1. Bosonic creation and annihilation operators $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$ were defined in class in terms of their respective matrix elements in the occupation-number basis of the bosonic Fock space $\mathcal{F}^B$:

$$\begin{aligned} \left\langle \{n'_\beta\}_\beta \right| \hat{a}_\alpha^\dagger \left| \{n_\beta\}_\beta \right\rangle &= \begin{cases} \sqrt{n_\alpha + 1} & \text{provided all } n'_\beta = n_\beta + \delta_{\alpha,\beta} \text{ and} \\ 0 & \text{otherwise;} \end{cases} \\ \left\langle \{n'_\beta\}_\beta \right| \hat{a}_\alpha \left| \{n_\beta\}_\beta \right\rangle &= \begin{cases} \sqrt{n_\alpha} & \text{provided all } n'_\beta = n_\beta - \delta_{\alpha,\beta} \text{ and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

This exercise is about the way these operators acts on coordinate-space wave functions of multi-particle states.

(a) Consider an N–boson state of the form $|(\alpha_1, \alpha_2, \ldots, \alpha_N)\rangle = \left|\{n_\beta\}_\beta\right\rangle$. Show that the coordinate-space wave function of this state has form

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \frac{1}{\sqrt{N! \prod_\beta n_\beta!}} \sum \varphi_{\alpha_1}(\mathbf{x}_{\nu_1}) \, \varphi_{\alpha_2}(\mathbf{x}_{\nu_2}) \cdots \varphi_{\alpha_N}(\mathbf{x}_{\nu_N}) \tag{2}$$

where the sum is over all $N!$ permutations $\mathbf{x}_{\nu_1}, \mathbf{x}_{\nu_2}, \ldots, \mathbf{x}_{\nu_N}$ of the particle positions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$. (In other words, $\nu_1, \nu_2, \ldots, \nu_N$ are summations indices running over all $N!$ permutations of the integers $1, 2, \ldots, N$.)

Hint: permuting the positions is equivalent to permuting the 1-particle wave functions $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_N}$, but watch for coincident terms on the right hand side of eq. (2).

(b) Now consider a generic N–particle state $|N, \Psi\rangle \in \mathcal{H}_N^B$ with a generic wave function $\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$; more precisely, $\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ must be totally symmetric with respect to the N positions $\mathbf{x}_1, \ldots, \mathbf{x}_N$ but otherwise, it is completely generic.

Show that a creation operator $\hat{a}_\alpha^\dagger$ acting on this state produces an $(N+1)$ particle state $|N+1,\Psi'\rangle \in \mathcal{H}^B_{N+1}$ with a totally symmetric wave function

$$\begin{aligned}\Psi'(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_{N+1}) \;&=\; \frac{1}{\sqrt{N+1}}\sum_{i=1}^{N+1}\varphi_\alpha(\mathbf{x}_i)\,\Psi(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots,\mathbf{x}_{N+1})\\ &\stackrel{\text{def}}{=}\; \frac{1}{\sqrt{N+1}}\Big[\varphi_\alpha(\mathbf{x}_1)\,\Psi(\mathbf{x}_2,\ldots,\mathbf{x}_{N+1})\;+\;\varphi_\alpha(\mathbf{x}_2)\,\Psi(\mathbf{x}_1,\mathbf{x}_3\ldots,\mathbf{x}_{N+1})\;+\;\cdots\\ &\qquad +\;\varphi_\alpha(\mathbf{x}_N)\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_{N+1})\;+\;\varphi_\alpha(\mathbf{x}_{N+1})\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\Big].\end{aligned} \tag{3}$$

Note that while Ψ is a function of N positions, Ψ' is a function of $N+1$ positions. In particular, for $N=0$, Ψ is simply a complex number but Ψ' is a 1-particle wave function, $\Psi'(\mathbf{x}_1)=\varphi_\alpha(\mathbf{x}_1)\times\Psi$.

Hint: First prove (3) for the wave-functions Ψ of the form (2) — and do not forget to verify the normalization of the resulting Ψ' — then use the fact that the states $|(\alpha_1,\alpha_2,\ldots,\alpha_N)\rangle$ constitute a basis of the $\mathcal{H}^B_N$.

