1. The first exercise is about first-quantized *v.* second-quantized forms of one-body and two-body operators acting on identical bosons. In class, we wrote the wave function of an N–particle state $|\alpha_1,\ldots,\alpha_N\rangle = \big|\{n_\beta\}\big\rangle$ as

$$\phi_{\alpha_1,\ldots,\alpha_N}(\mathbf{x}_1,\ldots,\mathbf{x}_N) \;=\; \sqrt{\frac{\prod_\beta n_\beta!}{N!}} \sum_{\substack{\text{distinct permutations}\\ (\tilde\alpha_1,\ldots,\tilde\alpha_N)\text{ of }(\alpha_1,\ldots,\alpha_N)}} \phi_{\tilde\alpha_1}(\mathbf{x}_1)\cdots\phi_{\tilde\alpha_N}(\mathbf{x}_N), \tag{1}$$

and we defined the annihilation operators $\hat a_\alpha$ according to

$$\hat a_\alpha \big|\{n_\beta\}\big\rangle \;=\; \sqrt{n_\alpha}\,\big|\{n'_\beta = n_\beta - \delta_{\alpha\beta}\}\big\rangle\,. \tag{2}$$

(a) Consider an N–particle state $|N,\Psi\rangle$ with a completely generic totally-symmetric wave function $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. Show that the $(N-1)$–particle state $|(N-1),\Psi'\rangle = \hat a_\gamma\,|N,\Psi\rangle$ has wave function

$$\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\mathbf{x}_N\,\phi^*_\gamma(\mathbf{x}_N)\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{3}$$

Hint: First verify this formula for Ψ of the form (1), and then generalize to arbitrary (but totally-symmetric) Ψ by linearity.

Now consider a one-body operator $\hat R_1$. In the first-quantized formalism $\hat R_{\rm tot}$ acts on N–particle states according to

$$\hat R^{(1)}_{\rm tot} \;=\; \sum_{i=1}^N \hat R_1(i\underline{^{\rm th}}\text{ particle}) \tag{4}$$

while in the second-quantized formalism it becomes

$$\hat R^{(2)}_{\rm tot} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat R_1\,|\beta\rangle\;\hat a^\dagger_\alpha \hat a_\beta\,. \tag{5}$$

(b) Use eq. (3) to verify that for any two N–particle states $\langle N,\Psi_1|$ and $|N,\Psi_2\rangle$

$$\langle N,\Psi_1|\,\hat R^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=\; \langle N,\Psi_1|\,\hat R^{(2)}_{\rm tot}\,|N,\Psi_2\rangle\,. \tag{6}$$

Hint: Use $\hat R_1 = \sum_{\alpha,\beta} |\alpha\rangle\,\langle\alpha|\,\hat R_1\,|\beta\rangle\,\langle\beta|$.

Next, consider a two-body operator $\hat{S}_2$ which acts in the first-quantized formalism according to

$$\hat{S}^{(1)}_{\text{tot}} \;=\; \tfrac{1}{2}\sum_{i\neq j} \hat{S}_2(i\underline{\text{th}} \text{ and } j\underline{\text{th}} \text{ particles}) \tag{7}$$

and in the second-quantized formalism according to

$$\hat{S}^{(2)}_{\text{tot}} \;=\; \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\otimes\langle\beta|\,\hat{S}_2\,|\gamma\rangle\otimes|\delta\rangle\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\,. \tag{8}$$

(c) Again, show that for any two N–particle states $\langle N,\Psi_1|$ and $|N,\Psi_2\rangle$

$$\langle N,\Psi_1|\,\hat{S}^{(1)}_{\text{tot}}\,|N,\Psi_2\rangle \;=\; \langle N,\Psi_1|\,\hat{S}^{(2)}_{\text{tot}}\,|N,\Psi_2\rangle\,. \tag{9}$$

(d) Finally, let $\hat{A}_1$ be a one-body operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be the corresponding second-quantized operators defined similar to eqs. (5) and (8).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\underline{\text{st}}}) + \hat{A}_1(2^{\underline{\text{nd}}})\right), \hat{B}_2\right]$ then $\hat{C} = [\hat{A},\hat{B}]$

Hint: First, calculate the commutator $[\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta,\hat{a}^\dagger_\mu\hat{a}_\nu]$.

2. The second problem is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$.

(a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\text{def}}{=} \exp\bigl(\xi\hat{a}^\dagger - \xi^*\hat{a}\bigr)\,|0\rangle$. Show that

$$|\xi\rangle \;=\; e^{-|\xi|^2/2}\,e^{\xi\hat{a}^\dagger}\,|0\rangle \quad\text{and}\quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,. \tag{10}$$

(b) Calculate the uncertainties Δq and Δp for a coherent state $|\xi\rangle$ and verify their minimality: $\Delta q\Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\text{def}}{=} \langle\hat{n}\rangle = |\xi|^2$.

Hint: use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$.

(c) Show that for $\xi(t) = \xi_0 e^{-e\omega t}$ the coherent state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}\,|\xi(t)\rangle = \hat{H}\,|\xi(t)\rangle$.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap $\langle\eta|\xi\rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{11}$$

for any given classical complex field $\Phi(\mathbf{x})$.

(f) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \stackrel{\text{def}}{=} \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{12}$$

(g) Let

$$\hat{H} \;=\; \int d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger\cdot\nabla\hat{\Psi} \;+\; V(\mathbf{x})\hat{\Psi}^\dagger\hat{\Psi}\right)$$

and show that for any classical field configuration $\Phi(\mathbf{x},t)$ that satisfies the classical field equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{13}$$

(h) Finally, show that the quantum overlap $|\langle\Phi_1|\Phi_2\rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.