The Lienard-Wiechert Potentials and their Fields

The Lienard-Wiechert potentials are the exact potentials generated by a point charge q with a position given by **R**(t), for which the charge density is $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r} - \mathbf{R}(t))$. The solution for the potential is then

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_o}\int d^3r' \frac{\delta^3(\mathbf{r}'-\mathbf{R}(t_{ret}))}{|\mathbf{r}-\mathbf{r}'|} \qquad (1)$$

where the source is evaluated at the retarded time, $t_{ret} = t - |\mathbf{r}-\mathbf{r}'|/c$. This is surprisingly awkward, for it introduces a second (implicit) dependence on **r**' in the delta function. In general, a delta function evaluates the integrand where the argument of the delta function is 0, but note that near $f(x_o)=0$,

$$\int \delta(f(x))dx = \int \delta(0 + (x - x_o)f'(x_o) + ...)dx \qquad (2)$$

the argument of the delta function becomes more complicated, shown by expanding f(x) in a Taylor series about x_o. With a change of variable to y, the argument of the second delta function above,

$$\int \delta(f(x))dx = \int \delta(y)\frac{dy}{f'(x_o)} \qquad (3)$$

Thus the integral over the delta function not only evaluates the integrand at the zero of the function, but also introduces the 1/f' factor. In this case, the argument of the three-dimensional delta function is $\mathbf{r}' - \mathbf{R}(t_{ret})$, and the integral goes to a small sphere about **R**. The generalization of f' in this case is the quantity $1 - (\mathbf{v}/c)\cdot\hat{\mathbf{n}}$ from the explicit and implicit dependencies on **r**'. Like several aspects of this calculation, this is subtle; see note at the end. Refer to the list below for the various derivatives that are used often in the following calculations. Applying this to the general solutions of the wave equation for the potentials of a moving point charge gives

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{c}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})} \qquad \mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathbf{v}}{c\,|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{(4)}$$

These look deceptively like the potentials from the static cases, with the minor addition of the $(c-\hat{\mathbf{n}}\cdot\mathbf{v})$ term, which looks as if it would vanish for v<<c. In fact, that is the term from which the radiation fields arise, and the simple appearance hides very considerable complexity, largely because all quantities are evaluated at the retarded time, which is defined by an implicit relation:

$$t_{ret} = t - \frac{|\mathbf{r}-\mathbf{R}|}{c} \qquad \text{where } \mathbf{R}(t_{ret}) \qquad \mathbf{v}(t_{ret}) = d\mathbf{R}/dt \qquad \hat{\mathbf{n}} = \frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|} \qquad (5)$$

When taking the derivatives to find the fields, one must note that the **r** dependence appears explicitly through both **r** and $\hat{\mathbf{n}}$ as well as implicitly through the t_{ret} dependence. Thus $\nabla = \nabla_{explicit} + (\nabla t_{ret})d/dt_{ret}$. The time dependence is also through t_{ret}, which appears in four places in ϕ and five in **A**. The algebra is an extensive exercise in the application of the chain rule. However, the basic physics remains. The charge q is the source of the potential, and the current q**v** is the source of the vector potential. Furthermore, since we have already shown that the potentials from the general solutions used above satisfy the Lorentz gauge condition, it is not necessary to check that again.

Because many of the expressions occur repeatedly, it is convenient to tabulate some results as preparation:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad \text{(used often in both directions)} \qquad (6)$$

$$\nabla_{\text{explicit}} \frac{1}{|\mathbf{r}-\mathbf{R}|} = \frac{-\hat{\mathbf{n}}}{|\mathbf{r}-\mathbf{R}|^2} \qquad \text{(from Coulomb's law calculations)} \qquad (7)$$

$$\nabla_{\text{explicit}} |\mathbf{r}-\mathbf{R}| = \hat{\mathbf{n}} \qquad \text{(from radiation calculations)} \qquad (8)$$

$$\nabla t_{\text{ret}} = \frac{-\hat{\mathbf{n}}}{c - \hat{\mathbf{n}}\cdot\mathbf{v}} \qquad \text{(from derivation of Lienard-Wiechert)} \qquad (9)$$

$$\frac{d|\mathbf{r}-\mathbf{R}|}{dt_{\text{ret}}} = \frac{d}{dt_{\text{ret}}} \sqrt{(x-R_x)^2 + (y-R_y)^2 + (z-R_z)^2} = -\hat{\mathbf{n}}\cdot\mathbf{v} \qquad (10)$$

