1. Consider coherent states of a harmonic oscillator.

(a) Show that for any complex number α,

$$|\alpha\rangle \stackrel{\text{def}}{=} \exp\bigl(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\bigr)\,|0\rangle \;=\; e^{-|\alpha|^2/2}\,e^{\alpha\hat{a}^\dagger}\,|0\rangle \quad \text{and} \quad \hat{a}\,|\alpha\rangle \;=\; \alpha\,|\alpha\rangle\,. \tag{1}$$

(b) Calculate the uncertainties Δq and Δp for a coherent state $|\alpha\rangle$ and verify their minimality: $\Delta q \Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\text{def}}{=} \langle\hat{n}\rangle = |\alpha|^2$.

Hint: use $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and $\langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha|$.

(c) Show that for any initial coherent state $|\alpha_0\rangle$,

$$|\psi(t)\rangle \;\equiv\; e^{-i\omega t/2}\,\bigl|\alpha = \alpha_0 e^{-i\omega t}\bigr\rangle \tag{2}$$

satisfies the time-dependent Schrödinger equation.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{3}$$

for any given classical complex field $\Phi(\mathbf{x})$.

(f) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \stackrel{\text{def}}{=} \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int\! d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{4}$$

(g) Let

$$\hat{H} \;=\; \int\! d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger\cdot\nabla\hat{\Psi} \;+\; v(\mathbf{x})\hat{\Psi}^\dagger\hat{\Psi}\right)$$

and show that for any classical field configuration $\Phi(\mathbf{x},t)$ that satisfies the classical field equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{5}$$

(h) Finally, show that the quantum overlap $|\langle\Phi_1|\Phi_2\rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.

2. Consider a complex relativistic field $\Phi(x)$ with a Lagrangian density

$$\mathcal{L} \;=\; \partial^\mu\Phi^*\,\partial_\mu\Phi \;-\; m^2\,\Phi^*\Phi \;-\; \tfrac{1}{4}\lambda\big(\Phi^*\Phi\big)^2. \tag{6}$$

This Lagrangian has a symmetry $\Phi(x) \mapsto e^{i\theta}\Phi(x)$. According to Noether theorem (which we shall study later in class), this symmetry gives rise to a conserved current

$$J^\mu \;=\; i\Phi^*\partial^\mu\Phi \;-\; i(\partial^\mu\Phi^*)\Phi. \tag{7}$$

(a) Write down classical field equations for $\Phi(x)$ and $\Phi^*(x)$ (treat them as independent fields!) and verify that indeed $\partial_\mu J^\mu = 0$.

Canonical quantization of the complex field yields non-hermitian quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$ and the Hamiltonian

$$\hat{H} \;=\; \int d^3\mathbf{x}\,\left(\hat{\Pi}^\dagger\hat{\Pi} \;+\; \nabla\hat{\Phi}^\dagger\cdot\nabla\hat{\Phi} + m^2\,\hat{\Phi}^\dagger\hat{\Phi} \;+\; \tfrac{1}{4}\lambda\,\hat{\Phi}^\dagger\hat{\Phi}^\dagger\hat{\Phi}\hat{\Phi}\right). \tag{8}$$

(b) Derive the Hamiltonian (8) and write down the equal-time commutation relations between the quantum fields $\hat{\Phi}(\mathbf{x})$, $\hat{\Phi}^\dagger(\mathbf{x})$, $\hat{\Pi}(\mathbf{x})$ and $\hat{\Pi}^\dagger(\mathbf{x})$.

Because of the non-hermiticity of the quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$, their respective plane-wave modes $\hat{\Phi}_{\mathbf{p}}$, $\hat{\Phi}_{\mathbf{p}}^\dagger$, $\hat{\Pi}_{\mathbf{p}}$ and $\hat{\Pi}_{\mathbf{p}}^\dagger$ are completely independent of each other *i.e.*, $\hat{\Phi}_{\mathbf{p}}^\dagger \neq \hat{\Phi}_{-\mathbf{p}}$ and $\hat{\Pi}_{\mathbf{p}}^\dagger \neq \hat{\Pi}_{-\mathbf{p}}$. Let us therefore define:

