According to the Noether theorem, a translationally invariant system of classical fields ϕ_a has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noeter}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{1}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noeter}} \;+\; \partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}, \tag{2}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

1. Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$,

$$\begin{aligned}\partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\text{Noether}} = \text{(hopefully)} =\; 0\\ P^\mu_{\text{net}} \;&\equiv\; \int\! d^3\mathbf{x}\,T^{0\mu} \;=\; \int\! d^3\mathbf{x}\,T^{0\mu}_{\text{Noether}}\,.\end{aligned} \tag{3}$$

For the scalar fields, real or complex, $T^{\mu\nu}_{\text{Noeter}}$ is properly symmetric and one simply has $T^{\mu\nu} = T^{\mu\nu}_{\text{Noeter}}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu,\partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{4}$$

where A_μ is a real vector field and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

2. Write down $T^{\mu\nu}_{\text{Noeter}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

3. The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\text{EM}} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{5}$$

   Show that this expression indeed has form (2) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

4. Write down the components of the stress-energy tensor (5) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Now consider the electromagnetic fields coupled to a conserved electric current J^μ.

5. Use Maxwell's equations to show that

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda} J_\lambda \,. \tag{6}$$

Eq. (6) suggests that any system of charged "matter" fields which carries the current J^μ should have its stress-energy tensor $T^{\mu\nu}_{\rm mat}$ obeying

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu}{}_{\lambda} J^{\lambda}_{\rm EM} \,. \tag{7}$$

Consequently, the combined stress-energy tensor $T^{\mu\nu}_{\rm EM} + T^{\mu\nu}_{\rm mat}$ should be divergence-less and thus lead to a conserved total energy and momentum.

Generally, testing eq. (7) for any particular system of charged "matter" fields makes use of fields' equations of 'motion' and also of the fact that *the covariant derivatives D_μ do not commute with each other*. Instead, when acting upon a field Φ_q of charge q, one has

$$(D_\mu D_\nu - D_\nu D_\mu)\Phi_q \;=\; iq\, F_{\mu\nu}\, \Phi_q \tag{8}$$

(in $c = \hbar = 1$ units).

6. Verify eq. (8).

Consider a specific example of the EM coupled to a complex scalar field of electric charge $q \neq 0$. The net Lagrangian density of the EM + scalar fields is

$$\mathcal{L} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \,. \tag{9}$$

7. Verify the gauge invariance of the electric current

$$J^\mu \;=\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{10}$$

by expressing it in terms of the scalar fields and their *covariant* derivatives $D^\mu\Phi$ and $D^\mu\Phi^*$.

8. Write down equations of motion for the scalar fields Φ and Φ^* and use them to verify the current conservation $\partial_\mu J^\mu = 0$.

9. Write down the Noether stress-energy tensor for this field system and show that

$$T^{\mu\nu}_{\text{net}} \equiv T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{mat}} = T^{\mu\nu}_{\text{Noeter}} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \tag{11}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is the same function of EM fields as in the free EM case [**3.**], $T^{\mu\nu}_{\text{EM}}$ is exactly as in eq. (5), and

$$T^{\mu\nu}_{\text{mat}} = D^\mu\Phi^* D^\nu\Phi + D^\nu\Phi^* D^\mu\Phi - g^{\mu\nu}\big(D_\lambda\Phi^* D^\lambda\Phi - m^2\Phi^*\Phi\big) \tag{12}$$

Hint: In the presence of an electric current J^μ, the $\partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (12) for the scalar field.

10. Use the scalar field's equations of motion and eq. (8) to verify eq. (7).