$$\frac{d}{dt_{\text{ret}}} \frac{1}{|\mathbf{r}-\mathbf{R}|} = \frac{\hat{\mathbf{n}}\cdot\mathbf{v}}{|\mathbf{r}-\mathbf{R}|^2} \qquad (11)$$

$$\frac{dt_{\text{ret}}}{dt} = \frac{c}{c - \hat{\mathbf{n}}\cdot\mathbf{v}} \qquad \left(\text{from} \quad dt_{\text{ret}} = dt - \frac{1}{c}\frac{d|\mathbf{r}-\mathbf{R}|}{dt_{\text{ret}}} dt_{\text{ret}}\right) \qquad (12)$$

$$\nabla_{\text{explicit}}(\hat{\mathbf{n}}\cdot\mathbf{v}) = \frac{\mathbf{v}}{|\mathbf{r}-\mathbf{R}|} - \frac{\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{v})}{|\mathbf{r}-\mathbf{R}|} = \frac{\mathbf{v} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{v})}{|\mathbf{r}-\mathbf{R}|} \qquad (13)$$

$$\frac{d}{dt_{ret}}(\hat{\mathbf{n}}\cdot\mathbf{v}) = \hat{\mathbf{n}}\cdot\mathbf{a} - \frac{v^2}{|\mathbf{r}-\mathbf{R}|} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2}{|\mathbf{r}-\mathbf{R}|} \qquad (\mathbf{a} = d\mathbf{v}/dt\text{; the acceleration}) \qquad (14)$$

(Although these are all straightforward, some require several lines of algebra to confirm.)

We can now proceed with the extensive algebra to compute the electric field.

$$E = \frac{-q}{4\pi\varepsilon_o} \left\{ \nabla \frac{c}{|\mathbf{r}-\mathbf{R}|(c - \hat{\mathbf{n}}\cdot\mathbf{v})} + \frac{\partial}{\partial t} \frac{\mathbf{v}/c}{|\mathbf{r}-\mathbf{R}|(c - \hat{\mathbf{n}}\cdot\mathbf{v})} \right\} \qquad \{ \text{I} + \text{II} \} \qquad (15)$$

$$\nabla = \nabla_{\text{explicit}} + (\nabla t_{\text{ret}}) d/dt_{\text{ret}} \qquad \text{and} \qquad \frac{\partial}{\partial t} = \frac{dt_{\text{ret}}}{dt} \frac{d}{dt_{\text{ret}}}$$

The two contributions in braces will be evaluated separately. For I, the full contributions from the first term in the denominator are given, followed by the contributions from the second:

$$\{\text{I}\} = \frac{-c\hat{\mathbf{n}}}{|\mathbf{r}-\mathbf{R}|^2 (c - \hat{\mathbf{n}}\cdot\mathbf{v})} + \frac{c\hat{\mathbf{n}}\cdot\mathbf{v}}{|\mathbf{r}-\mathbf{R}|^2 (c - \hat{\mathbf{n}}\cdot\mathbf{v})} \nabla t_{\text{ret}} + \frac{c}{|\mathbf{r}-\mathbf{R}|(c - \hat{\mathbf{n}}\cdot\mathbf{v})^2} \left[\frac{\mathbf{v} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{v})}{|\mathbf{r}-\mathbf{R}|} + \hat{\mathbf{n}}\cdot\mathbf{a}\, \nabla t_{\text{ret}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - v^2}{|\mathbf{r}-\mathbf{R}|} \nabla t_{\text{ret}} \right]$$

$$\{\text{I}\} = \frac{c\hat{\mathbf{n}}\cdot\mathbf{a}}{|\mathbf{r}-\mathbf{R}|(c - \hat{\mathbf{n}}\cdot\mathbf{v})^2} \nabla t_{\text{ret}} + \frac{1}{|\mathbf{r}-\mathbf{R}|^2} \left\{ \frac{-c\hat{\mathbf{n}}}{c - \hat{\mathbf{n}}\cdot\mathbf{v}} + \frac{c\ \hat{\mathbf{n}}\cdot\mathbf{v}}{(c - \hat{\mathbf{n}}\cdot\mathbf{v})} \nabla t_{\text{ret}} + c\, \frac{\mathbf{v} - (\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + [(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - v^2]}{(c - \hat{\mathbf{n}}\cdot\mathbf{v})^2} \nabla t_{\text{ret}} \right\}$$

