Note on QED Vertex Correction

As discussed in class, the one-loop correction to the tree-level QED vertex $\bar{u}(p')\gamma^\mu u(p)$ is given by the following monster of an integral:

$$\bar{u}(p')\,\delta_1\Gamma^\mu(p,p')\,u(p) \;=\; -2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\bar{u}(p')\,\mathcal{N}^\mu\,u(p)}{\left[\ell^2-\Delta+i0\right]^3} \tag{1}$$

where

$$\ell \;=\; k\,+\,xp\,+\,yp', \tag{2}$$

$$\Delta \;=\; (xp+yp')^2 \;=\; (1-z)^2m^2\,-\,xyq^2, \tag{3}$$

$$\mathcal{N}^\mu \;=\; \gamma^\nu(\not{k}+\not{p}'+m)\gamma^\mu(\not{k}+\not{p}+m)\gamma_\nu\,. \tag{4}$$

In this note I simplify the numerator (4) *in the context of the integral* (1).

The first step is obvious: Let us get rid of the γ^ν and γ_ν factors using $\gamma^\nu\not{a}\gamma_\nu=-2\not{a}$, $\gamma^\nu\not{a}\not{b}\gamma_\nu=4(ab)$ and $\gamma^\nu\not{a}\not{b}\not{c}\gamma_\nu=-2\not{c}\not{b}\not{a}$, thus

$$\mathcal{N}^\mu \;=\; -2m^2\gamma^\mu \;+\; 4m(2k+p+p')^\mu \;-\; 2(\not{k}+\not{p})\gamma^\mu(\not{k}+\not{p}'). \tag{5}$$

Next, we should re-express the right hand side here in terms of the Feynman's loop momentum ℓ rather than k using eq. (2); expanding the result in powers of ℓ, we get quadratic, linear and ℓ–independent terms. However, the $\int d^4\ell$ integral in eq. (1) and the denominator are both even with respect to $\ell\to-\ell$, so the numerator terms which are linear in ℓ would not contribute to the integral because of their oddness. Disregarding these terms then gives us

$$\begin{aligned}\mathcal{N}^\mu \;\cong\;& -2m^2\gamma^\mu \;+\; 4m(p+p'-2xp-2yp')^\mu\\ &-\;2\not{\ell}\gamma^\mu\not{\ell}\;-\;2(\not{p}-x\not{p}-y\not{p}')\,\gamma^\mu\,(\not{p}'-x\not{p}-y\not{p}')\\ =\;& -2m^2\gamma^\mu\;+\;4mz\,(p+p')^\mu\;+\;4m(x-y)q^\mu\;-\;2\not{\ell}\gamma^\mu\not{\ell}\\ &-\;2\left[z\not{p}'+(x-1)\not{q}\right]\gamma^\mu\left[z\not{p}+(1-y)\not{q}\right]\end{aligned} \tag{6}$$

where the second equality makes judicious use of $p'-p=q$ and $x+y+z=1$.

At this point, we can use the fact that the vertex correction Γ^μ — and hence the numerator $\mathcal{N}^\mu$ is always sandwiched between $\bar{u}(p')$ and $u(p)$. Consequently, on the last line of eq. (6) we may replace the $\not{p}'$ in the first factor with m since $\bar{u}(p')\,\not{p}' = \bar{u}(p')m$; likewise, in the last factor, we replace the $\not{p}$ with m since $\not{p}u(p) = mu(p)$. Thus, the last line of eq. (6) becomes

$$\begin{aligned}
&-2\left[z\,\not{p}' + (x-1)\,\not{q}\right]\gamma^\mu\left[z\,\not{p} + (1-y)\,\not{q}\right] \\
&\qquad\qquad\cong\ -2\left[mz - (1-x)\,\not{q}\right]\gamma^\mu\left[mz + (1-y)\,\not{q}\right] \\
&\qquad\qquad=\ -2z^2m^2\,\gamma^\mu\ +\ 2(1-x)(1-y)\ \not{q}\gamma^\mu\not{q} \\
&\qquad\qquad\qquad +\ 2z(1-x)m\,\not{q}\gamma^\mu\ -\ 2z(1-y)m\,\gamma^\mu\not{q} \\
&\qquad\qquad\cong\ -2\left[z^2m^2 + (1-x)(1-y)q^2\right]\gamma^\mu \\
&\qquad\qquad\qquad +\ 2z(y-x)mq^\mu\ +\ 2z(x+y-2)m \times i\sigma^{\mu\nu}q_\nu
\end{aligned} \tag{7}$$

where the last equality (or rather equivalence in the context of eq. (1)) follows from

$$\bar{u}(p')\,\not{q}\,u(p)\ =\ 0 \quad\Rightarrow\ \bar{u}(p')\big[\not{q}\gamma^\mu\not{q}\big]u(p)\ =\ -q^2 \times \bar{u}'\gamma^\mu u$$

as well as

$$\gamma^\mu\not{q}\ =\ q^\mu\ +\ i\sigma^{\mu\nu}q_\nu\,, \qquad \not{q}\gamma^\mu\ =\ q^\mu\ -\ i\sigma^{\mu\nu}q_\nu\,.$$

Substituting eq. (7) into eq. (6) and making use of the Gordon identity, we arrive at

$$\begin{aligned}
\mathcal{N}^\mu\ \cong\ & -2\,\not{\ell}\gamma^\mu\not{\ell}\ -\ 2\left[(1-4z+z^2)m^2 + (1-x)(1-y)q^2\right] \times \gamma^\mu \\
& -\ 4z(1-z)m^2 \times \frac{i\sigma^{\mu\nu}q_\nu}{2m}\ +\ 2(2-z)(x-y)\,mq^\mu.
\end{aligned} \tag{8}$$

Note that the last term here is odd under interchange $x \leftrightarrow y$ of the two Feynman parameters. Since everything else in the integral (1) is symmetric with respect to this interchange, the last term in eq. (8) integrates to zero and may be safely disregarded.

Finally, I would like to make use of the Lorentz symmetry of the integral (1) over the loop momentum ℓ. Because of this symmetry, the integral of a quadratic term $\ell^\alpha\ell^\beta$ should be proportional to the Lorentz invariant $g^{\alpha\beta}$, hence in four spacetime dimensions,

$$\ell^\alpha \ell^\beta \;\cong\; \tfrac{1}{4}\ell^2 g^{\alpha\beta} \tag{9}$$

and therefore

$$\not\ell\gamma^\mu\not\ell \;=\; \ell^\alpha\ell^\beta \times \gamma_\alpha\gamma^\mu\gamma_\beta \;\cong\; \tfrac{1}{4}\ell^2 \times \gamma_\alpha\gamma^\mu\gamma^\alpha \;=\; -\tfrac{1}{2}\ell^2\,\gamma^\mu. \tag{10}$$

Applying this equivalence to the first term in eq. (8) we arrive at

$$\mathcal{N}^\mu \;\cong\; [\ell^2 - 2(1-4z+z^2)m^2 - 2(1-x)(1-y)q^2] \times \gamma^\mu \;-\; 4z(1-z)m^2 \times \frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{11}$$

and at this point, I give up — it does not get any simpler than this.