Solving Laplace's Equation With MATLAB Using the Method of Relaxation

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Submitted on December 8th, 2010

Abstract

Programs were written which solve Laplace's equation for potential in a 100 by 100 grid using the method of relaxation. These programs, which analyze specific charge distributions, were adapted from two parent programs. The first instance analyzed was one such that the analytic solution was known, and the computed solution was seen to agree with the theoretical solution. Three other systems were analyzed. One included unusual boundary conditions, another calculated the potential due to a non pointlike charge distribution at the center of the grid, and the last being a crude model of an atom. The parent programs written are included in an appendix.

Background

Maxwell's derivation of Maxwell's equations marked an incredible acheivement where a set of equations can completely describe charges and electric current. Gauss's law,[1] one of these equations, describes electric fields in a vacuum with charge density ρ.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

One of the main reasons electrostatics is so widely studied is because it is a versatile way of finding the electric potential in an area. Coulomb's law (equation one)[2] for potential is a common way of solving for potential in an area with a known charge density.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\vec{r} - \vec{r'}} \rho\left(\vec{r'}\right) \mathrm{d}\tau' \tag{1}$$

Unfortunately, this integral is often extremely difficult to solve, and Poisson's equation (equation two, which arises from the fact that $\vec{E} = -\nabla V$) is an easier way to calculate the potential.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{2}$$

In cases where charge density is zero, equation two reduces to Laplace's equation, shown in equation three.

$$\nabla^2 V = 0 \tag{3}$$

Laplace's equation is a partial differential equation and its solution relies on the boundary conditions imposed on the system, from which the electric potential is the solution for the area of interest. Analytic solutions to this equation can be found using the method of separation of variables (provided the resulting integrals are possible). Often, even Laplace's equation is too difficult or unruly to solve analytically, and computational methods must be used.

Theory and Preliminary Calculation

I will begin by showing a specific example[3]. Approximating two infinite half plates as squares with length 100 cm and taking their separation to be 100 cm (with $V_0 = 1$ Volt), an analytical solution is possible given the following boundary conditions.

$$V = 1 \,\text{for}\, x = 0$$

$$V = 0 \,\text{for}\, y = 0$$

$$V = 0 \,\text{for}\, y = 10$$

$$V = 0 \,\text{for}\, x = 10$$

These boundary conditions are useful to test the validity of the relaxation method because the potential in this case can be solved for anaytically. In two dimensions, Laplaces equation appears as follows.

$$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = 0 \tag{4}$$

Solving this analytically requires the assertion that the potential can be represented as the product of two functions; one for the x direction, one for the y direction. This is called the method of separation of variables, and it is a Physicist's best friend when working with partial differential equations.

$$V(x, y) = F(x)G(y) \tag{5}$$

Substituting this into Laplace's equation yields

$$G(y)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x) + F(x)\frac{\mathrm{d}^2}{\mathrm{d}y^2}G(y) = 0, \tag{6}$$

and dividing equation six by equation five shows

$$\frac{1}{F(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x) + \frac{1}{G(y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2}G(y) = 0. \tag{7}$$

Because these two equations are both independent functions and their sum is equal to zero, they must both be equal to a constant.

$$\frac{1}{F(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x) = C_1 \tag{8}$$

$$\frac{1}{G(y)}\frac{\mathrm{d}^2}{\mathrm{d}y^2}G(y) = C_2 \tag{9}$$

With

$$C_1 + C_2 = 0. \tag{10}$$

Because of our particluar boundary conditions, we require $C_1 > 0$ and $C_2 < 0$.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x) = k^2 F(x) \tag{11}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}G(y) = -k^2 G(y) \tag{12}$$

The solution to these two ordinary differential equations is shown below.

$$F(x) = Ae^{kx} + Be^{-kx} \tag{13}$$

$$G(y) = C\sin(ky) + D\cos(ky) \tag{14}$$

From equation 5,

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right). \tag{15}$$

Because our boundary conditions require that the potential falls to zero for $x = 100$, and the potential is zero for $y = 0$ and $y = 100$, $A = D = 0$, and

$$V(x,y) = Ce^{-kx}\sin(ky). \tag{16}$$

B gets sucked up into C, because they both represent arbitrary constants. The remaining boundary condition allows us to solve for k, which yields

$$k = \frac{n\pi}{100}. \tag{17}$$

A solution of Laplace's equation will be the sum of all the possible values of $V(x,y)$, as shown in equation 18.