$$\{I\} = \frac{-c(\hat{\mathbf{n}}\cdot\mathbf{a})\hat{\mathbf{n}}}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} + \frac{1}{|\mathbf{r}\text{-}\mathbf{R}|^2}\left\{\frac{-c\hat{\mathbf{n}}}{c-\hat{\mathbf{n}}\cdot\mathbf{v}} - \frac{c(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{(c-\hat{\mathbf{n}}\cdot\mathbf{v})^2} + c\,\frac{\mathbf{v}-(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{(c-\hat{\mathbf{n}}\cdot\mathbf{v})^2} + c\,\frac{[\mathrm{v}^2 - (\hat{\mathbf{n}}\cdot\mathbf{v})^2]\hat{\mathbf{n}}}{(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3}\right\} \qquad (16)$$

where the first term is the far field radiation term and the second is the near field, non-radiating, contribution. Note, however, that this is not an approximation or expansion; these are <u>all</u> the terms. The second part is

$$\{II\} = \frac{c}{c-\hat{\mathbf{n}}\cdot\mathbf{v}}\left\{\frac{\mathbf{a}}{c\,|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}}{c\,|\mathbf{r}-\mathbf{R}|^2(c-\hat{\mathbf{n}}\cdot\mathbf{v})} + \frac{\mathbf{v}[\hat{\mathbf{n}}\cdot\mathbf{a} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2}{|\mathbf{r}-\mathbf{R}|} + \frac{\mathrm{v}^2}{|\mathbf{r}-\mathbf{R}|}]}{c\,|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^2}\right\}$$

$$\{II\} = \frac{\mathbf{a} + \frac{\mathbf{v}(\hat{\mathbf{n}}\cdot\mathbf{a})}{(c-\hat{\mathbf{n}}\cdot\mathbf{v})}}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^2} + \frac{1}{|\mathbf{r}-\mathbf{R}|^2(c-\hat{\mathbf{n}}\cdot\mathbf{v})^2}\left\{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} + \frac{[(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - \mathrm{v}^2]\mathbf{v}}{(c-\hat{\mathbf{n}}\cdot\mathbf{v})}\right\} \qquad (17)$$

Considering first the far field terms in I and II give

$$\frac{1}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3}\left\{-c(\hat{\mathbf{n}}\cdot\mathbf{a})\hat{\mathbf{n}} + (c - \hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{a} + (\hat{\mathbf{n}}\cdot\mathbf{a})\mathbf{v}\right\}$$

$$= \frac{1}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3}\left\{c\hat{\mathbf{n}} \times (\mathbf{a} \times \hat{\mathbf{n}}) + \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{a})\right\}$$

$$= \frac{1}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3}\left\{-\hat{\mathbf{n}} \times [(c\hat{\mathbf{n}} - \mathbf{v}) \times \mathbf{a}]\right\}$$

$$\mathbf{E}_{FF} = \frac{q}{4\pi\varepsilon_o}\frac{\hat{\mathbf{n}}\times[(c\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}]}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} \qquad (18)$$

Although radiation is modified as v approaches c, radiation remains absolutely proportional to acceleration. If the charge is not accelerated, there is no radiation field.

Returning to the more numerous near field terms, the near field terms in II are all proportional to **v** and may be combined as

$$II_{NF} = \frac{c(\hat{\mathbf{n}}\cdot\mathbf{v}) - \mathrm{v}^2}{|\mathbf{r}-\mathbf{R}|^2(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3}\mathbf{v}$$

$$I_{NF} = \frac{1}{|\mathbf{r}-\mathbf{R}|^2 (c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} \Big\{ c(c - \hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} + [-c(c - \hat{\mathbf{n}}\cdot\mathbf{v})^2 - c(\hat{\mathbf{n}}\cdot\mathbf{v})(c - \hat{\mathbf{n}}\cdot\mathbf{v}) - c(\hat{\mathbf{n}}\cdot\mathbf{v})(c - \hat{\mathbf{n}}\cdot\mathbf{v}) + c(v^2 - (\hat{\mathbf{n}}\cdot\mathbf{v})^2)]\hat{\mathbf{n}} \Big\}$$

Expanding the terms in brackets leads to extensive cancellations with the result

$$I_{NF} = \frac{1}{|\mathbf{r}-\mathbf{R}|^2 (c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} \Big\{ c(c - \hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} + c(v^2 - c^2)\hat{\mathbf{n}} \Big\}$$

Combining these results gives the complete expression for the exact electric field of a moving point charge