(c) Next, consider the annihilation operators $\hat{a}_\alpha$ and show that the wave function Ψ'' of the $N-1$ particle state $|N-1,\Psi''\rangle = \hat{a}_\alpha\,|N,\Psi\rangle \in \mathcal{H}^B_{N-1}$ can be written as

$$\Psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d\mathbf{x}_N\,\varphi^*_\alpha(\mathbf{x}_N)\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{4}$$

In particular, for $N=1$, Ψ is a 1–particle wave function while Ψ'' is a number, $\Psi''=\langle\alpha|\Psi\rangle$. For $N=0$, formula (4) degenerates to $\Psi''=0$ (since $\sqrt{(N=0)}=0$), which agrees with $\hat{a}_\alpha\,|0\rangle=0$ (and hence $\hat{a}_\alpha\,|N=0,\Psi\rangle=0$ for any Ψ), although in this case Ψ'' is rather ill-defined as a function.

2. Formulæ (4) and (3) allow for straightforward translation between first-quantized and second-quantized forms of various operators. In particular, consider an additive one-particle operator of the form

$$\hat{A}_{\text{tot}}(N\text{ particles}) \;=\sum_{i=1}^{N}\hat{A}_1(i^{\underline{\text{th}}}\text{ particle}). \tag{5}$$

As argued in class, the second-quantized form of such an operator is

$$\hat{A} \;=\sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha \hat{a}_\beta\,. \tag{6}$$

(a) Use formulæ (4) and (3) to explicitly calculate the wave function $\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ of the N particle state $|N,\Psi'\rangle = \hat{A}\,|N,\Psi\rangle$ (assume generic totally symmetric $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$) and show that in the first-quantized formalism, $\hat{A}_{\rm tot}$ acting on Ψ yields exactly same Ψ'.

Hint: Prove and use

$$\begin{aligned}\hat{A}_1(i^{\underline{\rm th}}\text{ particle})\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N) \\ =\sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\varphi_\alpha(\mathbf{x}_i)\int\! d\mathbf{x}'_i\,\varphi^*_\beta(\mathbf{x}'_i)\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots,\mathbf{x}_N).\end{aligned} \tag{7}$$

Now consider an additive two-particle interaction operator such as

$$\hat{V}_{\rm tot} \;=\; \tfrac{1}{2}\sum_{i\neq j} V(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j)$$

for some two-body potential $V(\mathbf{x}_i - \mathbf{x}_j)$. More generally, one has a two-particle operator $\hat{A}_2$ involving positions or other quantum numbers of two particles and the total $\hat{A}$ of an N particle system is given by

$$\hat{A}_{\rm total} \;=\; \tfrac{1}{2}\sum_{\substack{i,j=1,\ldots,N\\ i\neq j}} \hat{A}_2(i^{\underline{\rm th}}\text{ and }j^{\underline{\rm th}}\text{ particles}). \tag{8}$$

The second quantized form of such an additive two-particle operator is given by

$$\hat{A} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{A}_2\,|\gamma\otimes\delta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta \tag{9}$$

where $\langle\alpha\otimes\beta|$ and $|\gamma\otimes\delta\rangle$ are non-symmetrized two-distinct-particles states whose respective wave functions are simply $\varphi^*_\alpha(\mathbf{x}_1)\varphi^*_\beta(\mathbf{x}_2)$ and $\varphi_\gamma(\mathbf{x}_1)\varphi_\delta(\mathbf{x}_2)$.

(b) Verify by an explicit wave-function calculation that the operators (8) and (9) indeed produce identical results when acting on any N particle state $|N, \Psi\rangle$.

Note special cases of $N = 0$ or $N = 1$ where both (8) and (9) yield 0. For $N \geq 2$, use formulæ (4) and (3) and a suitable analog of eq. (7).

3. Finally, an exercise in using the bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha,\beta}\,. \tag{10}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$.

(b) Consider to one-particle operators $\hat{A}_1$ and $\hat{B}_1$ and let $\hat{C}_1$ be their commutator, $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$. Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be the second-quantized forms of the respective additive operators, *cf.* eq. (6).

Show that $[\hat{A}, \hat{B}] = \hat{C}$.

(c) Next, calculate the commutator $[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta]$.

(d) Finally, consider a one-particle operator $\hat{A}_1$, a two-particle operator $\hat{B}_2$ and a two-particle operator $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\underline{\text{st}}}) + \hat{A}_1(2^{\underline{\text{nd}}})\right), \hat{B}_2\right]$. Show that in this case the second-quantized $\hat{C}$ is again the commutator of the second-quantized $\hat{A}$ with the second-quantized $\hat{B}$.