The investigation of coherent manipulation of quantum state of matter and light has provided insights in many quantum phenomena and in quantum information processes. The realization of Bose-Einstein condensation (BEC) in dilute gases has provided a new tool for such investigations. Recently, it is found that the stability of quantum state has played a key role in many procedures for coherent manipulating and applying BEC in quantum information and quantum computation.

However, an important issue is still missing in the study of BEC, namely, the sensitivity of the quantum evolution of a BEC with respect to the small perturbation that may naturally arise from either the manipulation parameters or the interaction with environment. This type of stability of quantum motion is characterized by the so-called fidelity, or the Loschmidt echo, which is defined as the overlap of two states obtained by evolving the same initial state under two slightly different (perturbed and unperturbed) Hamiltonians. This quantity is of special interest in the fields of quantum information and quantum chaos.

In this Letter, we propose a system of two-component BEC trapped in a harmonic potential, subject to a periodic coupling (successive kicks) between the two components. Our aim is two-fold: (1) To investigate the instability of the BEC system with a small perturbation on its system parameters; (2) To propose a possible experiment to directly detect the fidelity decay. The system we propose is a two-component spinor BEC confined in a harmonic trap with two internal states coupled by a near resonant pulsed radiation field. Within the standard rotating-wave approximation, the Hamiltonian can be cast into the form:

$$\hat{H} = \mu(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) + g(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)^2 + K \delta_T(t)(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

where $K$ is the coupling strength between the two internal states, $g$ is the interaction strength, and $\mu$ is the difference between the chemical potentials of two components. $\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger$ are boson annihilation and creation operators for the two components, respectively. $\delta_T(t) = \sum_n \delta(t - nT)$ means that the radiation field is only turned on at certain discrete moments, i.e., integral multiples of the period $T$. Writing the above Hamiltonian in terms of the angular momentum operators, we have

$$\hat{H} = \mu \hat{L}_z + g \hat{L}_z^2 + K \delta_T(t) \hat{L}_x.$$ 

The Floquet operator depicting the quantum evolution in one period takes the following form,

$$\hat{U} = \exp[-i(\mu \hat{L}_z + g \hat{L}_z^2)T] \exp[-iK \hat{L}_x].$$

The Hilbert space is spanned by the eigenstates of $\hat{L}_z$, $|\ell\rangle$, with $\ell = -L, -L+1, \ldots, L$, where $L = N/2$ and $N$ is the total number of atoms. In the above expression and henceforth, the Planck constant is set to unit.

The above system has a classical counterpart in the limit $N \to \infty$, describing a spin on a Bloch sphere with $S_i = 1/2 < \hat{L}_i >, (i = x, y, z)$. The classical Hamiltonian takes the form

$$\hat{H} = \mu S_z + g_c S_z^2 + K \delta_T(t) S_x,$$

where $g_c = gL$. The equations $\dot{S}_i = [S_i, \hat{H}]_{cl}, (i = x, y, z)$ determine the motion of the centers of coherent quantum wavepackets and the quantum fluctuation is ignored, (i.e., equivalent to the mean-field Gross-Pitaevskii equation without considering a total phase).

They can be solved analytically: the free evolution between two consecutive kicks corresponds to a rotation around $S_z$ axis with the angle $(\mu + 2g_c)T$, and the periodic kicks added at times $nT$ give rotation around the $S_x$ axis with the angle $K$.

Dynamic motion of the classical system is classified by the magnification of its initial deviation. An exponential increase in time of the deviation means dynamical instability or chaotic motion, causing rapid proliferation of thermal particles. Quantitatively, one can calculate the (maximum) Lyapunov exponent,

$$\lambda = \lim_{t \to \infty} \frac{1}{T} \ln \frac{\langle |\Delta x(t)| \rangle}{|\Delta x(0)|}$$

with $|\Delta x(t)|$ denoting distance in
phase space. The exponent is positive for unstable motion, and tends to zero if the orbit is stable. Usually, phase space is a mixture of chaotic orbits and quasi-periodic (stable) orbits, as is shown in Fig. 1, where it is clearly seen one big island and four small islands; inside the islands motions are periodic or quasi-periodic, outside the islands motions are mainly unstable or chaotic. Here and in the following figures, \( \mu = T = 1 \).

The total relative area occupied by chaotic orbits, (as clearly seen in Fig.1), depends on system parameters \((g_c, K)\), and can be used to characterize the degree of mixture. It has been obtained by calculating the Lyapunov exponent of orbits with initial points randomly scattered in the whole phase space. The result is shown in Fig. 2. We see that the integrable cases mainly concentrate on the vertical line where the interaction strength vanishes, and on the horizontal lines where the coupling strength is a multiple of \( \pi \). This fact indicates that both nonlinearity term and the kick strength are essential in inducing chaos. The deep red areas in Fig. 2 give the parameter regime for the system where the phase space is full of unstable (chaotic) orbits.

