1. In homework#2 we developed Hamiltonian formalism for a massive vector field $A^\mu(x)$. Upon quantization, the 3–vector field $\mathbf{A}(x)$ and its canonical conjugate $-\mathbf{E}(x)$ become quantum fields subject to equal-time commutation relations

$$[\hat{A}^i(\mathbf{x}), \hat{A}^j(\mathbf{y})] \;=\; 0, \quad [\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \;=\; 0, \quad [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \;=\; -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1}$$

($\hbar = 1, c = 1$ units) governed by the free Hamiltonian

$$\hat{H} \;=\; \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 \;+\; \frac{(\nabla\cdot\hat{\mathbf{E}})^2}{2m^2} \;+\; \tfrac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 \;+\; \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 \right) \tag{2}$$

(we assume $J^\mu = 0$). For the non-dynamical A^0 field, its time-independent equation of motion becomes an operatorial identity

$$\hat{A}^0(x) \;=\; -\frac{\nabla\cdot\hat{\mathbf{E}}(x)}{m^2}\,. \tag{3}$$

The purpose of the present exercise is to expand fields in terms of creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$ where λ labels three different polarization states of a vector particle (spin $= 1$). Generally, bases for polarization states correspond to $\mathbf{k}$–dependent complex bases $\mathbf{e}_\lambda(\mathbf{k})$ for ordinary 3–vectors,

$$\mathbf{e}_\lambda(\mathbf{k})\cdot\mathbf{e}^*_{\lambda'}(\mathbf{k}) \;=\; \delta_{\lambda,\lambda'} \tag{4}$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k}\,\times$, namely

$$i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k}) \;=\; \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \qquad \lambda = -1, 0, +1. \tag{5}$$

By convention, the overall phases of the helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) \;=\; \frac{\mathbf{k}}{|\mathbf{k}|} \quad \mathbf{e}^*_\lambda(\mathbf{k}) \;=\; (-1)^\lambda\mathbf{e}_{-\lambda}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) \;=\; -\mathbf{e}^*_\lambda(+\mathbf{k}). \tag{6}$$

Combining Fourier transform with a basis decomposition, we have

$$\hat{\mathbf{A}}(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda}\,, \qquad \hat{A}_{\mathbf{k},\lambda} = \int d^3\mathbf{x}\,e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}(\mathbf{x}) \tag{7}$$

and ditto for the $\hat{\mathbf{E}}(\mathbf{x})$ fields and its $\hat{E}_{\mathbf{k},\lambda}$ modes.

(a) Show that $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$, and derive the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ operators.

(b) Show that

$$\hat{H} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\left(\frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} + \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right) \tag{8}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$ and $C_{\mathbf{k},\lambda} = 1+\delta_{\lambda,0}(\mathbf{k}^2/m^2)$.

(c) Define creation and annihilation operators according to

$$\hat{a}_{\mathbf{k},\lambda} = \sqrt{\frac{\omega_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}}\,\hat{A}_{\mathbf{k},\lambda} - i\sqrt{\frac{C_{\mathbf{k},\lambda}}{2\omega_{\mathbf{k}}}}\,\hat{E}_{\mathbf{k},\lambda}\,, \quad \hat{a}^\dagger_{\mathbf{k},\lambda} = \sqrt{\frac{\omega_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}}\,\hat{A}^\dagger_{\mathbf{k},\lambda} + i\sqrt{\frac{C_{\mathbf{k},\lambda}}{2\omega_{\mathbf{k}}}}\,\hat{E}^\dagger_{\mathbf{k},\lambda} \tag{9}$$

and verify that they satisfy bosonic commutation relations (at equal times).

(d) Show that

$$\hat{H} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\omega_{\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \text{const.} \tag{10}$$

(e) Next, consider the time dependence of the free vector field and show that

$$\hat{\mathbf{A}}(\mathbf{x},t) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0) + e^{+ikx}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{11}$$

(f) Write down a similar formula for the $\hat{A}^0(\mathbf{x},t)$ (use eq. (3)). Together with the previous result, you should get

$$\hat{A}_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{k}}} \sum_\lambda \left(e^{-ikx} f_\mu(\mathbf{k},\lambda)\, \hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx} f^*_\mu(\mathbf{k},\lambda)\, \hat{a}^\dagger_{\mathbf{k},\lambda}(0) \right)_{k^0=+\omega_\mathbf{k}} \tag{12}$$

where

$$f^\mu(\mathbf{k},\lambda) = \begin{cases} \big(0, \mathbf{e}_\lambda(\mathbf{k})\big) & \text{for } \lambda = \pm 1, \\ \left(\frac{|\mathbf{k}|}{m}, \frac{\omega_\mathbf{k}}{m}\frac{\mathbf{k}}{|\mathbf{k}|}\right) & \text{for } \lambda = 0. \end{cases} \tag{13}$$

Please note that the 4–vectors $f^\mu(\mathbf{k},\lambda)$ are nothing but purely-spatial vectors $\mathbf{e}_\lambda(\mathbf{k})$ Lorentz-boosted into the moving particle's frame. In particular, for all $(\mathbf{k},\lambda)$, $f^\mu f^*_\mu = -1$ and $f^\mu k_\mu = 0$.

