m_1 *and* m_2 *in phase, with the amplitude of* m_2 $\frac{-1+\sqrt{5}}{2}$ *times* m_1s*.* m_3 *and* m_4 *have the same motion out of phase with* m_1 *and* m_2*. The third mode shows* m_1 *and* m_4 *in phase and* m_2 *and* m_3 *(with amplitude* $\frac{1-\sqrt{5}}{2}$ *of* m_1s*) out of phase with* m_1 *and* m_4*. The fourth mode shows* m_1 *and* m_3 *out of phase with* m_2 *and* m_4*, with the amplitude of* m_2 *and* m_3 $\frac{1+\sqrt{5}}{2}$ *times the amplitude of* m_1*.*

A Little Anti-Symmetry

Since the program to graph the normal mode oscillation of the symmetric case was already written, it was almost trivial to change the parameters in question to show interesting modes of oscillation. Following this paragraph are a few of the most interesting modes.

The Limit Where $m_1 \to 2m$

The first situation analyzed was where m_1 was set to $2m$, and all other parameters were unchanged. The characteristic polynomial of this situation is shown in the following equation.

$$5k^4 - 24k^3m\omega^2 + 31k^2m^2\omega^4 - 14km^3\omega^6 + 2m^4\omega^8 = 0 \tag{26}$$

Unfortunately, the radical solutions to this equation are complex, so the solutions for the normal frequencies of the system are found using the NullSpace function built into Mathematica, which gives decimal approximations as solutions to the equation. The normal modes corresponding to these frequencies is shown in *Figure7*.

$$(\omega_1)^2 = \tfrac{k}{m}, \qquad (\omega_2)^2 = 0.327018\frac{k}{m} \tag{27}$$
$$(\omega_3)^2 = 3.46962\tfrac{k}{m}, \quad 2.20336\frac{k}{m}$$

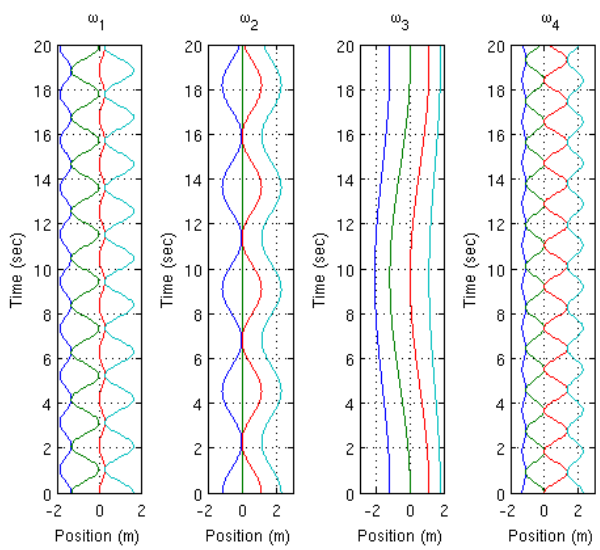

Figure 7: $m_1 = 2m$

The Limit Where $k_3 \gg k$

k_3 was set to $1000k$, and m_2 and m_3 were set to $\frac{1}{2}m$. The characteristic polynomial of this situation is shown in the following equation.

$$4001k^4 - 14006k^3m\omega + 10011k^2m^2\omega^2 - 2006km^3\omega^3 + m^4\omega^4 = 0 \tag{28}$$

The solutions of which are shown in the next equation.

$$\begin{aligned} (\omega_1)^2 = (2-\sqrt{2})\tfrac{k}{m}, \qquad & (\omega_2)^2 = (2+\sqrt{2})\frac{k}{m} \\ (\omega_3)^2 = (2002-\sqrt{4000002})\tfrac{k}{m}, \quad & (\omega_4)^2 = (2002+\sqrt{4000002})\frac{k}{m} \end{aligned} \tag{29}$$

