Complex Numbers and Algebra ©

The real numbers are complete for the operations addition, subtraction, multiplication, and division, or more suggestively, for the operations of addition and multiplication and their inverses. Complete simply means that the operation may be performed on any two real numbers and results in a real number. In contrast, real numbers are complete for powers, x^n, but <u>not</u> for the inverse operation; e.g. there is no real number for which $x^2 = -1$. This lack is of more than formal interest, for it implies that polynomial equations may lack solutions, and we are often interested in finding solutions to such equations. Complex numbers have the virtue of being complete for addition, multiplication, and powers, <u>and their inverses</u>. In this sense, they are a natural extension of real numbers, just as real numbers are an extension of integers. (The term imaginary is unfortunate; complex numbers are just as physically significant as vectors or any other mathematical construct which describes the universe.)

Complex numbers are based on the introduction of an i for which $i^2 \equiv -1$. Complex numbers have the form $z = x + iy$, with x and y real and $\mathrm{Re}(z) \equiv x$, $\mathrm{Im}(z) \equiv y$. Addition and subtraction follow the obvious rules for real and imaginary parts separately. Multiplication follows by simple expansion: $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1y_1 - x_1y_2 + i(x_1y_2 + x_2y_1)$. Complex conjugation is a useful operation with $z^* \equiv x - iy$, and division can then be written as $z_1/z_2 = z_1z_2^*/z_2z_2^* = [x_1y_1 + x_1y_2 + i(-x_1y_2 + x_2y_1)]/(x_2x_2 + y_2y_2)$, a number in the standard form for z in terms of real arithmetic. Note that <u>any</u> equation of the general form $z_1 = z_2$ is actually a pair of equations $x_1 = x_2$ and $y_1 = y_2$.

Since the real and imaginary parts of z are independent, it can be useful to regard a complex number as a two-dimensional vector in the x-y plane. (They are not truly vectors, for multiplication and division are not defined for vectors, but the dot product is related to $z_1z_2^*$.) In particular, an alternative to the representation $z = x + iy$ is in terms of the amplitude $|z|^2 \equiv x^2 + y^2 = zz^*$ and phase $\tan\phi = y/x$, the magnitude and direction for the vector. However, the expression for the phase must be used with care, for the phase must span the full range 0 to 2π to cover the plane, not just the principal part of arctan. (This is easily done on many computers, which include a library function $\tan^{-1}(y,x)$ of <u>two</u> variables to specify the full range.) Much of the utility of complex numbers arises from certain features of this representation.

The function e^x, like many advanced functions, is defined as a power series, from which its other properties are adduced. From that definition,

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n(\theta)^{2n} + i\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^n(\theta)^{2n+1}$$

$$= \cos\theta + i\sin\theta$$

based on the series definitions of those functions. One representation is therefore $z = |z|e^{i\theta}$, which is particularly convenient for multiplication and division. As an illustration, consider the trivial statement that $e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2}$ and compare it with the complicated trigonometric identities that result from examining the real and imaginary parts. This is the easiest way to derive the trigonometric sum formulas, but given a complicated expression of sines and cosines, simplifications may be far less apparent than if the complex exponential notation had been used.

Exercise: By writing $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, derive the identity for $\cos\alpha \cos\beta$.

In particular, a time dependence e^{pt} will have the same (factorable) time dependence in all derivatives, making manipulations much easier than sines or cosines, which mix for odd derivatives. This often simplifies and generalizes the solution of differential equations.

