Newton’s Law of Universal Gravitation

Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

\[ F = G \frac{m_1 m_2}{r^2} \]

- \( G \) is the universal gravitational constant = \( 6.673 \times 10^{-11} \) Nm\(^2\)/kg\(^2\)

\[ F = \frac{G M_{12}}{r^2} \]

Gravitational force is a field force that always exist between two particles regardless of the medium that separates them.

- \( F \) decreases rapidly with increasing separation.
- A finite-size spherically symmetric mass distribution produces the same gravitational force as if the entire mass were concentrated at its center.

Measuring the Gravitational Constant

Experiment by Henry Cavendish in 1798

- Two spheres (each of mass \( m \)) fixed to the ends of a light horizontal rod suspended by a thin metal wire.
- Two large spheres (each of mass \( M \)) are placed near the small ones.

- The attractive force between smaller and larger spheres causes the rod to rotate and twist the wire.
- The angle of rotation is measured by deflection of light beam for different masses at various separations.
Free-Fall Acceleration

$g$ and the Gravitational Force

- The magnitude of the force acting on a freely falling object of mass $m$ near the Earth’s surface:
  \[ mg = \frac{GM_m}{R_e^2} \quad \Rightarrow \quad g = \frac{GM_m}{R_e^2} \]

- For an object located a distance $h$ above the Earth’s surface:
  \[ F_y = \frac{GM_m m}{r^2} \quad \Rightarrow \quad g = \frac{GM_m}{(R_e + h)^2} \]

- $g$ decreases with increasing altitude!

Kepler’s Laws and the Motion of Planets

Kepler’s analysis of planetary motion is summarized in three statements:

1. All planets move in elliptical orbits with the Sun at one focus
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit

Kepler’s Laws and the Motion of Planets

First Law

All planets move in elliptical orbits with the Sun at one focus

- Eccentricity of an ellipse: $e = c/a$
  - Earth: 0.017; Pluto: 0.25; Comet Halley: 0.97

  - Aphelion: the point where the planet is farthest away from the Sun (aphelion for an object orbiting the Earth)
  - Perihelion: the point nearest the Sun (perihelion for an object orbiting the Earth)

Kepler’s Laws and the Motion of Planets

Second Law

- consequence of angular momentum conservation
- Torque of a central force: $\tau = \mathbf{r} \times \mathbf{F} = 0$
- Angular momentum: $L = r \times p = M \mathbf{r} \times \mathbf{v} = \text{const}$

\[ \frac{dA}{dt} = \frac{L}{2M_p} = \text{const} \]

Kepler’s Laws and the Motion of Planets

Third Law

The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit

- Newton’s 2nd law: $\frac{GM_m M_p}{r^2} - \frac{M_p v^2}{r} = 0$
- The orbital speed: $v = \frac{2\pi}{T} r$
- The period: $T^2 = \frac{4\pi^2}{GM_m} r^3 = Ke^3$

\[ T^2 = \frac{4\pi^2 r^3}{GM_m} = Ke^3 \]

Kepler’s Laws and the Motion of Planets

The Gravitational Field

- How objects interact when they are not in contact?
- Gravitational field exists at every point in space
- When a particle of mass $m$ is placed at a point where the gravitational field is $g$, the particle experiences a force $F = mg$
- Gravitational field is defined as $g = \frac{F}{m}$

The Earth’s gravitational field:

\[ g = \frac{F}{m} = \frac{GM_\text{Earth}}{r^2} \]
Gravitational Potential Energy

Any central force is conservative!

• Central force → directed along a radial line to a fixed center and has a magnitude that depends only on the radial coordinate \( r \)
• Conservative force → work it does on an object moving between two points is independent of the path

\[
dW = F \cdot dr = -F(r)dr
\]

\[
W = \int F(r)dr
\]

The work path is broken into a series of radial segments and arcs

• The work depends only on the initial and final values of \( r \) → force is conservative

\[
U(r) = -\frac{GMm}{r}
\]

Energy Considerations

Escape speed

• An object of mass \( m \) is projected vertically upward from the Earth’s surface
• Escape speed → the minimum speed the object must have in order to approach an infinite separation distance from the Earth

\[
v_f = \sqrt{\frac{2GM}{r}} = \frac{1}{r_{\text{esc}}}
\]

\[
\frac{v_f^2}{2} = \frac{GM}{r_{\text{esc}}}
\]

\[
2GM \left( \frac{1}{r_{\text{esc}}} \right)
\]

Independent of the mass of the object!

Energy Considerations

Black holes

• Very massive star → supernova → remaining central core continues to collapse → depending on its mass:
  - \(<1.4M_\odot \) → white dwarf
  - \(>1.4M_\odot \) → neutron star \( R=10 \text{ km} \)
  - \(>3M_\odot \) → black hole
• Escape speed exceeds the speed of light → the object appears to be black

Black holes are remains of stars that have collapsed under their own gravitational force
### SUMMARY

**Gravitation**

- Newton's law of universal gravitation: \( F = \frac{Gm_1m_2}{r^2} \)

- Kepler's laws of planetary motion:
  1. All planets move in elliptical orbits with the Sun at one focus
  2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals
  3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit

- The gravitational field: \( \vec{g} = \frac{F}{m} \)

- The gravitational potential energy: \( U(r) = -\frac{Gm_1m_2}{r} \)

- The total energy of the bound system: \( E = \frac{GMm}{2a} \)

- Escape speed: \( v_{esc} = \sqrt{\frac{2GM}{r_f}} \)