The Electromagnetic Stress Tensor

Matt Guthrie

March 18, 2013

In General Relativity,

$$T_{ab} = \frac{1}{4\pi}\left(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{cb}F_{cd}F^{cd}\right) \tag{1}$$

where

$$F = \left(F^{ab}\right) = \begin{pmatrix} 0 & E^{\intercal} \\ -E & B \end{pmatrix}, \tag{2}$$

with $Bp = (-B) \times p$.

Condense this to Special Relativity, with Minkowski coordinates.

$$g_{ab} = \eta_{ab},\ \ g^{ab} = \eta^{ab} \tag{3}$$

$$F_{cb} = \eta_{ac}\eta_{bd}F^{cd} = \eta_{ac}F^{cd}\eta_{bd} \tag{4}$$

Because $\eta_{bd} = \eta_{db}$,

$$F_{ab} = \eta_{ac}F^{cd}\eta_{db} = \left(\eta F \eta\right)_{ab}, \tag{5}$$

and thus,

$$\hat{F} = \left(F_{ab}\right) = \eta F \eta. \tag{6}$$

1.

$$F_{cd}F^{cd} = -F_{dc}F^{cd} = -\mathrm{tr}\left(\hat{F}F\right) \tag{7}$$

$$\hat{F}F = \left(\eta F \eta\right)F = \left(\eta F\right)^2 \tag{8}$$

Therefore, in Special Relativity,

$$\frac{1}{4}g_{ab}F_{cd}F^{cd} = \frac{1}{4}\eta_{ab}F_{cd}F^{cd} = -\frac{1}{4}\mathrm{tr}\left(\left(\eta F\right)^2\right)\eta_{ab} \tag{9}$$

2.

$$-\eta^{cd}F_{ac}F_{bd} = F_{ac}\eta^{cd}\left(-F_{bd}\right) = F_{ac}\eta^{cd}F_{db} = \left(\hat{F}\eta\hat{F}\right)_{ab} \tag{10}$$

$$\hat{F}\eta\hat{F} = \left(\eta F\eta\right)\eta\left(\eta F\eta\right) = \eta F\eta F\eta \tag{11}$$

Since $\eta^2 = I$,

$$\hat{F}\eta\hat{F} = \left(\eta F\right)^2\eta. \tag{12}$$

Write $T = (T_{ab})$. Therefore,

$$T = \frac{1}{4\pi}\left(\left(\eta F\right)^2 - \frac{1}{4}\mathrm{tr}\left(\left(\eta F\right)^2\right)I\right)\eta. \tag{13}$$

3.

$$\eta F = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & E^\intercal \\ -E & B \end{pmatrix} = \begin{pmatrix} 0 & E^\intercal \\ E & -B \end{pmatrix}. \tag{14}$$

Therefore,

$$\left(\eta F\right)^2 = \begin{pmatrix} 0 & E^\intercal \\ -E & B \end{pmatrix}\begin{pmatrix} 0 & E^\intercal \\ E & -B \end{pmatrix} = \begin{pmatrix} E^2 & -E^\intercal B \\ -BE & EE^\intercal + B^2 \end{pmatrix} \tag{15}$$

$-BE = B\times E = -E\times B$ $(-BE)^\intercal = -E^\intercal B^\intercal = -E^\intercal\left(-B\right) = E^\intercal B$
Therefore,

$$-E^\intercal B = \left(E\times B\right)^\intercal. \tag{16}$$

$$B^2p = B\left(-B\times p\right) = -B\times\left(-B\times p\right) \tag{17}$$

$$B^2p = \left(-B\cdot p\right)\left(-B\right) - \left(-B\right)\cdot\left(-B\right)p = BB^\intercal p - B^2Ip \tag{18}$$

Therefore,

$$B^2 = BB^\intercal - B^2I \tag{19}$$

and

$$\left(\eta F\right)^2 = \begin{pmatrix} E^2 & \left(E\times B\right)^\intercal \\ -\left(E\times B\right) & EE^\intercal + BB^\intercal - B^2I \end{pmatrix}. \tag{20}$$

4.
$\mathrm{tr}\left(\left(\eta F\right)^2\right) = E^2 + E^2 + B^2 - 3B^2 = 2\left(E^2 - B^2\right)$, so

$$O = (\eta F)^2 - \frac{1}{4}\mathrm{tr}\left((\eta F)^2\right) I = (\eta F)^2 - \frac{1}{2}\left(E^2 - B^2\right) I \tag{21}$$

$$O = \begin{pmatrix} E^2 & (E \times B)^\intercal \\ -E \times B & EE^\intercal + BB^\intercal - B^2 I \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\left(E^2 - B^2\right) & 0 \\ 0 & -\frac{1}{2}\left(E^2 - B^2\right) I \end{pmatrix} \tag{22}$$

$$O = \begin{pmatrix} \frac{1}{2}\left(E^2 + B^2\right) & (E \times B)^\intercal \\ -E \times B & EE^\intercal + BB^\intercal - \frac{1}{2}\left(E^2 + B^2\right) I \end{pmatrix} \tag{23}$$

5.

$$4\pi T = \begin{pmatrix} \frac{1}{2}\left(E^2 + B^2\right) & (E \times B)^\intercal \\ -E \times B & EE^\intercal + BB^\intercal - \frac{1}{2}\left(E^2 + B^2\right) I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} \tag{24}$$

Therefore,

$$T = \frac{1}{4\pi} \begin{pmatrix} \frac{1}{2}\left(E^2 + B^2\right) & -(E \times B)^\intercal \\ -(E \times B) & \frac{1}{2}\left(E^2 + B^2\right) I - EE^\intercal + -BB^\intercal \end{pmatrix}. \tag{25}$$

End result:

The energy density of the field, T_{00}, is

$$T_{00} = \frac{1}{8\pi}\left(E^2 + B^2\right). \tag{26}$$

And the momentum density of the field, $T_{\alpha 0}$, is

$$T_{\alpha 0} = -\frac{1}{4\pi} E \times B \tag{27}$$