Formalities: Although the exponential and trigonometric functions are introduced independent of calculus, the most compact, complete treatment of all properties uses calculus and the series definitions given above. The usual derivative properties follow immediate from the series: $d(e^x)/dx = e^x$; $d(\sin\theta)/d\theta = \cos\theta$; $d(\cos\theta)/d\theta = -\sin\theta$. The fact that e^x is a good notation for the function defined by the series, i.e. that $e^x e^y = e^{(x+y)}$, requires some thought, but application of the binomial formula for $(x+y)^n$ to the series expansion for $e^{(x+y)}$ will show equality. With these results, it is clear that $|e^{i\theta}| = 1$, $\sin^2\theta + \cos^2\theta = 1$, and sine and cosine have their usual role in a right triangle x,y,|z|. What remains to be established is the "scale" of θ, whether θ corresponds to radian measure. However, if one considers the unit circle for which $x = \cos\theta$, $y = \sin\theta$, $\sin^2\theta + \cos^2\theta = 1$ implies $ds^2 = d\theta^2$, the usual radian measure, and π is defined by the the requirement that $e^{i2\pi} = 1$. This is not an especially useful definition for evaluation of π, but we now have sufficient properties to define tan θ and hence $\tan^{-1}\theta$, from which one can take derivatives and form a Taylor series: $\pi/4 = \tan^{-1} 1 = 1 - 1/3 + 1/5 - 1/7 + 1/9$ ...... for example. Although this is not the manner in which these functions are normally defined or introduced, it is the easiest way retrospectively to establish all their properties.

Application to differential equations: A common use of complex algebra is to simplify the solution of differential equations, especially linear equations with constant (real) coefficients. If the equation(s) are regarded as complex and a complex solution x(t) is found, the real and imaginary parts of x(t) are each solutions. The general form $x(t) = Ae^{i\omega t}$ (with A being a complex constant in general) will always reduce linear equations with constant coefficients to simple algebraic equations for A and/or ω. For example, the damped harmonic oscillator becomes by this process a simple quadratic equation for the (complex) ω:

$$\frac{d^2x}{dt^2} + \omega_o^2 x + \frac{\gamma}{m}\frac{dx}{dt} = 0 \qquad \Rightarrow \qquad (\omega_o^2 - \omega 2 + i\omega\frac{\gamma}{m})Ae^{i\omega t} = 0$$

An alternative formulation for the use of complex numbers and functions to solve real equations, which is also applicable to nonlinear equations or equations with non-constant coefficients, is to write

$x(t) = Ae^{i\omega t} + A^*e^{-i\omega t}$, which is explicitly real. [This is necessary for nonlinear equations because, for example, $(\mathrm{Re}\, z)^2 \neq \mathrm{Re}(z^2)$; one cannot extract the real part of x(t) simply by taking the real part of the total equation.] A solution of his form must have the property that when it is substituted into the differential equation, the factor of $e^{i\omega t}$ and the factor of $e^{-i\omega t}$ must each be zero; otherwise the equation would not be satisfied for all times. If this form is used in linear equations with constant coefficients, those factors are simply the complex conjugates of one another, and the result is the same as using the simple $Ae^{i\omega t}$ directly. However, if the equation is nonlinear, a series

$$x(t) = \sum_{n=0}^{\infty} \left(A_n e^{in\omega t} + A_n^* e^{-in\omega t} \right)$$

may be required. A set of equations for the A_n will follow (to various orders n) from the requirement that the factors of each independent time function (constant, $e^{i\omega t}$, $e^{-i\omega t}$, $e^{i2\omega t}$, etc.) be zero. For example, a nonlinear term x^2 will generate terms $(A_1e^{i\omega_o t} + A_1^*e^{-i\omega_o t})^2 = A_1^2e^{i2\omega_o t} + A_1 A_1^* + (A_1^*)^2 e^{-i2\omega_o t}$) from the lowest-order oscillatory solution, a combination of constant and second harmonic terms. Similarly, a cubic nonlinearity x^3 [$(A_1e^{i\omega_o t} + A_1^*e^{-i\omega_o t})^3$] will generate a combination of ω_o and $3\omega_o$ terms.

Exercises

1. Plot each of the following complex numbers and their conjugates on the complex plane; express each in both $z = x + iy$ and $z = Ae^{i\theta}$ form: $1 + 2i$, $-2 + i$, $-3 - 4i$, $2e^{i\pi/3}$, $3e^{-2i\pi/3}$.

2. For each of the following pairs of complex numbers A, B, compute the following quantities: $A + B^*$; $\mathrm{Re}\{A - B\}$; AA^*; A^*/B; $\mathrm{Im}\{B^*B/A\}$:
   a) $A = 2 + i$; $B = 3 - 2i$.
   b) $A = -1 + 4i$; $B = 2e^{i\pi/6}$.
   c) $A = 3e^{5i\pi/6}$; $B = 5e^{-5i\pi/3}$

3. One usually writes $\sqrt[5]{1} = 1$, yet the equation $x^5 = 1$ is a (simple) fifth-order polynomial equation, for which one expects five roots. By writing $1 = e^{2n\pi i}$, for any integer n, obtain five roots to the equation, plot the roots in the complex plane, and write them in $x + iy$ form.

