1. According to Noeter theorem, a system of several classical fields $\phi_a$ has stress-energy tensor

$$T^{\mu\nu}_{\text{Noeter}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{1}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noeter}} \;+\; \partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}, \tag{2}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$,

$$\begin{aligned} \partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\text{Noether}} \;=\text{(hopefully)}=\; 0 \\ P^\mu_{\text{net}} \;&\equiv\; \int\! d^3\mathbf{x}\,T^{0\mu} \;=\; \int\! d^3\mathbf{x}\,T^{0\mu}_{\text{Noether}}\,. \end{aligned} \tag{3}$$

For the scalar fields, real or complex, $T^{\mu\nu}_{\text{Noeter}}$ is properly symmetric and one simply has $T^{\mu\nu} = T^{\mu\nu}_{\text{Noeter}}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{4}$$

where $A_\mu$ is a real vector field and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) Write down $T^{\mu\nu}_{\text{Noeter}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\text{EM}} \;=\; -F^{\mu\lambda}F^{\nu}{}_{\lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{5}$$

Show that this expression indeed has form (2) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

(d) Write down the components of the stress-energy tensor (5) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

2. Now consider electromagnetic fields coupled to some charged "matter" fields which carry an EM current $J^\mu$.

(a) Use Maxwell equations to show that

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda} J_\lambda \,. \tag{6}$$

Eq. (6) suggests that any system of charged "matter" fields should have its stress-energy tensor $T^{\mu\nu}_{\rm mat}$ obeying

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^\nu{}_\lambda J^\lambda_{\rm EM} \,. \tag{7}$$

Consequently, the combined stress-energy tensor $T^{\mu\nu}_{\rm EM} + T^{\mu\nu}_{\rm mat}$ should be divergence-less and thus lead to a conserved total energy and momentum.

Generally, testing eq. (7) for any particular system of charged "matter" fields makes use of fields' equations of 'motion' and also of the fact that *the covariant derivatives $D_\mu$ do not commute with each other.* Instead, when acting upon a field $\Phi_q$ of charge $q$, one has

$$(D_\mu D_\nu - D_\nu D_\mu)\Phi_q \;=\; iq\, F_{\mu\nu}\, \Phi_q \tag{8}$$

(in $c = \hbar = 1$ units).

(b) Verify eq. (8).

Now consider a specific example of EM coupled to a charged scalar field, with a combined Lagrangian density

$$\mathcal{L} \;=\; D^\mu \Phi^* D_\mu \Phi \;-\; m^2 \Phi^* \Phi \;-\; \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \,. \tag{9}$$

(c) Calculate the Noether stress-energy tensor for this field system and show that

$$T^{\mu\nu}_{\rm net} \;\equiv\; T^{\mu\nu}_{\rm Noeter} \;+\; \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \;=\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \tag{10}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is the same function of EM fields as in the free EM case (c), $T^{\mu\nu}_{\rm EM}$ is exactly as in eq. (5) and

$$T^{\mu\nu}_{\rm mat} \;=\; D^\mu\Phi^*\,D^\nu\Phi \;+\; D^\nu\Phi^*\,D^\mu\Phi \;-\; g^{\mu\nu}\bigl(D^\lambda\Phi^*\,D^\lambda\Phi \;-\; m^2\Phi^*\Phi\bigr) \tag{11}$$

Hint: In the presence of an electric current $J^\mu$, the $\partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (11) for the scalar field.

(d) Use the scalar field's equations of motion and eq. (8) to verify eq. (7).

3. Finally, let us quantize the (free) electromagnetic fields. Unlike the massive vector fields studied in previous homeworks, the Hamiltonian formalism for the massless EM fields suffers from redundancies associated with the gauge transformations. To ameliorate this problem, we would like to separate the gauge-invarian transverse polarizations of the vector field from the gauge-dependent longitudinal and scalar polarizations. The separation is best done in terms of the Fourier-transformed fields using the helicity basis defined in the previous homework.