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{100}} \sin\left(\frac{n\pi x}{100}\right) \tag{18}$$

The boundary condition at $x = 0$ requires that $V(0,y) = 1$.

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{100}\right) = 1 \tag{19}$$

C_n can be calculated using Fourier's trick as shown below:

$$C_n = \frac{2}{a}\int_0^{100} \sin\left(\frac{n\pi y}{100}\right)\mathrm{d}y, \tag{20}$$

with the integral being

$$C_n = \frac{2}{n\pi}(1 - \cos(n\pi)). \tag{21}$$

Equation 21 is $\frac{4}{n\pi}$ if n is odd, and 0 if n is even. The final solution for the potential is

$$V(x,y) = \sum_{n\,\text{odd}}^{\infty} \frac{4}{n\pi} e^{-\frac{n\pi x}{100}} \sin\left(\frac{n\pi y}{100}\right) \tag{22}$$

The infinite series in equation 22 converges to the analytic function of x and y shown in equation 24.

$$V(x,y) = \frac{2}{\pi} \arctan\left(\frac{\sin\left(\frac{\pi y}{100}\right)}{\sinh\left(\frac{\pi x}{100}\right)}\right) \tag{23}$$

A program was written to solve Laplace's equation for the previously stated boundary conditions using the method of relaxation, which takes advantage of a property of Laplace's equation where extreme points must be on boundaries. The program calculates the average between the four points closest to it, with the vital line of code being

```
V(iy,ix) = (V(iy-1,ix) + V(iy+1,ix) + V(iy,ix-1) + V(iy,ix+1))/4
```

Taking these averages over the grid many times (for some configurations, as many as 1000 iterations) reveals a solution which resembles the analytic solution determined by a given set of boundary conditions. Equation 23 plotted on the same axis as the computed value of potential using the method of relaxation is shown in the following figure, with equation 23 being the contour lines on the XY plane and the computed potential as the mesh.

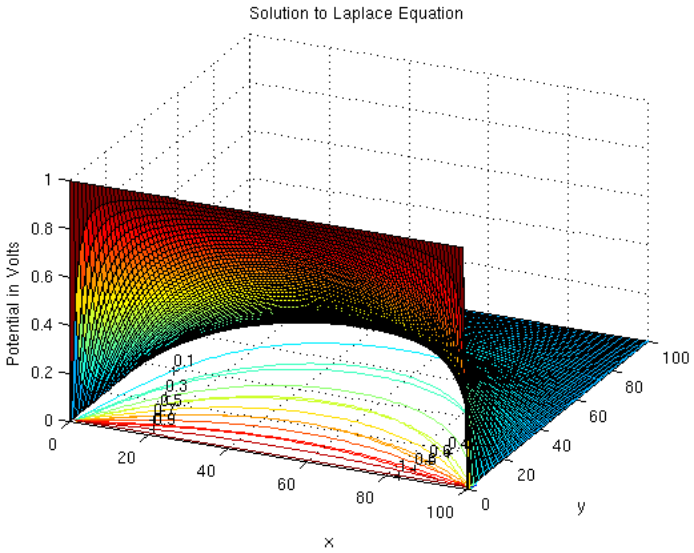

Figure 1: *The analytic solution (equation 23; contour lines), and the computed value of the potential (mesh). The agreement between computed and analytic values is very high, and the analytic plot is a bit different because of approximations taken with computation.*

The analytic and computed solutions for the previously stated boundary conditions are also shown in are shown in Figure 2.

