1. Consider a generic simple non-abelian Lie group G and its generators T^a. The (quadratic) Casimir operator $C_2 = \sum_a T^a T^a$ commutes with all the generators and hence for any irreducible representation (r) of the group, C_2 restricted to (r) is simply a unit matrix times a number $C(r)$. In other words, if $T^a_{(r)}$ is a matrix of the generator T^a in the representation (r), then $\sum_a T^a_{(r)} T^a_{(r)} = C(r) \times \mathbf{1}$. For example, for the isospin group $SU(2)$, the irreps are characterized by the isospin I and $C(I) = I(I+1)$.

   (a) By symmetry, for any complete representation (r) of the group,

$$\mathrm{tr}_{(r)}(T^a T^b) \;\equiv\; \mathrm{tr}\left(T^a_{(r)} T^b_{(r)}\right) \;=\; R(r)\delta^{ab} \tag{1}$$

   for some coefficient $R(r)$. Show that for any irreducible representation,

$$\frac{R(r)}{C(r)} \;=\; \frac{\dim(r)}{\dim(G)}\,. \tag{2}$$

   In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I+1)(2I+1)$.

   (b) Suppose the first three generators of G generate an $SU(2)$ subgroup. Show that if a representation (r) of G decomposes into several $SU(2)$ multiplets of isospins $I_1, I_2, \ldots, I_n$, then

$$R(r) \;= \sum_{i=1}^{n} \tfrac{1}{3} I_i(I_i+1)(2I_i+1). \tag{3}$$

   (c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting on the first two components of a complex N-vector. The fundamental representation (N) of the $SU(N)$ decomposes into one doublet and $(N-2)$ singlets of the $SU(2)$ subgroup, hence

$$R(N) \;=\; \tfrac{1}{2} \qquad \text{and} \qquad C(N) \;=\; \frac{N^2-1}{2N}\,. \tag{4}$$

   The *adjoint* representation of the $SU(N)$ group consists of traceless hermitian $N \times N$

matrices Φ_j^k which transform according to

$$\Phi' \;=\; U\Phi U^\dagger \qquad i.e.,\quad \Phi'^k_j \;=\; U_j^i \Phi_i^\ell U_\ell^{*k}\,, \tag{5}$$

thus $(\mathrm{adj}) + (1) = (N) \times (\overline{N})$ (where the singlet (1) corresponds to the missing trace part of Φ_j^k).

Show that the adjoint representation of the $SU(N)$ decomposes into one $SU(2)$ triplet, $2(N-2)$ doublets and $(N-2)^2$ singlets and hence

$$R(\mathrm{adj}) \;=\; C(\mathrm{adj}) \;\equiv\; C(G) \;=\; N. \tag{6}$$

(d) The symmetric and the anti-symmetric 2–index tensors from irreducible representations of the $SU(N)$ group. Find out the decomposition of these irreps under an $SU(2) \subset SU(N)$ and calculate their respective R factors.

2. For a field $\Phi^a(x)$ in the adjoint representation of an $SU(N)$ gauge symmetry, the covariant derivative

$$D_\mu \Phi^a(x) \;=\; \partial_\mu \Phi^a(x) \;-\; iA_\mu^c(x)[T^c_{\mathrm{adj}}]^{ab}\Phi^b(x) \tag{7}$$

can be written in matrix notations as

$$D_\mu \Phi(x) \;=\; \partial_\mu \Phi(x) \;-\; i[A_\mu(x), \Phi(x)]. \tag{8}$$

(a) Verify the covariance of this derivative.

(b) Show that $[\mathcal{D}_\mu, \mathcal{D}_\nu]\Phi = -i[F_{\mu\nu}, \Phi]$.

The non-abelian gauge field tension $F^a_{\mu\nu}(x)$ itself transforms in the adjoint representation of the gauge group, hence $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} - i[A_\lambda, F_{\mu\nu}]$, *etc., etc.*

(c) Prove the non-abelian Bianchi identity

$$D_\lambda F_{\mu\nu} \;+\; D_\mu F_{\nu\lambda} \;+\; D_\nu F_{\lambda\mu} \;=\; 0. \tag{9}$$

Now consider the classical Yang–Mills theory of the non-abelian gauge field $A^\mu(x)$ governed by the Lagrangian

$$\mathcal{L}_{\rm YM} \;=\; \frac{-1}{2g^2}\,\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big). \tag{10}$$

(d) Show that $\delta F^{\mu\nu}(x) = D^\mu \delta A^\nu(x) - D^\nu \delta A^\mu(x)$ and use this observation to write the classical YM field equations of motion as $D^\mu F_{\mu\nu} = 0$.

Next, let us add the fermionic fields and have

$$\mathcal{L} \;=\; \frac{-1}{2g^2}\,\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) \;+\; \overline{\Psi}(i\not{D} - m)\Psi. \tag{11}$$

(e) Show that in this case the classical YM field equations of motion become

$$D^\mu F_{\mu\nu} \;=\; -g^2 J_\nu \tag{12}$$

where

$$J^a_\nu \;=\; \overline{\Psi}\gamma_\nu \frac{\lambda^a}{2}\Psi. \tag{13}$$

Together, eqs. (9) and (12) serve as the non-abelian analogue of the Maxwell equations.

(f) Show that eq. (12) requires the fermionic current J_ν to be *covariantly conserved*, $D^\nu J_\nu = 0$.

(g) Use Dirac equations for the fermionic fields to verify that the current (13) is indeed covariantly conserved.

(h) Finally, consider the *second* variation of the Yang-Mills action expanded around some non-trivial solution $A^\mu(x)$ of the YM equations and show that

$$\delta_2\left[\int d^4x \mathcal{L}_{\rm YM}\right] \;=\; \frac{1}{g^2}\int d^4x\left[\mathrm{tr}\left(\delta A^\mu D^2 \delta A_\mu\right) \;+\; \mathrm{tr}\left((D_\mu \delta A^\mu)^2\right) \;+\; 4i\,\mathrm{tr}\left(F_{\mu\nu}\delta A^\mu \delta A^\nu\right)\right].$$