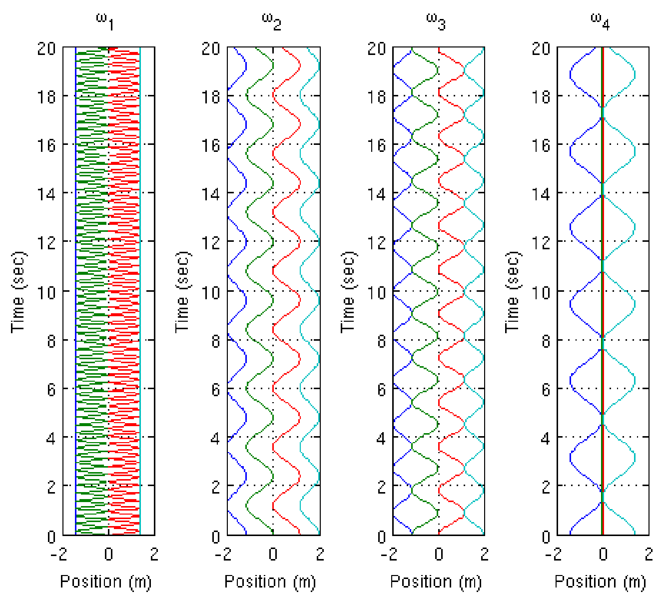

Figure 8: $k_3 \gg k$

The Limit Where $k_4 = k_2 \gg k$

All of the masses were set to $\frac{m}{2}$, and k_2 and k_4 were set to $1000k$. The characteristic polynomial is

$$3002000k^4 - 4005001k^3m\omega^2 + \frac{4012005}{4}k^2m^2\omega^2 - \frac{1001}{2}km^3\omega^6 + \frac{m^4\omega^8}{16} = 0 \tag{30}$$

Which has solutions as shown in the following equations.

$$(\omega_1)^2 = (2001 - \sqrt{4000001})\tfrac{k}{m}, \quad (\omega_2)^2 = (2003 - \sqrt{4000001})\frac{k}{m} \tag{31}$$
$$(\omega_3)^2 = (2001 + \sqrt{4000001})\tfrac{k}{m}, \quad (\omega_4)^2 = (2003 + \sqrt{4000001})\frac{k}{m}$$

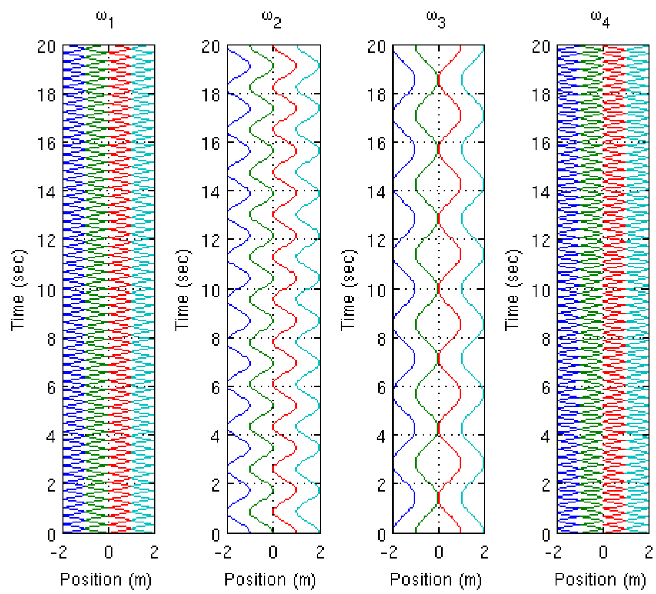

Figure 9: $k_2 = k_4 \gg k$

Examining Beat Modes

k_3 was set to $\frac{k}{20}$. The characteristic polynomial of the system is now

$$\frac{6k^4}{5} - \frac{67k^3}{10}m\omega^2 + \frac{23k^2}{2}m^2\omega^4 - \frac{61k}{10}m^3\omega^6 + m^4\omega^8 = 0 \tag{32}$$

which gives solutions of

$$\begin{aligned}(\omega_1)^2 &= \tfrac{(3-\sqrt{5})k}{2m}, \quad (\omega_2)^2 = \frac{(3+\sqrt{5})\,k}{2m} \\ (\omega_3)^2 &= \tfrac{(31-\sqrt{481})k}{20m}, \quad (\omega_4)^2 = \frac{(31+\sqrt{481})\,k}{20m}\end{aligned} \tag{33}$$

Normal frequency (ω_0) and ϵ solutions are easily found with decimal approximations, which are shown below.

$$\begin{aligned}(\omega_0)_1 &= 0.41769\tfrac{k}{m}, \quad (\omega_0)_2 = 2.63231\frac{k}{m} \\ \epsilon_1 &= 0.0357242\tfrac{k}{m}, \quad \epsilon_2 = 0.0142758\frac{k}{m}\end{aligned} \tag{34}$$

The graphs of these solutions are shown in *Figure10*.