4. Consider the quadratic equation $x^2 - 2x + c = 0$. For each of the following values of c, plot the solutions in the complex plane: -8, -3, 0, 1, 2, 5, 10.

5. Prove that $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$. (Suggestion: Express the right side in complex exponentials, expand the expressions, and reduce them to the left hand side.)

Three Ways to Skin a Cat

Consider the wave equation for a string driven at x = 0 and fixed at x = L:

$$\frac{\partial^2\psi}{\partial t^2} - c^2 \frac{\partial^2\psi}{\partial x^2} = 0 \qquad \text{Boundary conditions:} \quad \psi(0,t) = Y_o \cos \omega t \qquad \psi(L,t) = 0 \quad \text{and} \;\; k = \frac{\omega}{c}$$

The solution may be obtained in several ways, each with certain advantages. To illustrate the possibilities, three paths are worked out here.

I. Conventional real functions

This is a simple approach in that little more than conventional trigonometry is required, but it requires some "insight" (smart guessing) and algebra to obtain solutions, and it does not generalize well to more complicated cases, for example damped waves. The general solution to the wave equation with the ω time dependence and waves going both left and right is

$$\psi(x,t) = A \cos(-kx + \omega t + \phi_a) + B \cos(kx + \omega t + \phi_b)$$

The boundary conditions require

$$Y_o \cos \omega t = A \cos(\omega t + \phi_a) + B \cos(\omega t + \phi_b) \quad \text{and}$$

$$0 = A \cos(-kL + \omega t + \phi_a) + B \cos(kL + \omega t + \phi_b)$$

Although these may appear to be two equations in four unknowns, the fact that each holds for all values of t generates sufficient constraints. However, they are complicated trigonometric equations. Solution depends upon noting that only if |A| = |B| can one apply trigonometric addition formulas to obtain useful relations. In this case, it specifically clear that only if |A| = |B| can the sum of the cosine terms in the last equation be zero for all times. If the magnitudes were not equal, there would certainly be oscillations in time. Nevertheless, the weakness in this method of solution is that it depends on being able to impose this sort of simplification on the problem. It is not general. Assuming A = B, the first equation becomes

$$Y_o \cos \omega t = 2A \cos(\omega t + \frac{\phi_b + \phi_a}{2}) \quad \cos(\frac{\phi_b - \phi_a}{2})$$

Since the time variations must be the same on both sides of the equation -- for example, they must go through zero at the same points, $\phi_b + \phi_a = 0$. Then

$$A = \frac{Y_o}{2} \frac{1}{\cos(\frac{\phi_b - \phi_a}{2})}$$

The boundary condition at the fixed end requires

$$0 = \cos(-kL + \omega t + \phi_a) + \cos(kL + \omega t + \phi_b) = \cos(\omega t + \frac{\phi_b + \phi_a}{2}) \;\; \cos(kL + \frac{\phi_b - \phi_a}{2})$$

which can only be satisfied if the final cosine term is zero, that is

$$\frac{\phi_b - \phi_a}{2} = -kL + \frac{\pi}{2} \qquad\qquad \phi_b - \phi_a = -2kL + \pi$$

We now have two equations for the two phases $\phi_b = -kL + \pi/2$ and $\phi_a = kL - \pi/2$. Putting these values back into the general solution and the equation for A gives the full result

$$\psi(x,t) = A\,\{\cos[k(L-x) + \omega t + \pi/2] + \cos[k(x-L) + \omega t + \pi/2]\,\} \qquad A = \frac{Y_o}{2\,\sin kL}$$

It is a superposition of waves traveling to the left and right, and it has the expected resonances (the amplitude becomes infinite) when $kL = n\pi$, the same condition as the string with both ends fixed.

