1. Consider the quark-antiquark pair production in *Quantum ChromoDynamics* (QCD). Specifically, let us focus on the $u\bar{u} \to d\bar{d}$ process so there is only one tree-level diagram contributing to this process. Draw this diagram and calculate the amplitude, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles and calculate the total cross section. For simplicity, you may neglect the quark masses.

   Note that the $u\bar{u} \to d\bar{d}$ pair production in QCD is very similar to the $e^-e^+ \to \mu^-\mu^+$ pair production in QED, so the only new aspect of this problem is summing over the colors.

2. Now consider a scalar analogue of QCD or more generally a theory of Yang–Mills fields A^a_μ and complex scalars Φ_i in some representation (r) of the Gauge group G.

   (a) Write down the Lagrangian and the Feynman rules of this theory.

   Next, consider the annihilation process $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

   (b) Draw the diagrams and write down the tree-level annihilation amplitude.

   As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving* **a** *longitudinally polarized gauge boson vanish, provided all other gauge bosons are transversely polarized.* In other words,

$$\begin{aligned} \mathcal{M} \;&\equiv\; e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n}\, \mathcal{M}_{\mu_1\mu_2\cdots\mu_n}(\text{momenta}) \;=\; 0 \\ \text{when}\quad e_1^\mu \propto k_1^\mu \quad &\text{but}\quad e_2^\nu k_{2\nu} \;=\; \cdots \;=\; e_n^\nu k_{n\nu} \;=\; 0. \end{aligned}$$

   (c) Verify this identity for the scalar annihilation amplitude.

3. Finally, let us calculate the annihilation cross-section in Scalar QCD. Generally, the tree-level annihilation amplitude (*cf.* the previous problem) can be written in the form

$$\mathcal{M} \;=\; F\{T^a, T^b\}^i{}_j \;+\; iG[T^a, T^b]^i{}_j \tag{1}$$

   where j is the 'color' index of the scalar particle belonging to some representation (r) of the gauge group G, i is the color index of the scalar anti-particle belonging to the conjugate representation ($\bar{r}$), and a and b are the color indices of the gauge bosons belonging to the adjoint representation.

(a) Show that the annihilation amplitude indeed has form (1) and write down the coefficients F and G as explicit functions of the particles momenta and polarizations.

(b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalar colors. Show that

$$\frac{1}{\dim^2(r)}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)}\times\left(4C(r)\,|F|^2\,+\,C(\text{adj})\,(|G|^2-|F|^2)\right). \qquad (2)$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{N^2-1}{2N^2}\left(\frac{N^2-2}{N}\,|F|^2\,+\,N|G|^2\right). \qquad (3)$$

(c) Evaluate F and G in the center of mass frame. Take the gauge boson's polarization vectors $e^{\mu}_{1,2}$ to be purely spatial and transverse (to the vector particle's momenta).

(d) Finally, calculate the (polarized, partial) cross-section for the annihilation process.