Journal of the Physical Society of Japan
Vol. 73, No. 10, October, 2004, pp. 2612–2615

Gap Evolution in $\nu = 1/2$ Bilayer Quantum Hall Systems

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(Received May 24, 2004)

Fractional quantum Hall states in bilayer systems at the total filling fraction $\nu = 1/2$ are examined numerically within some ranges of layer separation and interlayer tunneling. It is shown that the ground state changes continuously from a two-component state to a one-component state as the interlayer tunneling rate is increased, while the lowest excited state changes discontinuously. This fact explains the unusual behavior of the observed activation energies which reveals an upward cusp as a function of interlayer tunneling. Some trial wave functions for the ground state and quasihole excited states are inspected.

KEYWORDS: fractional quantum Hall effect, bilayer system, tunneling, quasiparticles
DOI: 10.1143/JPSJ.73.2612

The fractional quantum Hall effect (FQHE)[1] occurs at the filling factor $\nu \equiv n\phi_0/B = p/(2p \pm 1)$ (p integer) in a two-dimensional electron system with the perpendicular magnetic field B, while the $\nu = 1/2$ effect has never been observed.[2] Here, $\phi_0 \equiv hc/e$ is the flux quantum and n is the electron density. In the physics of the FQHE system, the composite fermion picture is quite useful.[3] In this description, the $\nu = p/(2p \pm 1)$ FQHE state can be understood as the $\nu' = p$ integer quantum Hall effect of composite fermions which possess two-flux quanta $2\phi_0$.[3] Thus, $\nu = 1/2$ corresponds to a zero-field system of composite fermions and the ground state can be understood as a Fermi liquid state of composite fermions.[4]

On the other hand at $\nu = 5/2$, which is half-filling of the second Landau level, the fractionally quantized plateau of the Hall resistance was observed.[5] After some numerical investigations,[6,7] the ground state at $\nu = 5/2$ is believed to be a kind of p-wave paired state of composite fermions first discussed by Moore and Read:[8]

$$\Psi_{\mathrm{Pf}} = \mathrm{Pf}\left(\frac{1}{u_i v_j - v_i u_j}\right) \prod_{i<j} (u_i v_j - v_i u_j)^2. \tag{1}$$

Here, we use spherical geometry[9] for convenience; $(u_j, v_j) = (\cos(\theta_j/2)\mathrm{e}^{\mathrm{i}\phi_j/2}, \sin(\theta_j/2)\mathrm{e}^{-\mathrm{i}\phi_j/2})$ is the spinor coordinate of the j'th electron, and $\mathrm{Pf}[A]$ is the Pfaffian of the antisymmetric matrix A.

When an internal degree of freedom such as a spin or layer index is introduced, the physics becomes more colorful. The $\nu = 1/2$ FQHE was observed in a double-quantum-well (DQW) structure[10] and a wide-single-quantum-well (WSQW).[11] In a two-layer system without interlayer transfer, the ground state can be approximated by a two-component state proposed by Halperin:[12]

$$\Psi_{331} = \prod_{i<j} (u_i v_j - v_i u_j)^3 \prod_{i<j} (\eta_i \xi_j - \xi_i \eta_j)^3 \times \prod_{i,j} (u_i \xi_j - v_i \eta_j)^1, \tag{2}$$

within certain ranges of the ratio d/l of the layer separation and the magnetic length.[13] Here, (u_i, v_i) and (η_i, ξ_i) are complex spinor coordinates of electrons in the top and bottom layers, respectively. The d/l-dependence observed in the $1/2$ FQHE in DQW[10] fits the theoretical prediction very well.[13] On the other hand, the $1/2$ state measured in WSQW is more subtle,[11] since such a system possesses the duality of a bilayer and a thick single-layer system. In fact, both one-component[14] and two-component[15] theoretical models have been proposed. To determine the nature of the ground state in WSQW experimentally, Suen *et al.* measured activation energy as a function of interlayer tunneling.[16] At $\nu = 2/3$, the gap shows a downward cusp behavior which indicates a clear transition from a two-component state to a one-component state. However, at $\nu = 1/2$ such a characteristic was not observed. Since the gap first increases when tunneling amplitude Δ_{SAS} is decreased, they considered the two-component state, most likely Ψ_{331}-like state, as the ground state. They also concluded that the two-component state does not evolve to the one-component FQHE state but to a metallic state as Δ_{SAS} increases. At the center of the FQHE region, the gap shows a sharp upward cusp.