(g) Finally, verify that the vector field (12) satisfies the free equations of motion $\partial_\mu \hat{A}^\mu(x) = 0$ and $(\partial^2 + m^2)\hat{A}^\mu(x) = 0$.

2. Now consider the Feynman propagator for the massive vector field.

(a) First, a lemma: Show that

$$\sum_\lambda f^\mu(\mathbf{k},\lambda) f^{\nu *}(\mathbf{k},\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}. \tag{14}$$

(b) Next, show that

$$\begin{aligned} \langle 0|\, \hat{A}^\mu(x)\hat{A}^\nu(y)\, |0\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{k}} \left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} \right]_{k^0=+\omega_\mathbf{k}} \\ &= \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D(x-y). \end{aligned} \tag{15}$$

(c) Finally, the Feynman propagator: Show that

$$\begin{aligned} G_F^{\mu\nu} \equiv \langle 0|\, \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\, |0\rangle &= \int \frac{d^4\mathbf{k}}{(2\pi)^4} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i0} \\ &= \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D_F(x-y). \end{aligned} \tag{16}$$

3. The last exercise is about superfluid Helium. For $\mu = \lambda n > 0$, we expect a ground state with a non-zero expectation values of the creation/annihilations fields, $\langle\hat{\Psi}\rangle = \sqrt{n}$, up to an arbitrary phase. Let us therefore define shifted fields $\widetilde{\Psi}(x) = \hat{\Psi}(x) - \sqrt{n}$ and $\widetilde{\Psi}^\dagger(x) = \hat{\Psi}^\dagger(x) - \sqrt{n}$ and expand the Hamiltonian (or rather $\hat{H} - \mu\hat{N}$) into powers of shifted fields.

   (a) Show that

$$\hat{H} - \mu\hat{N} \;=\; \text{const} \;+\; \hat{H}_2 \;+\; \hat{H}_{\text{int}}$$

   where (in $\hbar = 1$ units)

$$\hat{H}_2 \;=\; \int d^3\mathbf{x} \left\{ \frac{1}{2M} \nabla\widetilde{\Psi}^\dagger \cdot \nabla\widetilde{\Psi} \;+\; \frac{\lambda n}{2} \left(\widetilde{\Psi}^\dagger\widetilde{\Psi}^\dagger + 2\widetilde{\Psi}^\dagger\widetilde{\Psi} + \widetilde{\Psi}\widetilde{\Psi} \right) \right\} \tag{17}$$

   while $\hat{H}_{\text{int}}$ comprises cubic and quartic terms with respect to the shifted fields.

   (b) Fourier-transform the shifted fields into shifted creation/annihilation operators $\tilde{a}^\dagger_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} - (2\pi)^3\sqrt{n}\,\delta^{(3)}(\mathbf{k})$ and $\tilde{a}_{\mathbf{k}} = \hat{a}_{\mathbf{k}} - (2\pi)^3\sqrt{n}\,\delta^{(3)}(\mathbf{k})$, then perform a canonical transform

$$\hat{b}_{\mathbf{k}} \;=\; \cosh(t_{\mathbf{k}})\,\tilde{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\,\tilde{a}^\dagger_{-\mathbf{k}}\,, \quad \hat{b}^\dagger_{\mathbf{k}} \;=\; \cosh(t_{\mathbf{k}})\,\tilde{a}^\dagger_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\,\tilde{a}_{-\mathbf{k}}\,. \tag{18}$$

   Show that for any $t_{\mathbf{k}} = t_{-\mathbf{k}}$, the operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ satisfy bosonic commutation relations. (At equal times, of course).