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_o} \Big\{ \frac{\hat{\mathbf{n}}x[(c\hat{\mathbf{n}}-\mathbf{v})x\mathbf{a}]}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} + \frac{(c\hat{\mathbf{n}}-\mathbf{v})(c^2 - v^2)}{|\mathbf{r}-\mathbf{R}|^2 (c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} \Big\}$$

which must certainly be the most lengthy calculation of the field produced by a stationary point charge, but nevertheless correct even in that limit. The corresponding magnetic field is simply

$$\mathbf{B} = \frac{\hat{\mathbf{n}} \text{ x } \mathbf{E}}{c}$$

a result one would have expected for the radiation field, but not necessarily for the total field. We shall confirm it by direct calculation for the far field terms but not complete the algebra for the near field parts.

$$\mathbf{B} = \nabla \text{ x } \mathbf{A} = \nabla_{explicit} \text{ x } \mathbf{A} + (\nabla t_{ret}) \text{ x } \frac{d\mathbf{A}}{dt_{ret}}$$

$$= \frac{q}{4\pi\varepsilon_o} \Big\{ \nabla_{explicit} \frac{1}{c|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})} \text{ x } \mathbf{v} + (\nabla t_{ret}) \text{ x } \frac{d}{dt_{ret}} \Big[\frac{\mathbf{v}}{c|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})} \Big] \Big\}$$

The calculation is straightforward, for we have already evaluated each of the derivatives that appear here. As before, the first term ($\nabla_{explicit}$) generates only near field terms.

$$\mathbf{B}_{FF} = \frac{q}{4\pi\varepsilon_o} \frac{-\hat{\mathbf{n}}}{c-\hat{\mathbf{n}}\cdot\mathbf{v}} \text{ x } \Big\{ \frac{\mathbf{a}}{c|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})} + \frac{\mathbf{v}(\hat{\mathbf{n}}\cdot\mathbf{a})}{c|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^2} \Big\}$$

$$\mathbf{B}_{FF} = \frac{q}{4\pi\varepsilon_o} \frac{-1}{c|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} \hat{\mathbf{n}} \text{ x } \Big[(c - \hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{a} + (\hat{\mathbf{n}}\cdot\mathbf{a})\mathbf{v} \Big]$$

By rewriting the brackets as

$$[\ ] = (c\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} - \hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{a} + (\hat{\mathbf{n}}\cdot\mathbf{a})(\mathbf{v} - \hat{\mathbf{n}})$$

where the final term ($\hat{\mathbf{n}}$) can be added because it does not contribute to the cross product, one can pair the terms as

$$[\ ] = \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{a}) - \hat{\mathbf{n}} \times (c\hat{\mathbf{n}} \times \mathbf{a}) = \hat{\mathbf{n}} \times [(\mathbf{v} - c\hat{\mathbf{n}}) \times \mathbf{a}]$$

which is the negative of the corresponding expression in **E**, canceling the negative sign in $\mathbf{B}_{FF}$ above and proving the relation for the far field of **B**. The calculation for the near field follows the same pattern, but has more terms.

Returning to the electric field,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_o}\left\{\frac{\hat{\mathbf{n}}\times[(c\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}]}{|\mathbf{r}-\mathbf{R}|(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3} + \frac{(c\hat{\mathbf{n}}-\mathbf{v})(c^2-v^2)}{|\mathbf{r}-\mathbf{R}|^2(c-\hat{\mathbf{n}}\cdot\mathbf{v})^3}\right\}$$

we note that the separation between near and far fields is less obvious than for an oscillating source, where $|\mathbf{r}| \gg \lambda$ defined the far field. Here the first term dominates roughly when

$$|\mathbf{r}| \gg c^2/|\mathbf{a}|$$

If $|\mathbf{a}|$ is the acceleration of gravity on earth, the distance is an astronomical 10^{16} m. On the other hand, if the charge is in a circular orbit R_o at v,

$$|\mathbf{r}| \gg \frac{c^2}{v^2} R_o$$

For a hydrogen atom, the first factor may be 10^5, but the far field dominates only a few microns away. (This contrast is really an illustration of the weakness of the gravitational force compared with electromagnetic forces.) Equally important, the radiated energy is carried only by the far field terms. (The caveat is that if there are structures -- boundary conditions -- in the region dominated by the near field, those interactions may dominate the energy flows.)

In the non-relativistic limit, the radiation field is, where θ is the angle between $\hat{\mathbf{n}}$ and **a**.