There are also some theoretical progresses. Based on the pairing picture of composite fermions, Halperin considered the continuous evolution of the ground state between Ψ_{331} and Ψ_{Pf}, and proposed a d–Δ_{SAS} phase diagram.[17] Pursuing this scenario, Ho argued interesting connections between these $1/2$ FQHE states and superfluid ^{3}He.[18] Namely, Ψ_{331} and Ψ_{Pf} correspond to the ABM state and A_1 state, respectively, and the introduction of Δ_{SAS} corresponds to the Zeeman splitting along the x-axis in ^{3}He superfluid. Here, the pseudospins $\uparrow$ and $\downarrow$ are assigned the electrons in the top and bottom layers, respectively. Although such a mean-field picture of composite fermion pairing possesses a beautiful structure, the relation to the experimental result mentioned above has been unclear and it stays only in theoretical curiosity. In this article, we perform a numerical investigation of the evolution of the $\nu = 1/2$ FQHE state as a function of interlayer tunneling and layer spacing. It is shown that the ground state evolves continuously, while the quasihole state evolves discontinuously between the two-component and one-component states. Based on these results, a reasonable explanation for the inscrutable upward cusp behavior of the activation energy is presented.

The Hamiltonian is given by

$$H = H_{\mathrm{SAS}} + H_{\mathrm{C}}. \tag{3}$$

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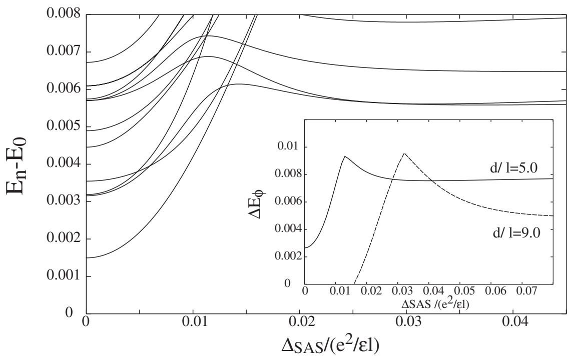

Fig. 1. Neutral excitation energies as a function of $\Delta_{\rm SAS}/(e^2/\epsilon l)$ at $d/l = 5.0$. Inset: Calculated ΔE_ϕ in units of $e^2/\epsilon l$ as a function of $\Delta_{\rm SAS}/(e^2/\epsilon l)$ for $d/l = 5.0$ and $d/l = 9.0$. The data show an upward cusp which is similar to the experimentally observed energy gap. The number of electrons is $N = 6$.

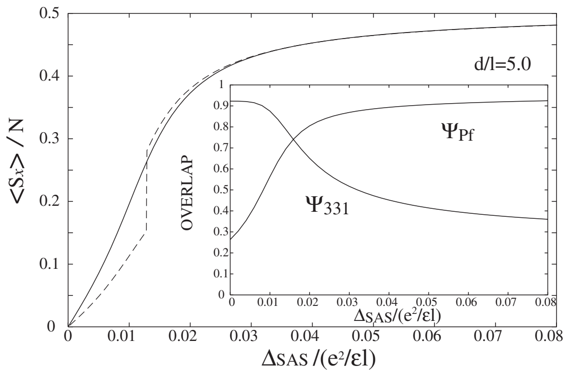

Fig. 2. Expectation value of the x-component of the pseudospin in the ground state (solid line) and the state with an extra flux quantum (dashed line) as a function of $\Delta_{\rm SAS}/(e^2/\epsilon l)$. Inset: Overlap between the ground state wave function and the Pfaffian, and 331 state trial function as a function of interlayer tunneling in the six-electron system at $d/l = 5.0$.