Now we turn to the quantum system and trace the fidelity (Loschmidt echo) \( M(t) \), defined as

\[
M(t = nT) = \langle \Phi_0 | \left( \hat{U}_t \right)^n \circ \left( \hat{U} \right)^n | \Phi_0 \rangle,
\]

where the initial state \( | \Phi_0 \rangle \) is chosen as a coherent state, \( | \Phi_0 \rangle = e^{\alpha^* \hat{L} - \alpha \hat{L}^*} | L \rangle \), with \( \alpha = \frac{\pi - \hat{\theta}}{\text{Planck constant}} e^{-i \varphi} \). A small perturbation on the Hamiltonian is added by changing \( K \rightarrow K + \varepsilon \), with \( \hat{U} \rightarrow \hat{U}_\varepsilon \). In this system, the effective Planck constant \( h_{\text{eff}} = 1/L \).

We discuss fidelity decay in three typical situations, in which the corresponding classical system is fully chaotic.
Figure 4: Fidelity decay in a classically nearly integrable case, with \( K = 2, g_c = 0.2, L = 100, \) and \( \epsilon = 0.003 \). Upper panel: Fidelity of four randomly chosen initial coherent states, with the smooth solid curve being the Gaussian fit to one of them. Lower panel: Averaged fidelity, with average performed over 50 initial coherent states.

Near-integrable, and mixed, respectively. The corresponding parameters are picked up from Fig.2.

For the parameters \( K = 2, g_c = 4 \), from Fig.2 we know that the phase space is fully chaotic. Because of the ergodicity of the chaotic orbits, fidelity decay is expected to be independent on the initial condition. However, it strongly depends on the perturbation strength. For a small perturbation, fidelity shows a slow Gaussian decay (upper panel in Fig.3). With increasing perturbation strength, one meets a border \( \epsilon_p \sim 1/L^{3/2} \), at which the typical transition matrix element of perturbation between quasi-energy eigenstates becomes larger than the average level spacing. With the intermediate perturbation above the border (lower panel in Fig.3), the fidelity decays in an exponential way, where the decay rate \( \Gamma \) is the function of the interaction strength and the classical action diffusion constant [10]. With strong perturbation, the fidelity decays faster and finally saturates at some perturbation-independent decay rate [13] (lower panel in Fig.3).

From the above discussions and calculations we see, in practical applications of the BEC, the perturbation border \( \epsilon_p \) gives a up-limit for the perturbation strength that is tolerable, in order to avoid low fidelity.

As we choose parameters as \( K = 2, g_c = 0.2 \), the classical system is nearly integrable where the phase space is full of periodic and quasi-periodic orbits. We found Gaussian decay for the fidelity of single initial coherent states, with a strong dependence of decaying rate on the choice of initial condition [12]. However, after averaging over the whole phase space, we found that the fidelity decay can be well fitted by a inverse power law \( 1/t \) (see Fig.4). In this case, for the quantum evolution of initial coherent states, high fidelity can be expected because the fidelity has a power law decay on average.

Now we turn to the mixed case, which is more complicated than the previous two cases. It is usually expected that fidelity decay of initial coherent states lying in regular regions would be similar to that in a nearly integrable system, and that from chaotic regions be similar to that in a chaotic system. However, we found that this naive picture is not exact. As shown in Fig.5 for initial states from both irregular and regular regions, the behavior of fidelity may be quite different from those in the fully chaotic case and in the nearly-integrable case as shown in Figs. 3 and 4. We concentrate our discussions on the case in which initial coherent states lie within the largest regular island. We found that their fidelity almost has no decay up to time \( t = 200 \), quite different from the initial-condition-dependent Gaussian decay shown in Fig.4 for a nearly integrable system. Note that the quantum perturbation strength is chosen to be in the intermediate perturbation regime in Figs. 4 and 5. This phenomenon of high fidelity cannot be explained by means of expand-
states are coherent states with corresponding $(S_z, \theta)$. If the beam is strong enough, so that the tunnelling between the two condensates and kick the internal state to $\chi_i$ ($i = 1, 2$) without important change of the external states $\psi_i$.

After certain numbers of kicks, the radiation field and the strong blue detuning laser beam are turned off simultaneously and two BECs begin to interfere. The visibility of the interference is governed by

$$I \propto |\psi_1\chi_1|^2 + |\psi_2\chi_2|^2 + 2Re(\psi_1^*\psi_2\chi_1\chi_2).$$

Clearly, high fidelity of the two internal states corresponds to high visibility of the interference.

The work was supported in part by a Faculty Research Grant of National University of Singapore, the Temasek Young Investigator Award of DSTA Singapore under Project Agreement POD0410553 (BL), the Natural Science Foundation of China No.10275011 (WGW), NSFC10474008, and the Natural Science Foundation of US.

\[1\] M. D. Lukin, Rev. Mod. Phys. 75, 457 (2003).