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_o}\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a})}{c^2|\mathbf{r}-\mathbf{R}|} \qquad \mathbf{S} = \frac{\mu_o q^2 a^2}{16\pi^2 c}\frac{\hat{\mathbf{n}}}{|\mathbf{r}-\mathbf{R}|^2}\sin^2\theta$$

There is no radiation in the direction of the acceleration; the radiation is peaked in the plane perpendicular to the acceleration. The total radiated power is obtained by integrating on a large sphere to give

$$P = \frac{\mu_o q^2 a^2}{6\pi c}$$

the Larmor formula for the power radiated by a nonrelativistic charge.

The relativistic case is somewhat more subtle. The direct expression for **S** is

$$\mathbf{S} = \frac{\mu_o q^2}{16\pi^2 c}\frac{c^4|\hat{\mathbf{n}}\times[(c\hat{\mathbf{n}}-\mathbf{v})\times\mathbf{a}]|^2}{|\mathbf{r}-\mathbf{R}|^2(c-\hat{\mathbf{n}}\cdot\mathbf{v})^6}\hat{\mathbf{n}}$$

If one were analyzing a synchrotron light source, for example, this would be the applicable expression. However, the expression most often quoted in the literature is not this radiation rate through a distant sphere but rather the rate at which radiated power leaves the charge, d/dt_{ret} rather than d/dt. This cancels one factor of $(c - \hat{\mathbf{n}} \cdot \mathbf{v})$ to give the conventional result, expressed in terms of solid angle, as

$$\frac{dP}{d\Omega} = \frac{\mu_o q^2}{16\pi^2 c} \frac{c^3 |\hat{\mathbf{n}}x[(c\hat{\mathbf{n}}-\mathbf{v})x\mathbf{a}]|^2}{(c-\hat{\mathbf{n}}\cdot\mathbf{v})^5}$$

A rather complicated integral over angles to obtain the total radiated power leaves

$$P = \frac{\mu_o q^2}{6\pi c} \frac{1}{(1 - v^2/c^2)^3} \left\{ a^2 - \frac{|\mathbf{v} \text{ x } \mathbf{a}|^2}{c^2} \right\} = \frac{\mu_o q^2 \gamma^6}{6\pi c} \left\{ a^2 - \frac{|\mathbf{v} \text{ x } \mathbf{a}|^2}{c^2} \right\}$$

These results have two strong physical implications for radiation from relativistic particles. For the angular distribution of power, the radiation peaks strongly for $\hat{\mathbf{n}}$ along **v**; that is where the denominator becomes small. The radiation becomes "focused" in the direction of motion. Likewise, the total radiated power rises strongly as v approaches c. The increase may be as much as γ^6, but in most applications **a** is perpendicular to **v** so that the gain is only γ^4.

Although delta functions are discussed extensively in many references, including the three-dimensional $\delta^3(\mathbf{r}-\mathbf{r}_o)$ in generalized (orthogonal) coordinates, the generalization required here is not easily found.

$$\int F(\mathbf{r})\delta^3(\mathbf{A}(\mathbf{r}))d^3\mathbf{r}$$

If $\mathbf{A}(\mathbf{r}_o)$ is 0, expansion of the argument of the delta function about $\mathbf{r}_o$ produces

$$F(\mathbf{r}_o)\int \delta^3(\mathbf{A}^i_{,j}\mathbf{r}^j))d^3\mathbf{r}$$

over a small sphere centered on **r**=0. In general, this would require some discussion of the vector (or tensor) form of a generalized function, but in this case, $\mathbf{A} = \mathbf{r} - \mathbf{R}(t_{ret})$, and we have

$$\mathbf{A}^i_{,j} = \delta^i_j - \mathbf{V}^i\hat{\mathbf{n}}_j / c$$

If we now choose Cartesian coordinates for which one axis lies along $\hat{\mathbf{V}}$, we can apply the one-dimensional result for each Cartesian coordinate, for which the factor $f' = 1 - (\mathbf{v}/c)\cdot\hat{\mathbf{n}}$ in the $\hat{\mathbf{V}}$ direction, and f'=1 in the other two directions.

A common alternative approach, which solves the present problem, albeit without resolving the general vector delta-function issue, and which is often hidden in a formal 4-vector calculation, is <u>not</u> to begin with Eq.(1) but rather return to the full Green's function solution, integrated over space and time. If the spatial integrals are done first, the remaining time integral has the form

$$\int dt'\delta(t - t' - \frac{|\mathbf{r} - \mathbf{R}(t')|}{c})$$

which again evaluates everything at t_{ret}, but allows direct application of Eq.(3) to the usual scalar delta function with the factor $f' = 1 - (\mathbf{v}/c)\cdot\hat{\mathbf{n}}$ [cf. Eq.(10)], leading again to Eqs.(4).