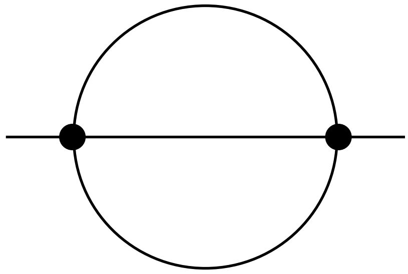
1. Consider the field-strength renormalization in the $\lambda\Phi^4$ theory. As discussed in class, the $\frac{1}{2}\delta_Z(\partial_\mu\Phi)^2$ counterterm does not show up at the one-loop level of the renormalized perturbation theory because the one-loop diagram yields a momentum-independent $\Sigma_{1\,\mathrm{loop}}$ amplitude. At the two-loop level however, there is a momentum-dependent 1PI diagram, namely



and hence the second-order $\delta_Z^{(2)} \neq 0$.

Your task is to evaluate this two-loop diagram and to calculate the counterterm $\delta_Z^{(2)}$.

This is a difficult calculation, so proceed with care. I suggest you first combine the three propagators using Feynman parameters and shift the two independent loop momenta to rewrite the loop integral as

$$\iiint dx\,dy\,dz\,\delta(x+y+z-1)\iint \frac{d^4\ell_1}{(2\pi)^4}\,\frac{d^4\ell_2}{(2\pi)^4}\,\frac{2}{\left[\alpha\ell_1^2+\beta\ell_2^2+\gamma p^2-m^2+i0\right]^3} \tag{1}$$

for some (x,y,z)–dependent coefficients α,β,γ. Second, Wick-rotate ℓ_1 and ℓ_2 into the Euclidean space, dimensionally regularize to $d<4$ and evaluate the momentum integrals. Third, for the purpose of calculating the δ_Z counterterm, take the derivative $d\Sigma/dp^2$ *before* integrating over Feynman parameters or taking the $d\to 4$ limit. Fourth, do take the $d\to 4$ limit and write the counterterm as

$$\delta_Z^{(2)} = \iiint dxdydz\,\delta(x+y+z-1)\,F(x,y,z)\times\left\{\frac{1}{\epsilon}+\mathrm{const}+\log G(x,y,z)\right\} \tag{2}$$

for some rational functions F and G of the Feynman parameters.

Finally, the simplest way of evaluating the integral (2) over the Feynman parameters is to use Mathematica. To help it along, replace the (x, y, z) variables to (w, ξ) according to $x = \xi w$, $y = (1-\xi)w$, $z = 1-w$, then integrate over the w variable first and over the ξ second. Here is a couple of integrals I did this way you might find useful:

$$\begin{aligned}
\iiint dxdydz\,\delta(x+y+z-1)\times\frac{xyz}{(xy+xz+yz)^3} &= \frac{1}{2}\,,\\
\iiint dxdydz\,\delta(x+y+z-1)\times\frac{xyz}{(xy+xz+yz)^3}\log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} &= -\frac{3}{4}\,.
\end{aligned} \tag{3}$$

2. Now consider the renormalization in the Yukawa theory. Specifically, solve **problem 10.2** of the *Peskin & Schroeder* textbook.