$$
\begin{aligned}
\hat{a}_{\mathbf{p}} &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{\mathbf{p}} + i\hat{\Pi}_{-\mathbf{p}}^\dagger}{\sqrt{2E_{\mathbf{p}}}}\,, \qquad & \hat{a}_{\mathbf{p}}^\dagger &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{\mathbf{p}}^\dagger - i\hat{\Pi}_{-\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\,, \\
\hat{b}_{\mathbf{p}} &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{-\mathbf{p}}^\dagger + i\hat{\Pi}_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\,, \qquad & \hat{b}_{\mathbf{p}}^\dagger &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{-\mathbf{p}} - i\hat{\Pi}_{\mathbf{p}}^\dagger}{\sqrt{2E_{\mathbf{p}}}}\,,
\end{aligned}
\tag{9}
$$

where

$$
E_{\mathbf{p}} \stackrel{\text{def}}{=} \sqrt{\mathbf{p}^2 + m^2}\,. \tag{10}
$$

(c) Verify the bosonic commutation relations (at equal times) between the annihilation operators $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}$ and the corresponding creation operators $\hat{a}_{\mathbf{p}}^\dagger$ and $\hat{b}_{\mathbf{p}}^\dagger$.

(d) Now, let us turn off the interactions (*i.e.*, set $\lambda = 0$). Show that the Hamiltonian of free charged fields is

$$
\begin{aligned}
\hat{H}_{\text{free}} &\stackrel{\text{def}}{=} \int d^3\mathbf{x} \left(\hat{\Pi}^\dagger\hat{\Pi} + \nabla\hat{\Phi}^\dagger \cdot \nabla\hat{\Phi} + m^2\,\hat{\Phi}^\dagger\hat{\Phi}\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, E_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{p}} + \hat{b}_{\mathbf{p}}^\dagger\hat{b}_{\mathbf{p}}\right) + \text{const.}
\end{aligned}
\tag{11}
$$

(e) Next, consider the electric charge operator $\hat{Q} = \int d^3\mathbf{x}\,\hat{J}_0(\mathbf{x})$. Show that for the system at hand

$$
\hat{Q} = \int d^3\mathbf{x} \left(\tfrac{i}{2}\{\hat{\Pi}^\dagger, \hat{\Phi}^\dagger\} - \tfrac{i}{2}\{\hat{\Pi}, \hat{\Phi}\}\right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{p}} - \hat{b}_{\mathbf{p}}^\dagger\hat{b}_{\mathbf{p}}\right). \tag{12}
$$

Actually, the classical formula (7) for the current $J_\mu(x)$ determines eq. (12) only up to ordering of the non-commuting operators $\hat{\Pi}(\mathbf{x})$ and $\hat{\Phi}(\mathbf{x})$ (and likewise of the $\hat{\Pi}^\dagger(\mathbf{x})$ and $\hat{\Phi}^\dagger(\mathbf{x})$). The anti-commutators in eq. (12) provide a solution to this ordering ambiguity, but any other ordering would be just as legitimate.

The net effect of changing operator ordering in $\hat{J}_0$ amounts to changing the total charge $\hat{Q}$ by an infinite constant (prove this!). The specific ordering in eq. (12) provides for the neutrality of the vacuum state.

Finally, consider the stress-energy tensor for the complex field $\Phi(x)$. Classically, Noether theorem gives

$$T^{\mu\nu} \;=\; \partial^\mu\Phi^*\,\partial^\nu\Phi \;+\; \partial^\mu\Phi\,\partial^\nu\Phi^* \;-\; g^{\mu\nu}\mathcal{L}. \tag{13}$$

Quantization of this formula is straightforward (modulo ordering ambiguity); for example, $\hat{\mathcal{H}} \equiv \hat{T}^{00}$ is precisely the integrand on the right hand side of eq. (8).

(f) Consider the total mechanical momentum operator of the fields $\hat{P}^i_{\rm mech} = \int d^3\mathbf{x}\,\hat{T}^{0i}(\mathbf{x})$ and show that in terms of creation and annihilation operators

$$\hat{\mathbf{P}}_{\rm mech} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathbf{p}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;+\; \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right) \tag{14}$$

Physically, eqs. (14), (11) and (12) show that a complex field $\Phi(x)$ describes both a particle and its antiparticle; they have exactly the same rest mass m but exactly opposite charges ± 1.