(a) Show that in terms of the $A_{\mathbf{k},\lambda}(t)$ and $A^0_{\mathbf{k}}$ modes of the classical fields, the free EM Lagrangian becomes

$$L \;\equiv\; \int\! d^3\mathbf{x}\mathcal{L} \;=\; L^\perp \;+\; L^\parallel \tag{12}$$

where

$$L^\perp \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm1}\left(\frac{1}{2}\,\dot{A}^*_{\mathbf{k},\lambda}\dot{A}_{\mathbf{k},\lambda} \;-\; \frac{\mathbf{k}^2}{2}\,A^*_{\mathbf{k},\lambda}A_{\mathbf{k},\lambda}\right). \tag{13}$$

and

$$L^\parallel \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{2}\left|\dot{A}_{\mathbf{k},0} + ikA^0_{\mathbf{k}}\right|^2. \tag{14}$$

(b) Show that the transverse ($\lambda=\pm1$) modes $A_{\mathbf{k},\lambda}$ are gauge invariant while the longitudinal and scalar modes transform according to

$$A_{\mathbf{k},0}(t) \;\to\; A_{\mathbf{k},0}(t) \,+\, ik\Lambda_{\mathbf{k}}(t), \qquad A^0_{\mathbf{k}}(t) \;\to\; A^0_{\mathbf{k}}(t) \;-\; \dot{\Lambda}_{\mathbf{k}}(t) \tag{15}$$

for arbitrary, independent $\Lambda_{\mathbf{k}}(t)$.

(c) Write down the Hamiltonian $H^{\perp}$ for the transverse modes and their canonical conjugates. Quantize the transverse modes and write down the commutation relations for the transverse $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$.

(d) Construct the creation and annihilation operators for the transverse modes and show that

$$\hat{H}^{\perp} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm 1} \omega_k \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \text{const} \tag{16}$$

with $\omega_k = k \equiv |\mathbf{k}|$. The massless particles created by the $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$ operators and annihilated by the $\hat{a}_{\mathbf{k},\lambda}$ are called *photons*.

As to the longitudinal and scalar modes $A_{\mathbf{k},0}$ and $A^0_{\mathbf{k}}$, they make for rather redundant dynamical variables in light of gauge transforms (15). The Hamiltonian formalism abhors such redundancy, so we need to fix a gauge. Let us therefore impose the so-called *transverse* gauge condition $A_{\mathbf{k},0} \equiv 0$ — or equivalently $\nabla \cdot \mathbf{A}(\mathbf{x}, t) = 0$.

(e) Show that in the transverse gauge, the scalar modes satisfy time-independent equations of motion $\mathbf{k}^2 A^0_{\mathbf{k}} = 0$.
Also show that for EM fields coupled to an electric current $J^{\mu}$, the scalar modes satisfy $\mathbf{k}^2 A^0_{\mathbf{k}}(t) = -J^0_{\mathbf{k}}(t)$, or in the field language, the scalar potential $A^0(\mathbf{x}, t)$ is the instantaneous Coulomb potential for the charge density $J^0(\mathbf{x}, t)$.
For that reason, the transverse gauge is also known as the Coulomb gauge.

Utimately, for the free EM fields, the longitudinal sector contains no dynamics and thus cab be simply thrown out of the quantum theory. Consequently, the free quantum EM theory is the theory of photons with transverse polarizations only, $\lambda = \pm 1$.

(f) Finally, assemble the time-dependend free quantum EM fields $\hat{A}^{\mu}(x)$ as

$$\hat{A}^{\mu}(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{\sqrt{2|\mathbf{k}|}} \sum_{\lambda=\pm 1} \Big( e^{-ikx} f^{\mu}(\mathbf{k},\lambda)\, \hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx} f^{*\mu}(\mathbf{k},\lambda)\, \hat{a}^{\dagger}_{\mathbf{k},\lambda}(0) \Big)_{k^0=+|\mathbf{k}|} \tag{17}$$

where *in the transverse gauge* $f^0(\mathbf{k},\lambda) \equiv 0$ and $\mathbf{f}(\mathbf{k},\lambda) = \mathbf{e}_{\lambda}(\mathbf{k})$.