Basic Logic and Set Theory

This short overview is designed to lay down the rudimentary aspects of sets and their associated notation. Set theory provides a foundation for virtually all of modern mathematics, and as such it is helpful to have at least an intuitive grasp of it. Fortunately, intuition generally does not fail us when it comes to the basics of set theory.

- A *set*, usually denoted by capital letters such as A, B, X, and so on, is a collection of *elements*, which may be anything so long as it is unambiguously defined. We mean that x is an element of the set A when it is written $x \in A$. If x is not an element of A, we write $x \notin A$.
  - For example, let A denote the set of people in a room, B denote the set of past and present U.S. Presidents, and Ω denote the set of tree species found in North America (assume these definitions may be tweaked so as to satisfy our requirement of unambiguity). If we let x denote George Washington, then clearly $x \in B$ while $x \notin A$. Let α denote the trees displaced by the Dell Medical School's construction, and let β denote the Live Oak species *Quercus fusiformis*. Then $\beta \in \Omega$, but $\alpha \notin \Omega$.
  - Practically any mathematical object we refer to is a set. For instance, we often speak of the set of natural numbers $\mathbb{N}$, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, the set of real numbers $\mathbb{R}$, and the set of complex numbers $\mathbb{C}$. Thus, $-4 \notin \mathbb{N}$, while $-4 \in \mathbb{Z}$. We may speak of the set of all circles in the Euclidean plane, the set of all continuous functions defined on the interval $[0, 1]$ or the set of all polynomials with rational coefficients.
  - Clearly, we will never be able to list all of the elements in most sets we ever consider. However, this is unnecessary as long as the set is clearly defined so that we may immediately note whether or not any particular element is a member of the set.
  - On a side note, we are restricted in what we can call a set. Particularly, sets cannot be "too big" (e.g. there is no such thing as the set of all sets).
- We say A is a *subset* of B (or A is contained in B, or A is included in B, or B contains A), written $A \subseteq B$, if $x \in B$ whenever $x \in A$. That is, all of the elements of A are also in B. We say A is a *proper subset* of B if $A \subseteq B$ and also there exists $x \in B$ with $x \notin A$; that is, B contains everything in A and also something more. In this case we write $A \subset B$ if we wish to emphasize that A is properly contained in B. We write $A \nsubseteq B$ to mean A is not a subset of B.
  - Again, if A denotes the set of all people in a lecture hall, and B denotes the set of all people in that lecture hall who are awake, we probably observe that $B \subseteq A$ (and hopefully it is not the case that $B \subset A$). If Ω is described as above, and Ω_0 is the set of all tree species native to Texas, then we see $\Omega_0 \subseteq \Omega$ and indeed we may say this inclusion is proper: $\Omega_0 \subset \Omega$.
  - You may be interested to see that we have the following chain of sets of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

- Before moving on, it will be useful to introduce some common logical symbols:
  - The colon ":" means "such that."
  - $p \Rightarrow q$ is the implication symbol which reads "the statement p implies the statement q" (i.e. if p is true, then q is also true).

- $p \Leftrightarrow q$ is the equivalence (or biconditional) symbol, which reads "the statement p if and only if the statement q" (i.e. p is true if q is true, and p is false if q is false).
- $\forall$ is the *universal quantifier*, meaning "for all." e.g. as the square of any real number x is nonnegative, we may write $\forall\, x \in \mathbb{R}$, $x^2 \geq 0$ (for any real number x, x^2 is greater than or equal to 0).
- $\exists$ is the *existential quantifier*, meaning "there exists." e.g. in any subset (more about this soon) X of integers, there exists a largest element, so we may write $\forall\, X \subseteq \mathbb{Z}, \exists\, x \in X : \forall\, y \in X \Rightarrow x \geq y$.
- $\exists!$ is the *uniqueness quantifier*, meaning "there exists exactly one." e.g. any natural number $n \in \mathbb{N}$ has a unique factorization into prime numbers. Therefore, if Y is the set of all prime numbers (this has been proved to be an infinite set), then we can write:

  $$\forall\, n \in \mathbb{N}\ \exists!\, \{p_1, p_2, ..., p_\ell\} \subset Y : n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_\ell^{\alpha_\ell} \text{ for some } \{\alpha_1, \alpha_2, ..., \alpha_\ell\} \subset \mathbb{N}$$

  To illustrate: if $n = 5880$, then the unique set of prime factors is $\{2, 3, 5, 7\} \subset Y$, and $5880 = 2^3 \cdot 3 \cdot 5 \cdot 7^2$, where $\{3, 1, 1, 2\} \subset \mathbb{N}$ is the corresponding set of their powers.

