1. First, a little exercise about Weyl spinors. As discussed in class, a Dirac spinor field $\Psi(x)$ is physically equivalent to two left-handed Weyl spinor fields $\chi(x)$ and $\varphi(x)$. In Weyl basis,

$$\Psi(x) \;\equiv\; \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \;=\; \begin{pmatrix} \chi(x) \\ -\sigma^2\varphi^*(x) \end{pmatrix}. \tag{1}$$

   (a) Show that in terms of the Weyl spinor fields, the Dirac Lagrangian becomes

$$\mathcal{L} \;\equiv\; \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \;=\; i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \;+\; i\varphi^\dagger\bar{\sigma}^\mu\partial_\mu\varphi \;+\; m\left(\varphi^\top\sigma^2\chi \,+\, \varphi^\dagger\sigma^2\chi^*\right) \tag{2}$$

   (up to a total derivative). Note that $\chi(x)$, $\varphi(x)$, $\chi^*(x)$ and $\varphi^*(x)$ are *fermionic* fields, so in the classical limit they *anticommute* with each other rather than commute. Thus, $\varphi^\top\sigma^2\chi = +\chi^\top\sigma^2\varphi$ even though the σ^2 matrix is antisymmetric.

   (b) Verify that each term in the Lagrangian (2) is separately invariant (up to a total derivative) under the combined Parity + Charge Conjugation symmetry $\widehat{\mathcal{CP}}$.

2. The rest of this homework concerns Dirac spinor fields and their discrete symmetries. We begin with the charge-conjugation properties of Dirac bilinears $\hat{\overline{\Psi}}\Gamma\hat{\Psi}$.

   (a) Show that $\hat{\mathcal{C}}\hat{\overline{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{C}} = \hat{\overline{\Psi}}\Gamma^c\hat{\Psi}$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.
   Hint: Mind the anticommutativity of the fermionic fields.

   (b) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are $\mathcal{C}$–even and which are $\mathcal{C}$–odd.

   (c) Consider a bound state of a fermion and an antifermion, *e.g.* a positronium state or a neutral meson. As argued in class, the parity of such bound state is $P = -(-1)^L$. Show that the C-parity of this state is $C = (-1)^S(-1)^L$ and use this fact to explain why the decay mode and the lifetime of a 1S positronium state depend on its spin.
   Hint: $\hat{a}^\dagger\hat{b}^\dagger = -\hat{b}^\dagger\hat{a}^\dagger$.

3. The time-reversal symmetry involves an *anti-linear*, *anti-unitary* operator $\hat{\mathcal{T}}$ that invert directions of all particle momenta and spins,[⋆]

$$\hat{\mathcal{T}}\,|\text{particle type}, \mathbf{p}, s\rangle \;=\; (\text{phase})\;|\text{same particle type}, -\mathbf{p}, -s\rangle\,. \tag{3}$$

The phase factor here here combines an arbitrary but fixed overall phase η with a phase factor inherent in spin reversal: $\widehat{\mathcal{SR}}\,|m_s\rangle \;=\; i^{2m_s}\,|-m_s\rangle$ in the $\hat{S}_z$ eigenbasis, or more generally,

$$\widehat{\mathcal{SR}}\,|\xi\rangle \;=\; i^{2S}e^{-\pi i\hat{S}_y}\,|\xi\rangle^*\,. \tag{4}$$

In particular, for spin $\frac{1}{2}$ particles, $\widehat{\mathcal{SR}}\,|\xi\rangle = \sigma_2\,|\xi\rangle^*$, in agreement with the rule $\xi_{-s} = \sigma_2\xi^*_s$ we used in class. Note however that reversing the spin twice results in a rotation by 2π, $\widehat{\mathcal{SR}}^2 = e^{-2\pi i\hat{S}_y} = (-1)^{2S}$, which is trivial for integral spins but changes the overall sign of the spin state for half-integral spins. Consequently, the time-reversal operator in the Fock space satisfies

$$\hat{\mathcal{T}}^2 \;=\; (-1)^F \;\equiv\; \text{rotation by } 2\pi. \tag{5}$$

In this exercise, we consider time-reversal of the Dirac spinor field. First, we need a lemma:

$$i^{2m_s}u^*(-\mathbf{p},-m_s) \;=\; -i\gamma^1\gamma^3u(+\mathbf{p},+m_s), \qquad i^{2m_s}v^*(-\mathbf{p},-m_s) \;=\; -i\gamma^1\gamma^3v(+\mathbf{p},+m_s). \tag{6}$$

(a) Prove this lemma.

In terms of electronic and positronic creation and annihilation operators, eq. (3) means

$$\begin{aligned}
\hat{\mathcal{T}}\hat{a}^\dagger(\mathbf{p},m_s)\hat{\mathcal{T}}^{-1} &= (\mp i)\,i^{2m_s}\,\hat{a}^\dagger(-\mathbf{p},-m_s),\\
\hat{\mathcal{T}}\hat{b}^\dagger(\mathbf{p},m_s)\hat{\mathcal{T}}^{-1} &= (\pm i)\,i^{2m_s}\,\hat{b}^\dagger(-\mathbf{p},-m_s),\\
\hat{\mathcal{T}}\hat{a}(\mathbf{p},m_s)\hat{\mathcal{T}}^{-1} &= (\pm i)\,i^{2m_s}\,\hat{a}(-\mathbf{p},-m_s),\\
\hat{\mathcal{T}}\hat{b}(\mathbf{p},m_s)\hat{\mathcal{T}}^{-1} &= (\mp i)\,i^{2m_s}\,\hat{b}(-\mathbf{p},-m_s),
\end{aligned} \tag{7}$$

where the spin-independent phase factors $\pm i$ make for a consistent time-reversal of the Dirac spinor field.

⋆ Please see J. J. Sakurai *Modern Quantum Mechanics*, §4.4 for a discussion of time reversal in general and spin reversal in particular.

(b) Use lemma (6) and eqs. (7) to show that

$$\hat{\mathcal{T}}\hat{\Psi}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} \;=\; \pm\gamma^1\gamma^3\,\Psi(\mathbf{x},-t). \tag{8}$$

(c) Next, consider the Dirac bilinears $\hat{\overline{\Psi}}\Gamma\hat{\Psi}$ and show that $\hat{\mathcal{T}}\hat{\overline{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{T}}^{-1} = \hat{\overline{\Psi}}\Gamma^t\hat{\Psi}$ where $\Gamma^t = \gamma^3\gamma^1\Gamma^*\gamma^1\gamma^3$.

(d) Calculate Γ^t for all 16 independent matrices Γ and find out which Dirac bilinears are $\mathcal{T}$–even and which are $\mathcal{T}$–odd.

(e) Verify the $\mathcal{T}$–invariance of the Dirac action.

4. Finally, consider the combined $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ symmetry of the Dirac field and verify that for any bilinear operator $\hat{\mathcal{O}}(x) = \hat{\overline{\Psi}}(x)\Gamma\hat{\Psi}(x)$,

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\,\hat{\mathcal{O}}(x)\,[\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}]^{-1} \;=\; \hat{\mathcal{O}}^\dagger(-x)\times(-1)^{\#\text{Lorentz indices in }\hat{\mathcal{O}}}. \tag{9}$$

Actually, eq. (9) holds for any physically measurable operator $\hat{\mathcal{O}}(x)$ in any legitimate quantum field theory — this is the famous CPT theorem — but the exercise is limited to the Dirac bilinear operators only.