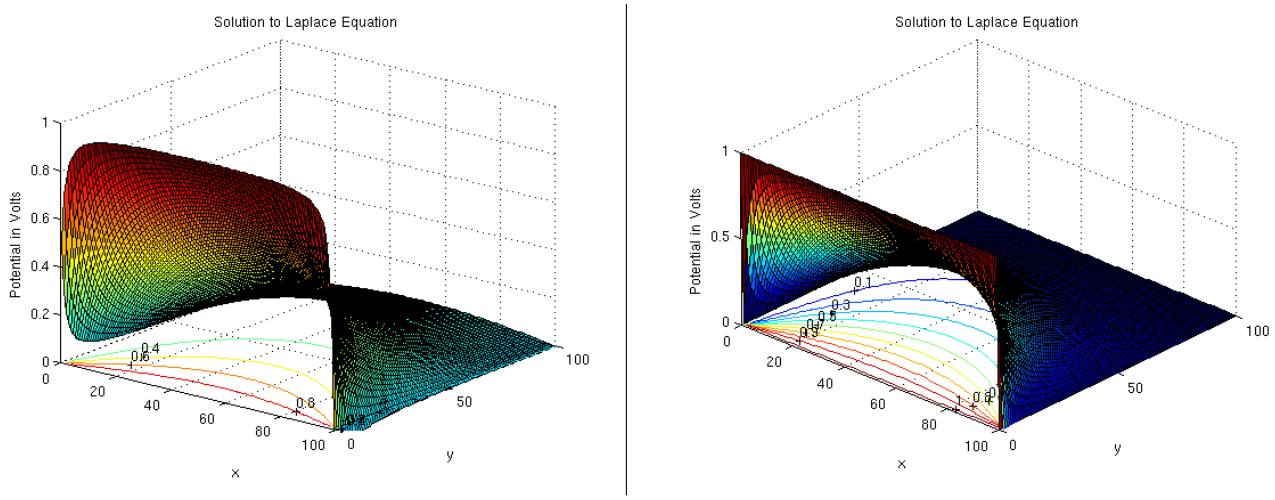

Figure 2: *The plot on the left is the analytic solution (equation 23), and the plot on the right is the computed solution. The agreement between computed and analytic values for functions should be perfect, while the analytic plot is a bit different because of the approximation of the infinite half-planes as 100 by 100 planes.*

The main reason I decided to compute the solution to these simple boundary conditions was to test the validity of my method of relaxation program. Because the two solutions are so similar, I can conclude that the program is working as can be expected and I can now move on to solve more complicated problems.

Computation

Unusual Boundary Conditions

The program created to solve considering boundary conditions only is incredibly versatile. Any single variable function can be used as a boundary, and the potential in the grid can be calculated. Many different configurations were tested, although the following boundary configuration is one of the most interesting.

$$V(x) = \sin\left(\frac{\pi x}{100}\right) \text{ for } y = 0\,;\; V(x) = \sin\left(\frac{\pi x}{100}\right) \text{ for } y = 100$$

$$V(y) = \sin\left(\frac{\pi y}{100}\right) \text{ for } x = 0\,;\; V(y) = \sin\left(\frac{\pi y}{100}\right) \text{ for } x = 100$$

The computed solution is shown in Figure 3.