   (c) Show that for a suitable choice of $t_{\mathbf{k}}$,

$$\hat{H}_2 \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\omega_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;+\; \text{const} \tag{19}$$

   where

$$\omega_{\mathbf{k}} \;=\; |k|\sqrt{\frac{\lambda n}{M} + \frac{k^2}{4M^2}}\,. \tag{20}$$

Please note that the ground state of the $\hat{H}_2$ is the state $|\Omega_2\rangle$ annihilated by all the $\hat{b}_{\mathbf{k}}$ operators. To construct this ground state, we start with the coherent state $|\text{coh}\rangle$ — which is annihilated by all the shifted $\tilde{a}_{\mathbf{k}}$ — and then modify according to

$$|\Omega_2\rangle \;=\; e^{\hat{F}}\,|\text{coh}\rangle\,, \qquad \hat{F} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{t_{\mathbf{k}}}{2}\left(\tilde{a}_{\mathbf{k}}\tilde{a}_{-\mathbf{k}} - \tilde{a}^\dagger_{\mathbf{k}}\tilde{a}^\dagger_{-\mathbf{k}}\right). \tag{21}$$

(∗) Optional exercise: Show that $\hat{b}_{\mathbf{k}} = e^{\hat{F}}\tilde{a}_{\mathbf{k}}e^{-\hat{F}}$, $\hat{b}^\dagger_{\mathbf{k}} = e^{\hat{F}}\tilde{a}^\dagger_{\mathbf{k}}e^{-\hat{F}}$, and hence $\hat{b}_{\mathbf{k}}\,|\Omega_2\rangle = 0$ as well as automatic bosonic commutation relations for the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators.

The excited states of the $\hat{H}_2$ Hamiltonian can be constructed by applying the $\hat{b}^\dagger_{\mathbf{k}}$ operators to the ground state $|\Omega_2\rangle$. Thus, one can say that *the $\hat{b}^\dagger_{\mathbf{k}}$ operators create quasiparticles and the the $\hat{b}_{\mathbf{k}}$ operators annihilate them*; from this point of view, the $|\Omega_2\rangle$ ground state is the *quasiparticle vacuum.*

(d) Show that the net mechanical momentum of the superfluid Helium is

$$\hat{\mathbf{P}} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}\,\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,, \tag{22}$$

thus quasiparticles do have well-defined momenta $\mathbf{k}$.

On the other hand, the quasiparticles do not have well-defined atomic numbers. This is related to the spontaneous breakdown of the phase symmetry, which is generated by the atom number operator $\hat{N}$. Physically, the quasiparticles interpolate between phonons in the superfluid (for small k) and atoms knocked out of the Bose condensate (for large k) — note the appropriate limits of the dispersion relation (20).

Actually, in the real helium with a finite-range interatomic potential $V_2(\mathbf{x}-\mathbf{y})$, the dispersion relation is a bit more complicated than eq. (20) — *e.g.*, there is a so-called 'roton dip' at intermediate values of the quasiparticle momenta k — but the small–k and the large–k limits work exactly as in this exercise. In particular, there is a positive lower bound on quasi-particle phase velocities, $\forall\mathbf{k},\ \omega_{\mathbf{k}} \geq v_0|\mathbf{k}|$. This fact plays a crucial role in superfluidity.

(e) Now consider the superfluid in a state of uniform motion with velocity $\mathbf{v}$. Use Galilean invariance to argue that quasiparticles in moving Helium are governed by the

$$\hat{H}'_2 \;=\; \hat{H}_2 \;+\; \mathbf{v}\cdot\hat{\mathbf{P}} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,(\omega_{\mathbf{k}} + \mathbf{v}\mathbf{k})\,\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{23}$$

Therefore, as long as $|\mathbf{v}| < v_0$, all excitations have positive energies, hence there is no spontaneous decay of the flowing “ground” state and no energy dissipation! This is the physical origin of superfluidity.

On the other hand, when the Helium flows too fast, $|\mathbf{v}| > v_0$, some quasiparticle modes acquire negative energies, which leads to spontaneous quasiparticle production, hence energy dissipation and loss of superfluidity.

The *critical velocity* v_0 is governed by the dispersion relation for the quasiparticles: $v_0 = \min(\omega_k/k)$. For the superfluid, $v_0 > 0$. In comparison, the ideal gas has $\omega_k = k^2/2m$, thus $v_0 = 0$ and no superfluidity at any velocity.

Actually, under most experimental conditions, there is an additional mechanism for losing superfluidity beyond a much smaller critical velocity than the v_0 obtaining from the microscopic theory. Specifically, turbulence leads to spontaneous generations of vortex rings, which move much slower than the quasiparticles and hence quench superfluidity at much slower speeds. In very thin capillaries however, the vortex rings do not form because of size limitations and the superfluidity persists until the microscopic critical velocity v_0.