The single-particle Hamiltonian;

$$H_{\rm SAS} = -\frac{1}{2}\Delta_{\rm SAS}\sum_m\sum_\sigma c^\dagger_{m\sigma}c_{m-\sigma} = -\Delta_{\rm SAS}S_x \qquad (4)$$

describes electron transfer between the layers. Here, the total pseudospin operator is defined using Pauli matrices as $\boldsymbol{S} = (1/2)\sum_m c^\dagger_{m\sigma}\boldsymbol{\sigma}_{\sigma\sigma'}c_{m\sigma'}$. On the other hand, $H_{\rm C}$ represents a Coulomb interaction within and between the layers:

$$H_{\rm C} = \frac{1}{2}\sum_{m_1\sim m_4}\langle m_1, m_2|V_{\sigma\sigma'}|m_3, m_4\rangle \times c^\dagger_{m_1\sigma}c^\dagger_{m_2\sigma'}c_{m_3\sigma'}c_{m_4\sigma}. \qquad (5)$$

To model a bilayer system, we consider not only the layer spacing d but also the thickness w. They are treated using the following form of Coulomb interaction: $V_{\uparrow\uparrow} = V_{\downarrow\downarrow} = e^2/\epsilon\sqrt{r^2 + w^2}$ and $V_{\uparrow\downarrow} = V_{\downarrow\uparrow} = e^2/\epsilon\sqrt{r^2 + d^2}$. Since we are interested in changing d, the thickness is fixed at the physically motivated value $w = 3.8l$ for a WSQW in this article. The eigenstates of the Hamiltonian are obtained exactly with the use of the exact diagonalization. In Fig. 1, we show the neutral and charged (inset) excitation energies. We first consider the quantity given as $\Delta E_\phi = E(N_\phi + 1, N) + E(N_\phi - 1, N) - 2E(N_\phi, N)$ in the $N = 6$ electron system as a function of interlayer tunneling where the number of flux N_ϕ is equal to $2N - 3 = 9$ for the ground state with p-wave pairing.[13,20] As shown in Fig. 1, the upward cusp behavior is found at $d/l = 5.0$, and 9.0. The charged excitations are quasiholes or vortices, which effectively contain a half quantum of flux because of pairing. Thus, to obtain the activation energy, one should divide the value ΔE_ϕ by 2 after subtracting the interaction energy for two $e^* = \pm\nu e/2$ charges. Here, note that the flux $\phi_0 = hc/e$ corresponds to the quasihole with the charge νe at the filling ν.[9] In consideration of this fact, the upward cusp behavior of the activation energy observed experimentally is reproduced in our simple theoretical model.

Our statement is as follows: The ground state changes continuously contrary to the case of $\nu = 2/3$, while the quasihole excited state changes discontinuously as a function of $\Delta_{\rm SAS}$. Since the neutral excitations do not show a minimum but cusp also at this point, the ground state evolution is not a second ordere transition. The calculated expectation value $\langle S_x\rangle$ indicates a continuous evolution from a two-component state ($\langle S_x\rangle = 0$) to a one-component ($\langle S_x\rangle = N/2$) state as shown in Fig. 2. Now we show the calculated overlaps between the exact ground state and the trial states as a function of $\Delta_{\rm SAS}$ at $d/l = 5.0$ in Fig. 2. The data show that Ψ_{331} and $\Psi_{\rm Pf}$ are relevant for small $\Delta_{\rm SAS}$ and large $\Delta_{\rm SAS}$, respectively. We found that the crossing point of these two quantities corresponds to the point of the upward cusp in the quantity ΔE_ϕ at each value of d/l. Thus contrary to the conclusion in ref. 15 or ref. 16, we found that the upward cusp is related to the crossover between the one-component and two-component states.