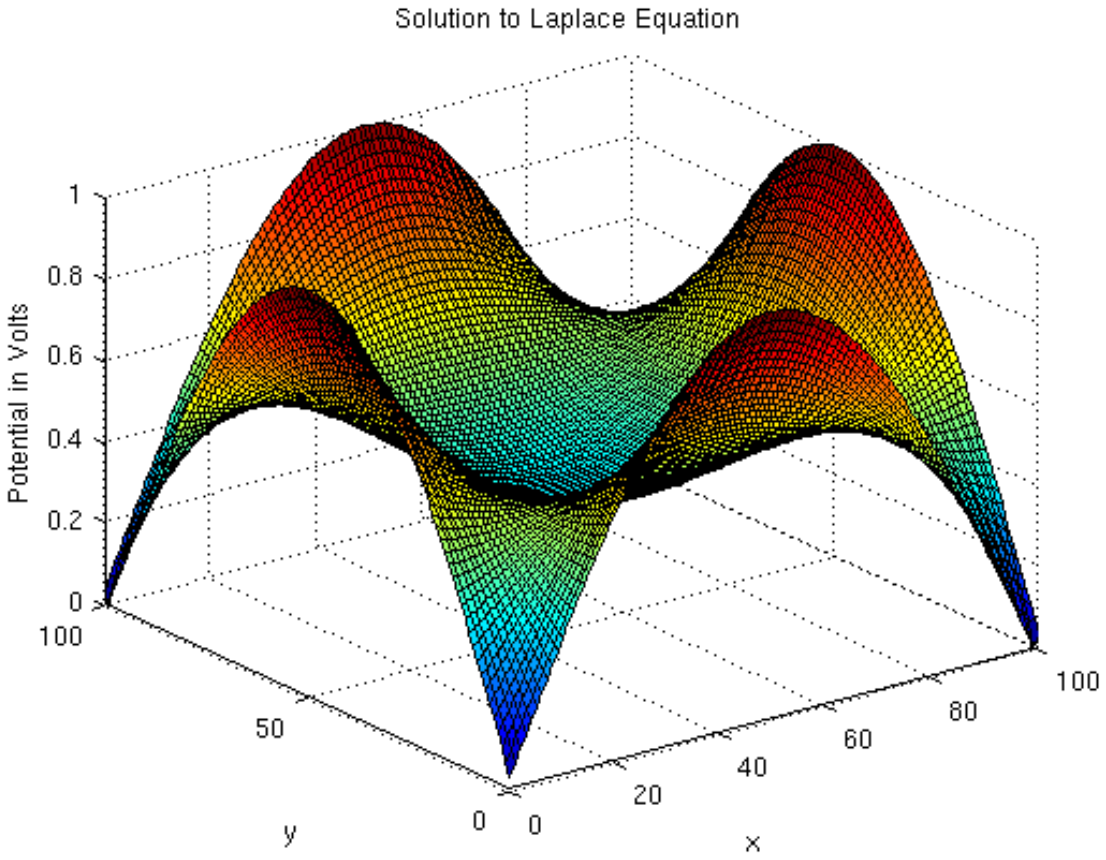

Figure 3: *A solution for interesting boundary conditions, which shows sinusoidal boundaries.*

The interesting thing is that, from this figure, it would seem that it is possible to violate Earnshaw's theorem. Earnshaw's theorem states that it is impossible to create an equilibrium point using only charges, that a stable equilibrium for potential energy is merely a local minimum in the electric potential. The potential energy can be calculated by

$$U(x, y) = qV(x, y),$$

where q is the charge of a particle in the grid. The local minimum at the center of the graph is an unstable equilibrium. What looks like a minimum is in fact a saddle point, and any small perturbation will send the particle on a path to shoot out of the grid.

Potential Due to a Distribution of Charge

Calculating the potential of a point charge was a matter of creating a small circle of charge with a potential equal to one volt. This is an interesting system because it does not involve boundary conditions, but a constant charge at the center of the grid. This program took a large amount of iterations to converge to show a smooth graph.

