PHY 101L - Experiments in Mechanics

introduction to error analysis

What is Error?

In everyday usage, the word *error* usually refers to a mistake of some kind. However, within the laboratory, "error" takes on a specific definition, and mistakes are only a small part of laboratory error. In the physics lab, error (or uncertainty) is inherent in what you are doing. Commonly, error is divided into three categories: *random* (or statistical) error, *systematic* error, and *mistakes*. In 101L, you will need to consider all three as you proceed through the experiments.

Random error results from the impossibility of knowing a quantity with complete accuracy. For example, measuring a piece of string twice could lead to two "different" lengths of the string. This does not mean the string changed length, but that the string can only be measured to within a particular accuracy. This range of values for the length of the string is what is generally referred to as the error.

Systematic error usually arises from a controllable cause, such as a ruler with a worn edge, or the reaction delay in turning off a stop watch. Systematic error causes your results to be systematically larger or smaller than they would be if the source of error were not present. In these labs, you should not be too concerned about measuring systematic error, but you should always keep in mind ways to reduce your systematic error.

Mistakes - some people call this *human error* - are often made in student laboratories, but by diligence on your part, these can often be avoided. If you suspect you have made a mistake taking a measurement you should take the measurement again if at all possible. However, if it is not possible, you should indicate your anomalous results in your report. You should never use the phrase "human error," it is vague and gives no information.

In many labs, you will be asked to answer questions or perform calculations related to a specific error situation, usually one that is well demonstrated in that particular lab. This does not mean that other error cannot occur. You should always consider all possible sources of error when writing a report.

Error Analysis

Error analysis is the practice of evaluating all sources of error when reporting your results; typically, this analysis is highly mathematical. In some ways, it is the most important part of the lab to consider. For example, if you are asked to use a pendulum and a stopwatch to find g, the acceleration due to gravity close to Earth's surface, you may end up with a result of 11.5 m/s^2, while g is normally accepted as 9.8 m/s^2. Have you shown that the equations which lead to $g = 9.8\ m/s^2$ are wrong and should be revised? Probably not. If you completed the lab carefully, and discussed how to avoid systematic error and implemented the results of that discussion, error analysis will most likely show that 11.5 m/s^2 is consistent with the accepted value of g, when experimental error is taken into account.

On the other hand, if your analysis of the random error in the experiment doesn't show that your results were consistent, you should discuss why this might be the case. It might be that there is a source of systematic error you have overlooked, or there might be another cause for the discrepancy. If this is the case, you should still state your results in your report. However, you should also include a paragraph which explains why you think your results are not consistent. An example might be:

> Using a stopwatch, the period of oscillation for a pendulum was measured, and from this data, the acceleration due to gravity was calculated to be $g = 11.5\ m/s^2$. This result is inconsistent with the accepted value; however, we believe this is due to a systematic error resulting from consistently failing to start the stopwatch as soon as the pendulum was released. This would give a shorter period, which would result in a higher calculated value for g.

Don't worry about the reasoning behind this particular example. Instead, note that it includes:

1. A brief outline of the procedure
2. The group's results
3. A statement that the results were not what the group expected
4. An explanation of why the group believes the results were not consistent
5. A short reasoning of why the group's explanation would result in the inconsistency

If you ever need to write a paragraph like this, try to include each of these points.

Examples

Take the time to work through at least some of the following examples. They'll help to clarify the different types of error, and you'll get some experience manipulating numbers.

Example 1: Error in a single measured value

You measure a piece of string with a meter stick that has marks every half centimeter (at 0 cm, 0.5 cm, 1 cm, etc) and find the length of the string is 21.3 cm. What is the error in this measurement?

The questions you must ask yourself are these: How certain am I that the measurement was exactly 21.3 cm? Am I sure it wasn't 21.2 or 21.4 cm? You are probably fairly certain that it wasn't 21.0, the string didn't look like it was exactly on the line. It was definitely smaller than 21.5, which is the halfway point between 21.0 and 22.0, and thus, fairly easy to estimate. You could easily say your measurement was

$$21.3 \pm 0.2\ cm.$$

Think a bit about what this means. The measurement has two parts. The first part, 21.3 is called the best value, and if you've only made one measurement, it is the number you measured. The second part, ± 0.2, is the uncertainty in your measurement. This part says that while you're pretty sure the string is 21.3 cm, it could be as long as 21.5 (or $21.3 + 0.2$) or as short as 21.1 (or $21.3 - 0.2$).

This sort of uncertainty comes up any time you take a direct measurement, such as measuring the length of the string, or finding the mass of an object on a triple beam balance. A similar situation also occurs when you read a measurement off a digital display. In this case, you can assume the error is $\pm$ one unit in the last decimal place of the display. If the display reads 23, the the full measurement would be 23 ± 1. If the display reads 23.4, the measurement would be 23.4 ± 0.1, and so on.

Example 2: Error in several measurements of the same quantity

In the first example, we had to find the error of the measurement based on a single reading. Sometimes this is unavoidable, because the measurement is unrepeatable due to experimental limitations. More frequently, though, a measurement can be taken more than once, and this leads to a better estimate of error than a single measurement.