Next we inspect Ho's $\boldsymbol{d}$-vector description given by[18] $\boldsymbol{d}(\theta) = (0, -{\rm i}\sin\theta, \cos\theta)$ within the middle range of tunneling. The p-wave paired state can be written as

$$\Psi_{\rm H}[\boldsymbol{d}(\theta)] = {\rm Pf}\left(\frac{\chi_{\sigma_i\sigma_j}[\boldsymbol{d}(\theta)]}{u_iv_j - v_iu_j}\right)\prod_{i<j}(u_iv_j - v_iu_j)^2, \qquad (6)$$

where the 2×2 matrix:

$$\chi = {\rm i}[(\boldsymbol{d}(\theta)\cdot\boldsymbol{\sigma})\sigma_y] = \begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix} \qquad (7)$$

is related to the order parameter of the triplet pairing,[18,19] and σ_j is the layer index of the j-th electron. Note that $\theta = 0$ and $\theta = \pi/4$ correspond Ψ_{331} and $\Psi_{\rm Pf}$, respectively. We constructed the wave function eq. (6) explicitly for several values of θ, and compared with the exact ground state in a six electron system. The optimized θ, which has the largest overlap with the ground state wave function at each value of $\Delta_{\rm SAS}$, is plotted in Fig. 3. The trial state eq. (6) with the optimized θ has a large overlap of about 0.9. Note that in triplet pairing the pseudospin is given as $\boldsymbol{S} \propto {\rm i}\boldsymbol{d}\times\boldsymbol{d}^*$. However, the pseudospin operator does not commute with the total Hamiltonian, thus the above value does not correspond to the exact value calculated in Fig. 2. THis is different from the mean-field theory of composite fermions.[18,19]

As shown in Fig. 2, $\langle S_x\rangle$ for the state with one extra flux quantum (dashed line) has a leap in the line at $\Delta_{\rm SAS}/(e^2/\epsilon l) = 0.013$, which indicates a level crossing between two-component and one-component in the quasihole states. As we mentioned above, the elementary charged excitations are described as the half quantum vortex $\phi_0/2$ which is

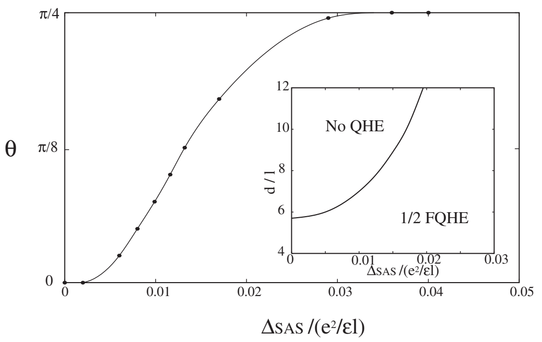

Fig. 3. Optimized value of θ which has largest overlap $\langle\Psi_{\rm H}[\mathbf{d}(\theta)]|\Psi_{\rm exact}\rangle$ under the change of $\Delta_{\rm SAS}/(e^2/\epsilon l)$ at $d/l = 5.0$. Inset: Calculated quantitative phase diagram of $\nu = 1/2$ bilayer system. At the boundary the energy gap collapses.

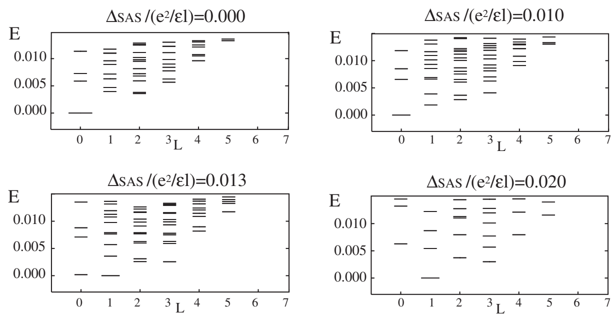

Fig. 4. Energy eigenvalues (measured in units of $e^2/\epsilon l$) in a six electron system with $N_\phi = 10$ (extra flux quantum). There is a level crossing when $\Delta_{\rm SAS}/(e^2/\epsilon l)$ increases.