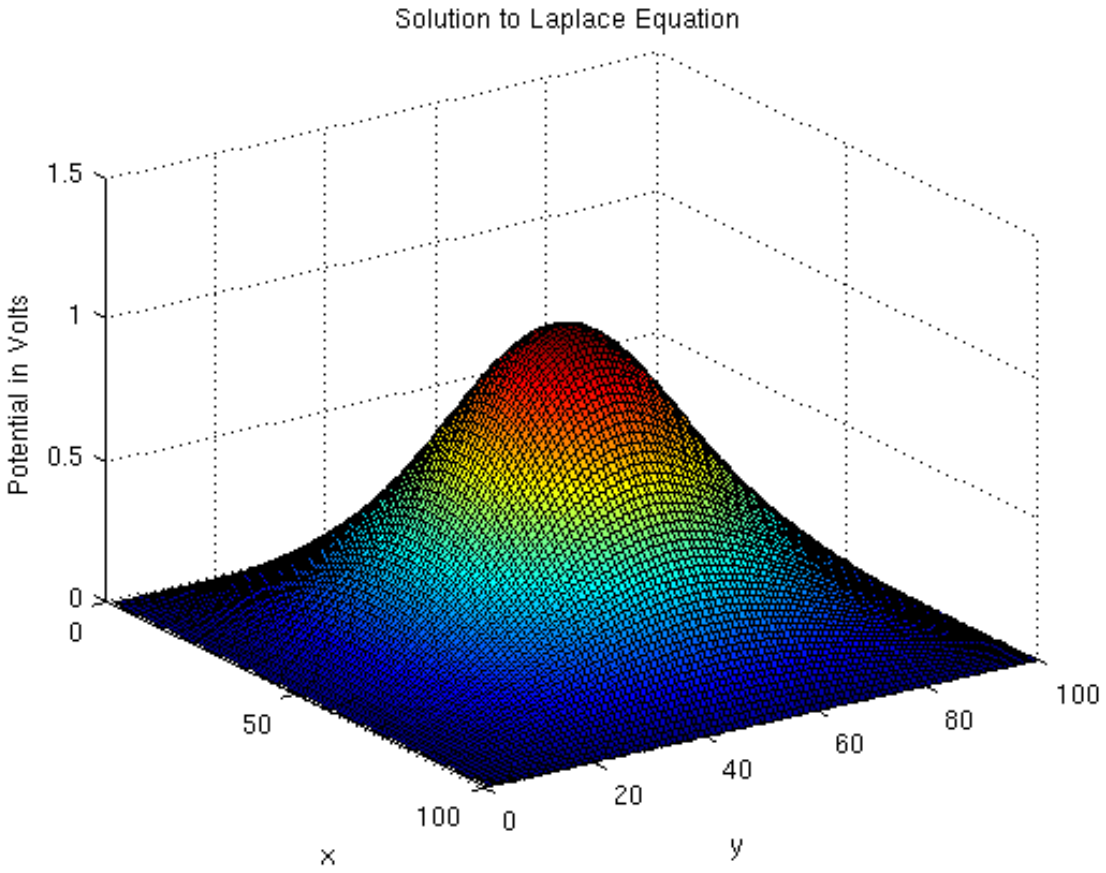

Figure 4: *The potential due to a non pointlike charge distribution at the center of the grid.*

It is interesting to note that the potential due to this charge distribution falls as $\frac{1}{r}$. This agrees with theory, matching the equation due to a point charge,

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}. \tag{24}$$

A common practice in electrostatics is to approximate a distribution of charge around its expected value (similar to finding the center of mass of a rigid body). The potential in Figure 4 behaves exactly like if it had been a point charge centered at its expected value.

Model of an Atom

By creating a large positive blob of charge at the center of the grid, and a larger blob of negative charge around the center blob shows a crude model of an atom.