II. Solution using complex functions

In this case, the driving boundary condition is written as $\mathrm{Re}\{Y_o e^{-i\omega t}\}$ and the general wave solution as

$$\psi(x,t) = \mathrm{Re}\left\{ A e^{i(kx-\omega t)} + B\, e^{i(-kx-\omega t)} \right\}$$

(One could choose either $e^{\pm i\omega t}$ for the time dependence, but all terms must have the same choice, and the general solution must include waves traveling both directions.) The boundary condition at x = 0 is

$$\mathrm{Re}\{Y_o e^{-i\omega t}\} = \mathrm{Re}\{ A e^{-i\omega t} + B\, e^{-i\omega t}\} \quad\Rightarrow\quad Y_o = A + B$$

And the condition at x =L is

$$0 = \mathrm{Re}\left\{ A e^{i(kL-\omega t)} + B\, e^{i(-kL-\omega t)} \right\} = \mathrm{Re}\left\{ (A e^{ikL} + B\, e^{-ikL})\, e^{-i\omega t}\right\} \quad\Rightarrow\quad 0 = A e^{ikL} + B\, e^{-ikL}$$

$$B = -A e^{2ikL} \qquad\qquad Y_o = A(1 - e^{2ikL})$$

$$A = \frac{Y_o}{1 - e^{2ikL}} = \frac{Y_o e^{-ikL}}{e^{-ikL} - e^{ikL}} = \frac{Y_o e^{-ikL}}{-2i\,\sin kL}$$

The full solution can then be written as

$$\psi(x,t) = \mathrm{Re}\left\{\frac{Y_o}{2}\,\frac{i}{\sin kL}\left[e^{i(kx-kL-\omega t)} + e^{i(-kx+kL-\omega t)}\right]\right\}$$

$$= \frac{Y_o}{2\,\sin kL}\left\{\sin[k(L-x)-\omega t] - \sin[k(x-L)-\omega t]\right\}$$

which clearly has the same amplitude and resonances as the result from section **I**, and can in fact be converted to the same form using trig identities. This version using complex functions is computationally simpler and can be generalized.

III. Alternate Complex Solution

This approach is very similar to the previous one. It looks somewhat more complicated, but it dispenses with the use of Re{} operations, which permits certain types of mathematical argument that you will find important in future work. For the moment, it is merely an alternative. The boundary condition at x = 0 is written

$$\psi(0,t) = \frac{Y_o}{2}\,\{ e^{i\omega t} + e^{-i\omega t}\}$$

and the full general wave solution is

$$\psi(x,t) = A_1 e^{i(kx-\omega t)} + A_2 e^{-i(kx-\omega t)} + B_1 e^{-i(kx+\omega t)} + B_2 e^{i(kx+\omega t)}$$

Applying the boundary conditions at x = 0 gives

$$\frac{Y_o}{2}\{ e^{i\omega t} + e^{-i\omega t}\} = A_1 e^{-i\omega t} + A_2 e^{i\omega t} + B_1 e^{-i\omega t} + B_2 e^{i\omega t}$$

This will be true only if the coefficients of each of the time dependencies match:

$$\frac{Y_o}{2} = A_1 + B_1$$

$$\frac{Y_o}{2} = A_2 + B_2$$

The equation at x = L is

$$0 = A_1 e^{i(kL-\omega t)} + A_2 e^{-i(kL-\omega t)} + B_1 e^{-i(kL+\omega t)} + B_2 e^{i(kL+\omega t)}$$

Again the coefficients of the $e^{i\omega t}$ and $e^{-i\omega t}$ terms must each be zero.

$$0 = A_1 e^{ikL} + B_1 e^{-ikL} \qquad B_1 = -A_1 e^{2ikL}$$

$$0 = A_2 e^{-ikL} + B_2 e^{ikL} \qquad B_2 = -A_2 e^{-2ikL}$$

Combining these with the equations above for the A and B terms produces equations analogous to those of section **II** for A:

$$A_1 = \frac{Y_o/2}{1 - e^{2ikL}} = \frac{Y_o e^{-ikL}}{-4i \sin kL}$$

$$A_2 = \frac{Y_o/2}{1 - e^{-2ikL}} = \frac{Y_o e^{ikL}}{4i \sin kL}$$

If these expressions for the A's and B's are substituted back into the equation for $\psi(x,t)$, the same sum of sines will be obtained. There are more terms to be summed than in section II, but they combine appropriately to form a purely real result without further work.