1. Consider the quark-antiquark pair production in *Quantum ChromoDynamics* (QCD). Specifically, let us focus on the  $u\bar{u} \rightarrow d\bar{d}$  process so there is only one tree-level diagram contributing to this process. Draw this diagram and calculate the amplitude, then sum/average the  $|\mathcal{M}|^2$  over both spins and *colors* of the final/initial particles and calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the  $u\bar{u} \to d\bar{d}$  pair production in QCD is very similar to the  $e^-e^+ \to \mu^-\mu^+$  pair production in QED, so the only new aspect of this problem is summing over the colors.

- 2. Now consider a scalar analogue of QCD or more generally a theory of Yang–Mills fields  $A^a_{\mu}$  and complex scalars  $\Phi_i$  in some representation (r) of the Gauge group G.
  - (a) Write down the Lagrangian and the Feynman rules of this theory.

Next, consider the annihilation process  $\Phi + \Phi^* \rightarrow 2$  gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

(b) Draw the diagrams and write down the tree-level annihilation amplitude.

As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: On-shell Amplitudes involving **a** longitudinally polarized gauge boson vanish, provided all other gauge bosons are transversely polarized. In other words,

$$\mathcal{M} \equiv e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n} \mathcal{M}_{\mu_1 \mu_2 \cdots \mu_n} (\text{momenta}) = 0$$
  
when  $e_1^{\mu} \propto k_1^{\mu}$  but  $e_2^{\nu} k_{2\nu} = \cdots = e_n^{\nu} k_{n\nu} = 0.$ 

- (c) Verify this identity for the scalar annihilation amplitude.
- 3. Finally, let us calculate the annihilation cross-section in Scalar QCD. Generally, the treelevel annihilation amplitude (*cf.* the previous problem) can be written in the form

$$\mathcal{M} = F\{T^{a}, T^{b}\}_{j}^{i} + iG[T^{a}, T^{b}]_{j}^{i}$$
(1)

where j is the 'color' index of the scalar particle belonging to some representation (r) of the gauge group G, i is the color index of the scalar anti-particle belonging to the conjugate

representation  $(\bar{r})$ , and a and b are the color indices of the gauge bosons belonging to the adjoint representation.

- (a) Show that the annihilation amplitude indeed has form (1) and write down the coefficients F and G as explicit functions of the particles momenta and polarizations.
- (b) Next, let us sum the  $|\mathcal{M}|^2$  over the gauge boson's colors and average over the scalar colors. Show that

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\dim(r)} \times \left(4C(r) |F|^2 + C(\mathrm{adj}) \left(|G|^2 - |F|^2\right)\right).$$
(2)

In particular, for scalars in the fundamental representation of the SU(N) gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{N^2 - 1}{2N^2} \left( \frac{N^2 - 2}{N} |F|^2 + N|G|^2 \right).$$
(3)

- (c) Evaluate F and G in the center of mass frame. Take the gauge boson's polarization vectors  $e_{1,2}^{\mu}$  to be purely spatial and transverse (to the vector particle's momenta).
- (d) Finally, calculate the (polarized, partial) cross-section for the annihilation process.