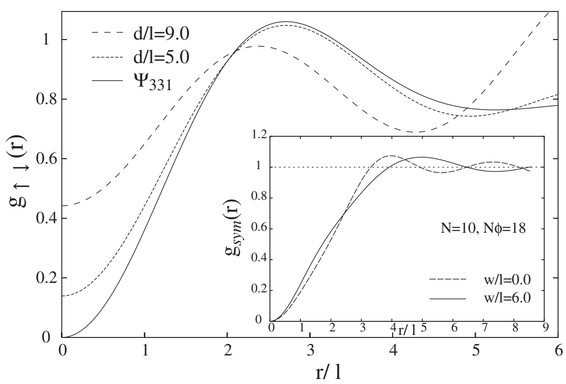

Fig. 5. Interlayer pair-correlation function at $\Delta_{\rm SAS} = 0$, $d/l = 5.0$, 9.0 and that in the Ψ_{331} state. The number of electrons is $N = 6$. Inset: Pair-correlation function in the symmetric sector in the case of $d = w = 0$ and $d = w = 6.0l$ at $\Delta_{\rm SAS} \to \infty$.

called the quasihole. However, it is impossible to construct numerically the one-half flux state. Thus, we consider the one extra flux quantum state which should correspond to the two-quasihole state. The quasiholes in the Pfaffian state are thought to obey non-abelian statistics[8)] which cause the Berry phase matrices when some of the quasiholes are interchanged contrary to the ordinary Laughlin quasiholes. For $2n$ quasihole states, the 2^{n-1} degeneracy is needed to possess non-abelian statistics. Read and Rezayi[21)] confirmed this nature in an exact diagonalization investigation with a three-body interaction which is the parent Hamiltonian for the Pfaffian state and its quasihole excited states.[20)] One of the two-quasiholes states is the Laughlin-type quasihole[9)] state: $\prod_{i=1}^{N} v_i\Psi_{\rm Pf}$ with the total angular momentum $L = 3$ for a six-electron system which corresponds to the two-quasihole state with a zero relative angular momentum. Because of the long-range nature of the Coulomb interaction, such a state with a higher total angular momentum should be higher in energy. The quantum number of the relative angular momentum of the two-quasiholes must be even because of their statistics. So the state with smaller energy has $L = 1$ and the trial function for the two-quasihole states is given by[8,21)]

$$\Psi_{\rm Pf}^{2qh} = {\rm Pf}\left(\frac{u_i v_j + v_i u_j}{u_i v_j - v_i u_j}\right)\prod_{i<j}(u_i v_j - v_i u_j)^2. \tag{8}$$

One of the quasihole is on the north pole and the other is on the south pole. We expect such a state to become relevant in a bilayer system with large $\Delta_{\rm SAS}$. Actually $\langle\Psi_{\rm Pf}^{2qh}|\Psi\rangle = 0.93929$ at $\Delta_{\rm SAS} = 0.08(e^2/\epsilon l)$ and $d/l = 5.0$. In the opposite limit of $\Delta_{\rm SAS} \to 0$, the quasihole states for Ψ_{331}, which are given by $\prod_{i=1}^{N_\uparrow} v_i\Psi_{331}$ and $\prod_{i=1}^{N_\downarrow} \xi_i\Psi_{331}$, might be relevant. The lowest energy state with two quasiholes should have a zero total angular momentum. In Fig. 4 energy eigenvalues for some values of $\Delta_{\rm SAS}$ in a six-electron system with an extra flux are shown against the total angular momentum L. The state with $L = 0$ is the lowest in the absence of tunneling. As $\Delta_{\rm SAS}$ increases, the energy of the state with $L = 1$ decreases and becomes lowest when $\Delta_{\rm SAS}$ exceeds $0.013(e^2/\epsilon l)$. The latter state can be approximated by $\Psi_{\rm Pf}^{2qh}$. The level crossing point exactly corresponds to the cusp point in Fig. 1. In other words, the upward cusp is a sign of the transition between abelian and non-abelian statistics.

