The Origin of the Lagrangian

By Matt Guthrie

Motivation

During my first year in undergrad I would hear the upperclassmen talk about the great Hamiltonian and Lagrangian formulations of classical mechanics. Naturally, this led me to investigate what all the fuss was about. My interest led to fascination, an independent study of the subjects, and ultimately I developed a sort of intuition for Lagrangian mechanics in my Junior mechanics class. One of the biggest frustrations I had while studying the subject was not from something traditionally frustrating, but from the fact that the Lagrangian,

$$L = T - U, \tag{1}$$

is never *derived.* Everywhere I had seen it, the equation was *assumed.* For example, from Taylor:[1]

> "The **Lagrangian function**, or just **Lagrangian**, is defined as
>
> $$\mathcal{L} = T - U$$
>
> ...You are certainly entitled to ask why the quantity $T - U$ should be of any interest. There seems to be no simple answer to this question except that it is, as we shall see directly."

Another example, from d'Inverno:[2]

> "We start from the Lagrangian functional... defined in terms of the metric by
>
> $$L = \left[g_{ab}(x)\dot{x}^a\dot{x}^b\right]^{\frac{1}{2}}."$$

What struck me most is this is from a book on General Relativity, any reader would absolutely understand the differential geometry required for a proper derivation of the Lagrangian.

One last example is from Boas[3], in her book on Mathematical Methods in physics:

> "...this assumption is called *Hamilton's principle.* It says that any particle or system of particles always moves in such a way that
>
> $$I = \int_{t_1}^{t_2} L\,\mathrm{d}t$$
>
> is stationary, where $L = T - V$ is called the *Lagrangian...*"

One should not expect a book on math methods to expound upon every subject presented, but again, this book introduces all the required differential geometry used in my derivation.

The main purpose for this writeup is to show that, not only is there a perfectly sound derivation for $L = T - U$, the derivation is not that complicated. The method I use does require some knowledge of differential geometry, but in a future version of this paper I will derive L without the use of much differential geometry, if at all. I am positive it is possible.

The Covariant form of Newton's Second Law

By using generalized acceleration components, it is possible to state Newton's second law for a single particle as

$$Q_k = ma_k, \tag{2}$$

with

$$a_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ki}\dot{q}^i\right) - \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j, \tag{3}$$

where g_{ij} is the metric tensor for the manifold containing the coordinates q. The components Q_k are the covariant components of the force, but they are usually called generalized force components. The derivation of a_k is outside of the scope of this paper, but its origin is somewhat important. Because the generalized force is a vector, it is valid in all coordinate systems. Using the general line element,

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}q^i\,\mathrm{d}q^j, \tag{4}$$

the kinetic energy of a particle can be expressed as a quadratic form in the generalized velocities:

$$T = \frac{1}{2}m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j. \tag{5}$$

Calculation of $\frac{\partial T}{\partial q^i}$ is easier than it may seem because the kinetic energy is a function of the generalized coordinates only through the dependence of the metric tensor on the coordinates:

$$\frac{\partial T}{\partial q^k} = \frac{1}{2}m\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j. \tag{6}$$

The metric tensor does not depend on the velocities, so

$$\frac{\partial T}{\partial \dot{q}^k} = mg_{ki}\dot{q}^i. \tag{7}$$

The factor of $\frac{1}{2}$ disappears because there is a double sum in equation (5)[1]. With these two relations known, it is now possible to express Newton's second law, equation (2), in the convenient form

$$Q_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k}. \tag{8}$$

Lagrange's Equations

Let a particle that is moving in three dimensions without constraints be acted on by a conservative force. The particle then has three independent coordinates or degrees of freedom. Recall that the generalized force components obey the covariant transformation law:

[1] Write out the chain rule yourself if you have trouble seeing this. Remember that $g_{ij} = g_{ji}$.

$$Q_k = \frac{\partial q'_i}{\partial q_k} Q'_i \tag{9}$$

Suppose that the q'_i are rectangular coordinates. Then $Q'_i = F_i$ are the rectangular components of force and

$$Q_k = \frac{\partial x_i}{\partial q_k} F_i = \frac{\partial x_i}{\partial q_k}\left(-\frac{\partial U}{\partial x_i}\right) = -\frac{\partial U}{\partial q_k}. \tag{10}$$

Thus, equation (8) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k}. \tag{11}$$

Moreover, U does not depend on velocity[2], so it is possible to rewrite equation (11) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial (T-U)}{\partial \dot{q}_k}\right) - \frac{\partial (T-U)}{\partial q_k} = 0 \tag{12}$$

or, equivalently:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \tag{13}$$

where $L = T - U$. It is obvious from this representation that L is a clever change of variable to make equation (12) and related equations in the Lagrangian formulation look a little neater.

Conclusion

Once you've derived the Lagrangian from Newton's laws and established that Lagrangian mechanics is an alternative interpretation of Newtonian mechanics, it's interesting to note how, beyond that, the formalism doesn't really care where the Lagrangian comes from. The Lagrangian is simply an assignment of a number to every point on a manifold. This frees up the possibly for theorists to postulate any sort of functional form for the Lagrangian. The core of the whole mechanics machinery is contained in this association of a vector field for every function (the field satisfying that function's Euler-Lagrange equation). As a result, this whole strange business of $\frac{1}{2}mv^2$ vanishes because all we're talking about is curves on some sort of generalized surface. This makes the Euler-Lagrange equation tremendously powerful! More generally, I would like to emphasize that the lack of derivation of the Lagrangian leaves the impression on students that it's magical and mystical when really it can be quite accessible.

[2] In the odd case where U *does* depend on velocity, the correction is trivial and resembles equation (8) (and the Euler-Lagrange equation remains the same).

Bibliography

[1] John R. Taylor. *Classical Mechanics*. University Science Books, 2005.

[2] Ray d'Inverno. *Introducing Einstein's Relativity*. Oxford University Press, 1995.

[3] Mary L. Boas. *Mathematical Methods in the Physical Sciences*. John Wiley & Sons, 2006.