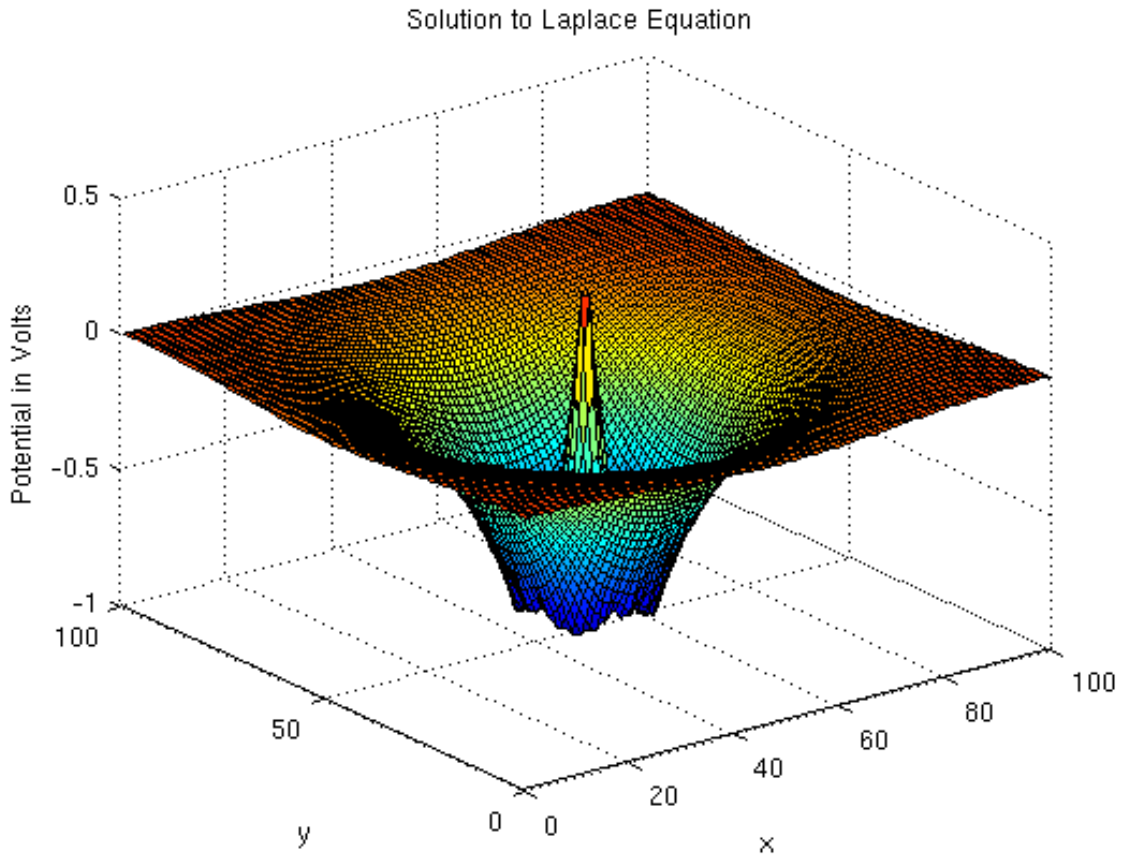

Figure 5: *A model of the potential due to an atom. Notice how the potential goes to zero as r increases.*

This is an interesting case where the charge in the center (the nucleus) is very small, and the electron cloud around the center charge is larger. It seems that (from this graph) the radial dependence on potential for an atom can be modeled[1] using concentric cylinders. This may not be of any use, but at least the concentric cylinder problem has an analytic solution. More importantly, though, the computed potential is radially symmetric with the potential far from the charge distributions averaging to zero.

[1] A model of a model!

Conclusion

It was shown that solving Laplace's equation using the method of relaxation will yield a potential that resembles the actual potential of the system. The programs written are incredibly versatile, and can solve for any combination of boundary conditions. A lot of work went into making the programs user friendly and self-explanatory with many comments, although it is important to keep in mind that some combinations of boundary conditions and charge distributions may require many iterations to converge.

References

[1] Chad Jeffrey Ohlandt. Basic laws - coulomb, ampere, faraday. `http://www-personal.umich.edu/~chadjo/research/embasic/node2.html`.

[2] R. Nave. Continuous charge distributions. `http://hyperphysics.phy-astr.gsu.edu/hbase/electric/conchg.html`.

[3] David J. Griffiths. *Introduction to Electrodynamics*. Pearson, 3rd edition, 1999. p. 127-132.

