A One-Dimensional PDE Boundary Value Problem

This is the wave equation in one dimension. The equation states that the second derivative of the height of a string ($u(x,t)$) with respect to time (t) is equal to the speed of the propagation of the wave (c) in the medium it's in multiplied by the second derivative of the height of the string with respect to position.

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t) \tag{1}$$

The partial differential equation basically says that the acceleration of the height of the string is equal to the speed of the string squared multiplied by the concavity of the graph. To start solving, we need to apply some assumptions. These are necessary to make sure our model make physical sense.

$$0 \leq x \leq L$$
$$t > 0$$

Boundary conditions are how the solution is defined at the endpoints of the system.

$$u(0,t) = 0$$
$$u(L,t) = 0$$

These conditions state that the endpoints are fixed at all times.

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

These conditions state the initial value of the height of the string and its slope. Without these known, it is impossible to find the solution of $u(x,t)$.

To solve the partial differential equation in equation (1), we use the method of separation of variables, which is the assumption that the solution $u(x, t)$ can be described by two independent functions of x and t. This is valid for a whole mess of functions (every one I've ever had to solve for, even in cylindrical and spherical coordinates), although I doubt that it is valid for every function. Mathematicians can come up with some pretty weird stuff.

$$u(x, t) = F(x)G(t) \tag{2}$$

The partial differential equation in (1) becomes the following pair of ordinary differential equations when we substitute the assumption we made of u shown in (2).

$$F(x)\frac{d^2}{dt^2}G(t) = c^2\frac{d^2}{dx^2}F(x)G(t) \tag{3}$$

Because these are differential equations that are equal to each other, they must be simultaneously equal to a constant, and we can rewrite them in the following manner.

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = -\lambda \tag{4}$$

The following equations are the individual differential equations for $F(x)$ and $G(t)$ (we also can assume that $\lambda > 0$, or else weird stuff happens.

$$F''(x) + \lambda F(x) = 0$$

$$G''(t) = -\lambda c^2 G(t)$$

Solving for $F(x)$ first.

The solution of the differential equation is a linear combination of sine and cosine functions.

$$F(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

Now we can apply some boundary conditions.

$$F(0) = 0 = c_1$$
$$F(x) = c_2 \sin\left(\sqrt{\lambda}x\right)$$
$$F(L) = 0 \Rightarrow 0 = c_2 \sin\left(\sqrt{\lambda}L\right)$$
$$0 = c_2 \sin(n\pi) = c_2 \sin\left(\sqrt{\lambda}L\right)$$
$$\sqrt{\lambda} = \frac{n\pi}{L}$$

If it isn't clearly evident why this is valid, plot the following equation on the domain $0 \leq x \leq 1$ for different values of n (and with $L = 1$ for simplicity). Coincidentally, this is (most of) the solution for $F(x)$.

$$F(x) \sim \left[\sin\left(\frac{n\pi x}{L}\right)\right] \tag{5}$$

The $\sim$ and the square brackets mean that we're not quite finished with the solution for $F(x)$, which will be revisited soon. Now, we'll try solving for $G(t)$.

$$G''(t) = -\lambda c^2 G(t) \tag{6}$$

$$\frac{G''(t)}{G(t)} = -\lambda c^2$$

Again, sines and cosines.

$$G(t) = c_3 \cos\left(\sqrt{\lambda}ct\right) + c_4 \sin\left(\sqrt{\lambda}ct\right) \tag{7}$$

The solutions to $u(x,t)$ involve Fourier series, as does every PDE I've ever solved for in more than one dimension (not a PDE).

$$u_n(x,t) = F(x)G(t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \tag{8}$$

Using the initial condition (substitute $t = 0$ for the above equation),

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Next,

$$\frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty}\left[-A_n\frac{n\pi c}{L}\sin\left(\frac{n\pi ct}{L}\right) + B_n\frac{n\pi c}{L}\cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial}{\partial t}u(x,0) = g(x) = \sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)$$

and

$$B_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

The solution for $u(x,t)$ is the preceeding two calculations substituted into equation eight. You can use the same kind of technique for higher dimensions (separate the variables, apply boundary conditions).

The solutions for the circular shape are a bit more difficult, involving circular Bessel functions, but the solutions for the square (two spatial dimensions) aren't much more difficult than the solution for one spatial dimension.