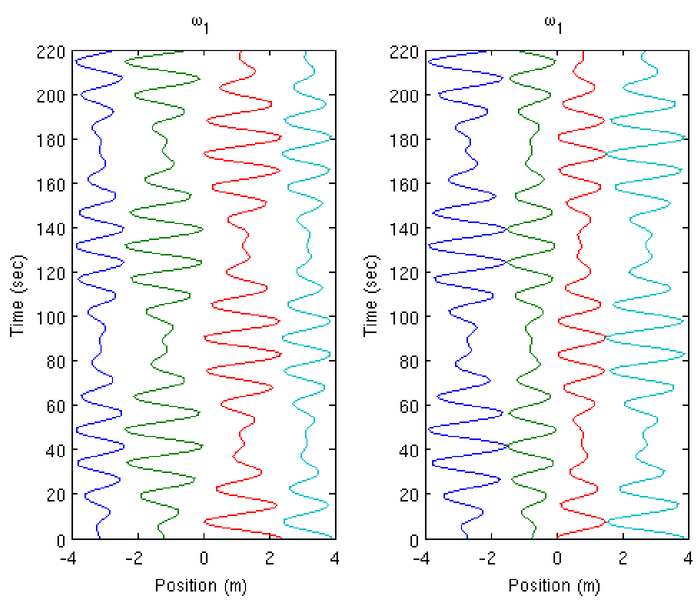

Figure 10: *Beat frequencies for an extremely weak k_3.*

k_2 and k_4 were then set to $\frac{k}{2}$, which has a characteristic polynomial as seen below.

$$\frac{7k^4}{4} - 9k^3 m\omega^2 + 12k^2 m^2 \omega^4 - 6km^3\omega^6 + m^4\omega^8 = 0 \tag{35}$$

Characteristic frequencies are now shown

$$(\omega_1)^2 = \tfrac{\left(2-\sqrt{2}\right)k}{2m}, \quad (\omega_2)^2 = \frac{\left(4-\sqrt{2}\right)k}{2m}$$
$$(\omega_3)^2 = \tfrac{\left(2+\sqrt{2}\right)k}{2m}, \quad (\omega_4)^2 = \frac{\left(4+\sqrt{2}\right)k}{2m} \tag{36}$$

As well as ω_0 and ϵ solutions of

$$(\omega_0)_1 = 0.792893\tfrac{k}{m}, \quad (\omega_0)_2 = 2.20711\frac{k}{m}$$
$$\epsilon_1 = 0.5\tfrac{k}{m}, \qquad \epsilon_2 = 0.5\frac{k}{m} \tag{37}$$

And finally, the motion for this case is shown in *Figure11*.

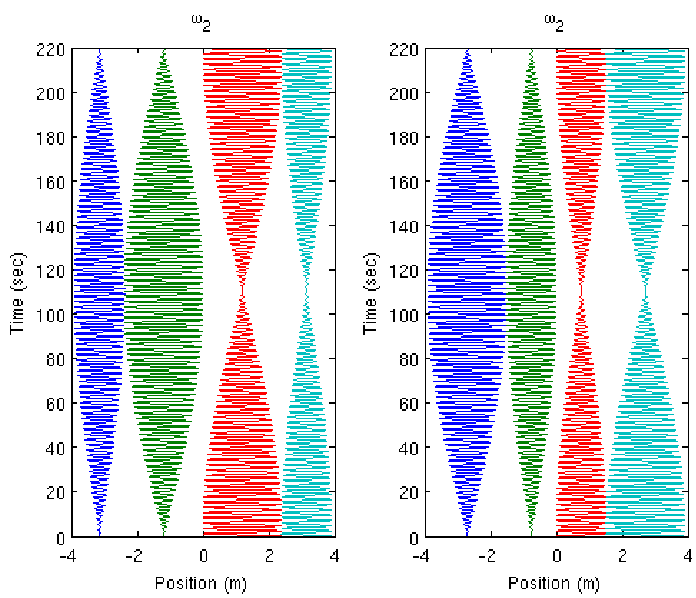

Figure 11: *Beat frequencies for* k_2 *and* k_4 *set to* $\frac{k}{2}$. *This case is interseting because the individual frequencies are very high and the beat frequencies are clearly visible.*