Now we study a larger d/l region. The $\nu = 1/2$ FQHE state is observed in the WSQW with large $d/l > 7$. This fact is far from what was originally expected.[13)] Figure 5 shows the interlayer pair-correlation function $g_{\uparrow\downarrow}(r)$ at $\Delta_{\rm SAS} = 0$. In a crude sense, the Ψ_{331} state can be understood as two Laughlin 1/3 states locked together so that the electrons in one layer are bound to correlation holes in the other. We find that the correlation hole or locking between the layers declines and the system goes into the uncorrelating phase when d/l becomes large as shown in Fig. 5. Thus the FQHE state is not realized at $d/l > 7$ if $\Delta_{\rm SAS} = 0$. On the other hand, at large $\Delta_{\rm SAS}/(e^2/\epsilon l)$, our quantitative phase diagram depicted in the inset of Fig. 3 indicates the 1/2 FQHE. Actually as shown in Fig. 6, the ground state at large $\Delta_{\rm SAS}$ can be understood as the Pfaffian state, while around the cusp point, the optimized value of the Ho's parameter is $\theta = \pi/6$. The two-quasihole state also shows level crossing when the tunneling rate $\Delta_{\rm SAS}$ is increased. As shown in the inset of Fig. 6, at the large $\Delta_{\rm SAS}$ region, the two-quasihole state of Moore and Read is relevant.

As a supplement, we comment on the experiment, at large $\Delta_{\rm SAS}$. In the experiment the gap becomes smaller when $\Delta_{\rm SAS}$ is increased further, and vanishes at $\Delta_{\rm SAS}/(e^2/\epsilon l) > 0.08$.[16)] To realize this larger tunneling region, the magnetic field is reduced, and not only d/l but also w/l becomes small ($w/l \sim 2.5$). That is, the well width w should be appreciated as the third consequent parameter. When w and d are significantly small and tunneling rate is large, the system corresponds to a flat single-layer system, and the Rezayi–

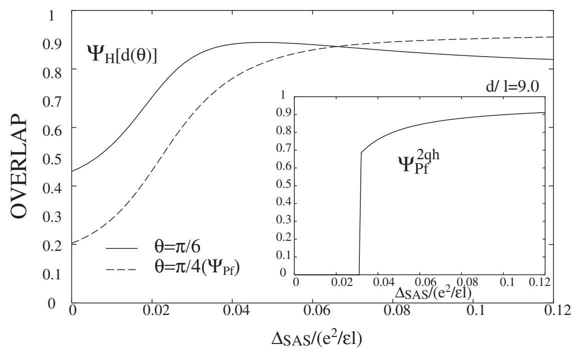

Fig. 6. Overlap with the trial function eq. (6) with $\theta = \pi/6$ and $\pi/4$ (Ψ_{Pf} state) as a function of $\Delta_{SAS}/(e^2/\epsilon l)$ at $d/l = 9.0$. Inset: Overlap with the two-quasihole state of Moore and Read in the added extra quantum flux system.

Read state[22] describing the Fermi liquid state of composite fermions[4] might be relevant, and quantized plateau does not appear. In fact, in the limit of $\Delta_{SAS} \to \infty$ and $d = w \to 0$, the two-correlation function in the symmetric sector (dashed line) shows a '$2k_F$-like oscillation'[22] at $N = 10$ and $N_\phi = 18$ as we see in Fig. 5. Contrary, at $d = w = 6.0l$, the short range repulsion is reduced and the oscillation disappears, indicating Cooper instability. Thus, to observe the one-component $1/2$ FQHE, a sample with a sufficiently wide well and large tunneling rate must be utilied.

In this letter, we investigated the evolution of the $\nu = 1/2$ bilayer FQHE state. We showed that the ground state evolves continuously as the tunneling rate is changed, while the quasihole state reveals a level crossing from a two-component to a one-component. The fact that the energy gap becomes maximum through the transition is really unusual within the physics of quantum phase transition. Perhaps similar singularity might be observed in transport phenomena such as the Hall resistance of drag current or pseudospin current.

Acknowledgment

This work is supported by Grants-in-Aid for Scientific Research (C) 10640301 and 14540294 from the Japan Society for the Promotion of Science.

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