- Two sets A and B are said to be *equal*, written $A = B$, if they contain exactly the same elements. By this definition of equality, we see that $A = B$ if and only if it is true that $A \subseteq B$ and $B \subseteq A$. Note that this gives a different characterization of a proper subset: A is a proper subset of B if it is contained in B but not equal to B.
  - If Ω' is the set of all tree species native to Austin, TX, and Ω'' is the set of all tree species native to San Marcos, TX, then it is most likely the case that $\Omega' = \Omega''$. With almost equal certainty, we may say that $\Omega' \subseteq \Omega_0$ but $\Omega' \neq \Omega_0$ (one need only find a tree species found in Texas but not in Austin), so this inclusion is a proper one.
  - Let $2\mathbb{Z}$ denote the set of all even integers, and let X denote the set of all numbers which take the form $2n - 12$ for $n \in \mathbb{Z}$. It is not hard to prove that $2\mathbb{Z} = X$.
- The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements.
  - Let A be the set of all corpses in a lecture hall. Hopefully $A = \emptyset$.
  - Let B be the set of all even numbers which take the form $2n + 1$ for $n \in \mathbb{Z}$. Then $B = \emptyset$.
- A set is often defined by describing the properties common to the elements contained in it. The notation used is $\{x : x$ has so-and-so property$\}$ (the set of all x such that x has so-and-so property). We may also define a set by listing a few of its elements and hoping the pattern is obvious to the reader.
  - $A = \{x : x$ is or has been a U.S. President$\} = \{$George Washington, John Adams, Thomas Jefferson, ...$\}$
  - $\Omega = \{x : x$ is a tree species native to North America$\}$ (using the list definition for this set is probably not the wisest choice)
  - $\mathbb{N} = \{0, 1, 2, ...\}$
  - $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
  - $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\} = \{..., -4, -2, 0, 2, 4, ...\}$

- The *union* of two sets A and B is denoted by $A \cup B$. This is defined by the set containing all of the elements in A, or in B, or in both. Using the above notation, we have $A \cup B = \{x : x \in A \text{ or } x \in B\}$. We may likewise define the union of more than two sets. If $A_1, A_2, ...$ are various sets, then the union of A_i is denoted $A_1 \cup A_2 \cup ... \cup A_n$ (if there are n sets of which the union is constructed) or $\cup_{i=1}^{n} A_i$ to mean the set containing elements that are in at least one of the A_i.
  - Let Ω_1 be the set of all tree species native to one U.S. state, Ω_2 be the set of all tree species native to another state, and so on until Ω_{50}. Then
  $$\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_{50} = \bigcup_{i=1}^{50} \Omega_i = \tilde{\Omega},$$
  where we let $\tilde{\Omega}$ be the set of all tree species native to the United States.
  - If we let $2\mathbb{Z}+1 = \{..., -5, -3, -1, 1, 3, 5, ...\}$ denote the set of all odd integers, then $2\mathbb{Z} \cup (2\mathbb{Z} + 1) = \mathbb{Z}$. Note this merely states that every integer is either even or odd (or both).
- The *intersection* of two sets A and B is denoted by $A \cap B$, and is defined to be the set containing all of the elements in both A and B; that is, $A \cap B = \{x : x \in A \text{ and } x \in B\}$. Similarly, we may generalize this to the intersection of n or even an infinite number of sets, denoted by $A_1 \cap A_2 \cap ... \cap A_n$ or $\cap_{i=1}^{n} A_i$ in the finite case, which is meant to denote the set whose elements are in all of the A_i.
  - Consider the set $\Omega_1 \cap \Omega_2 \cap ... \cap \Omega_{50}$. Is this an empty set? That is, are there any tree species which are native to all 50 states?
  - Let $3\mathbb{Z} = \{3n : n \in \mathbb{Z}\} = \{..., -9, -6, -3, 0, 3, 6, 9, ...\}$ denote the set of all integers divisible by 3. Then we have $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z} = \{6n : n \in \mathbb{Z}\}$, the set of all integers divisible by 6.
- The *complement* of a set A relative to another set U is denoted by $U \backslash A$, or simply A^c or A' when the set U is understood from the context. This is the set of all elements in U that are not in A, defined as $\{x : x \in U \text{ and } x \notin A\}$. When $A \subset U$, this gives us $U = A \cup A^c$.
  - Let U be the set of all tree species on the planet, so that Ω^c is the set of all tree species which are not native to North America.
  - Consider the set of irrational real numbers $\mathbb{R} \backslash \mathbb{Q}$, or $\mathbb{Q}^c$ when it is understood that our "universe" U is the set of real numbers $\mathbb{R}$.
  - Similarly, if the "universe" is $U = \mathbb{Z}$, then $(2\mathbb{Z})^c = 2\mathbb{Z} + 1$. That is,
  $$(2\mathbb{Z})^c = \{x : x \in U \text{ and } x \notin 2\mathbb{Z}\} = \{x : x \text{ is an integer and is not even}\} = 2\mathbb{Z} + 1.$$