Let's say everyone in the group measured the string independently, and got lengths of 21.3, 21.6, and 21.2. No one got the same length for the string, so how can a "best" value be chosen? The answer is simple, and probably evident. The best value is simply the average value, or mean, of the measurements.

$$\frac{21.3 + 21.6 + 21.2}{3} = 21.4 \ cm$$

One thing to note: when you do this on your calculator, you will get an answer of 21.366666..., so how do we choose where to round? If you have taken a chemistry class, you probably spent quite a bit of time figuring out significant figures. We won't require you to be quite as rigorous, but it is important to keep in mind that you shouldn't have more places in your best value than the error. As we will see shortly, the error in the above calculation is ± 0.2 cm, so having a best value of 21.36667 doesn't make sense. If the tenths place isn't known to be exactly three, how can the other digits past the tenths place possibly be known? Instead, round to the decimal place that corresponds to your error, rounding down if the next digit is less than 5, and up if it is 5 or greater.

Now, how do we find the error in these measurements? First, each measurement must be subtracted from the best value.

$$21.4 - 21.3 = 0.1 \ cm$$

$$21.4 - 21.6 = -0.2 \ cm$$

$$21.4 - 21.2 = 0.2 \ cm$$

Then, find the average value of the differences, while taking their absolute value:

$$\frac{0.1 + 0.2 + 0.2}{3} = 0.2\ cm.$$

This value is the error in the measurement, so in this case, the whole measurement would be written

$$21.4 \pm 0.2\ cm.$$

It is always best to make more than one measurement when you can. A good system to use is for every direct measurement you make which you will need an error for, have each group member make the measurement independently. It does not take much longer, for example, for a mass to be measured three times than once, and your error analysis will be considerably better.

Example 3: Propagation of Error

Many times, the quantity we are interested in is not a directly measured value, but a value calculated from a measured value. Each measured value has an error associated with it, so there must be some error in the calculated value. The question is, how do you find that error?

A complete answer to that question would involve multivariable calculus. This analysis requires math knowledge that most people taking this class won't have acquired yet. Fortunately, there are some basic rules derived from the multivariable method that work with the kind of equations that show up in this lab.

A short summary of propagation of error rules:

- $f(x, y, z)$ is a function f that depends on the variables x, y, and z. In terms of error, x, y, and z are the measured values and f is the calculated value. The errors in x, y, and z will be represented by δx, δy, and δz, and the error in f will be shown as δf. a, b, and c are simply constant coefficients.

- If the measured values are **added** or **subtracted** from one another, as in

  $$f(x, y, z) = ax + by - cz,$$

  then the propagated error is

  $$\delta f = \sqrt{(a \cdot \delta x)^2 + (b \cdot \delta y)^2 + (c \cdot \delta z)^2}.$$

- If the measured values are **multiplied** or **divided** by one another, as in

  $$f(x, y, z) = \frac{(ax)(by)}{cz},$$

  then the rule for propagated error is

$$\frac{\delta f}{f} = \sqrt{\left(\frac{a \cdot \delta x}{x}\right)^2 + \left(\frac{b \cdot \delta y}{y}\right)^2 + \left(\frac{c \cdot \delta z}{z}\right)^2}.$$

You can find δf by multiplying both sides by f.

- If the measured value is **taken to a power**, as in

$$f(x, y) = \sqrt{y} \cdot x^2,$$

the rule for propagated error is

$$\frac{\delta f}{f} = \sqrt{\left(2\frac{\delta x}{x}\right)^2 + \left(\frac{1}{2}\frac{\delta y}{y}\right)^2}$$

One place where the propagation of error becomes necessary is in the calculation of kinetic energy. The equation for kinetic energy is

$$K(m, v) = \frac{1}{2}mv^2,$$

where m is the mass of an object, and v is its velocity. Because you have measured m and v, you have best values for each of them, but propagation of error must be used to find the error in the kinetic energy.

The most difficult part of applying the propagation of error rules is the first decision you have to make: deciding which rules apply to your equation. In this example, grouping $1/2mv^2$ into $(1/2m)\,(v^2)$ shows that the equation contains terms from two of the rule groups above - $1/2m$ fits in to the multiplication rule and v^2 fits into the power rule. The error equation you would write is

$$\frac{\delta K}{K} = \sqrt{\left(\frac{1}{2}\frac{\delta m}{m}\right)^2 + \left(2\frac{\delta v}{v}\right)^2}.$$

Let's say the mass of your object was $0.497 \pm .005\ kg$, and its velocity was $1.50 \pm .03\ m/s$. The best value would then be

$$K = 1/2 \cdot 0.497\ kg \cdot 1.5^2\ m^2/s^2 = 0.559\ J.$$

The propagated error would be

$$\delta K = 0.56\ J \cdot \sqrt{\left(\frac{1}{2}\frac{.005\ kg}{0.497\ kg}\right)^2 + \left(2\frac{.03\ m/s}{1.5\ m/s}\right)^2} = 0.02\ J,$$

and the final value for kinetic energy would be given as $0.56 \pm 0.02\ J$. Note the rounding of the best value: because the error was in the hundredths place, the best value was rounded to the same place.