Program Appendix

Calculating the Potential Using Boundary Conditions

```
%This program calculates the solution to Laplace's equation, which is
%defined as the divergence of the gradient of a certian potential (or
%steady state heat flow). The program calculates the solutions to Laplace's
%equation using the relaxation method, and does so for a square grid that
%has a 100 unit principal length.

clear all; close all; clc

x = 0:1:100; %defines the physical space for the model
y = 0:1:100; %(creates the grid to plot solution values on)

V(101,101) = 0;    % Set all points to zero.

for i=1:101
V(i,1) = 10; % Set one boundary to a certain function (in Volts).
end
for i=1:101
V(1,i) = 0; % Set one boundary to a certain function (in Volts).
end
for i=1:101
V(i,101) = 0; % Set one boundary to a certain function (in Volts).
end
for i=1:101
V(101,i) = 0; % Set one boundary to to a certain function (in Volts).
end

n=1000;     %sets the number of iterations, needs to be quite high to get a
            %good representation of the solution

for i=1:n; %for loop using the relaxation method to calculate the potential
    for ix = 2:100;     % all internal points
       for iy = 2:100;  % (N - 2 total internal points for each dimension)
          V(iy,ix) = (V(iy-1,ix) + V(iy+1,ix) + V(iy,ix-1) + V(iy,ix+1))/4;
       end
    end
end

V;   % Print out values for the electric potential (uncomment if you really
     % want to see this, but it's 1000 numbers)

[X,Y] = meshgrid(x,y);     % plots a mesh of the solution

subplot(2,2,1); surf(X,Y,V)  %plots a surface plot of the solution
xlabel('x'); ylabel('y'); zlabel('Potential in Volts');

subplot(2,2,2);contour(X,Y,V)%shows a (shaded) contour plot of the solution
[potential] = contour(V);
xlabel('x'); ylabel('y');clabel(potential);

subplot(2,2,3); contour3(X,Y,V)  %shows a contour plot with colored lines
xlabel('x'); ylabel('y'); zlabel('Potential in Volts');

subplot(2,2,4); meshc(X,Y,V)    %shows a mesh plot of the solution
xlabel('x'); ylabel('y'); zlabel('Potential in Volts');
```

Calculating the Potential Due to a Concentrated Charge Distribution

```
%This program solves Laplace's equation for a non pointlike charge
%distribution at the center and another non pointlike charge (with opposite
%sign) also at the center. This program is a crude model of an atom with
%the nucleus having the same charge as the electron cloud, and can
%calculate the potential of the combination of charges.

clear all; clc; close all

n=100;                              %set length of the grid
V=zeros(n); V(n,n)=0;               %initializing the grid as zero

h = waitbar(0,'Please wait...');

for i=1:1000              %calculating the solution using relaxation method

 for k=n/2-10:n/2+10          %sets up a non pointlike distribution of
  for j=n/2-10:n/2+10         %charge at the center of the grid
    r=(k-n/2)^2+(j-n/2)^2;        %which is negative
    if r<=100 && r>=95
      V(k,j)=10;
    end
  end
 end

%V=(V(:,[n,(1:(n-1))])+V(:,[(2:n),1])+V([n,(1:(n-1))],:)+V([(2:n),1],:))/4;

for f = 1:n; %for loop using the relaxation method to calculate the potential
    for fx = 2:99;     % all internal points
       for fy = 2:99;  % (N - 2 total internal points for each dimension)
          V(fy,fx) = (V(fy-1,fx) + V(fy+1,fx) + V(fy,fx-1) + V(fy,fx+1))/4;
       end
    end
end

    j=i/1000;
    waitbar(j)

%the previous line is the relaxation method in all its glory, calculating
%the potential using averages.

 V(1,1:n)=zeros(1,n); V(n,1:n)=zeros(1,n); V(1:n,1)=zeros(n,1);
 V(1:n,n)=zeros(n,1); %resetting the boundary values

end

close(h)

% for i=1:n
%     re(x,y)=1/sqrt(xi^2+yi^2);
% end

x=(1:n);
y=(1:n);
surf(x,y,V)
xlabel('x'); ylabel('y'); zlabel('Potential in Volts');
title('Solution to Laplace Equation')
```