1. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and π^0, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2) \cong \mathrm{Spin}(4)$ which would be exact if the two lightest quark flavors u and d were exactly massless; in reality, the current quark masses m_u and m_d do not exactly vanish but are small enough to be treated as a perturbation. Exact or approximate, the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken $\mathrm{Spin}(4)/\mathrm{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and π^0.

   QCD is a rather complicated theory, so it is often convenient to describe the physics of the spontaneously broken chiral symmetry in terms of a simpler effective theory such as the linear sigma model. This model has 4 real scalar fields; in terms of the unbroken isospin symmetry, we have an isosinglet $\sigma(x)$ and an isotriplet $\underset{\sim}{\pi}(x)$ comprising $\pi^1(x)$, $\pi^2(x)$ and $\pi^3(x)$ (or equivalently, $\pi^0(x) \equiv \pi^3(x)$ and $\pi^\pm(x) \equiv \left(\pi^1(x) \pm i\pi^2(x)\right)/\sqrt{2}$). The Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu \sigma)^2 \;+\; \tfrac{1}{2}(\partial_\mu \underset{\sim}{\pi})^2 \;-\; \frac{\lambda}{8}\left(\sigma^2 + \underset{\sim}{\pi}^2 - f^2\right)^2 \;+\; \beta\sigma \tag{1}$$

   is invariant under the $SO(4)$ rotations of the four fields, except for the last term which we take to be very small. (In QCD $\beta \sim \frac{m_u + m_d}{2f}\left\langle \overline{\Psi}\Psi \right\rangle$ which is indeed very small because the u and d quarks are very light.)

   In class, we discussed this theory for $\beta = 0$ and showed that it has $SO(4)$ spontaneously broken to $SO(3)$ and hence 3 massless Goldstone bosons. In this exercise, we let $\beta > 0$ but $\beta \ll \lambda f^3$ to show how this leads to massive but light pions.

   (a) Show that the scalar potential of the linear sigma model with $\beta > 0$ has a unique minimum at

$$\langle \underset{\sim}{\pi} \rangle \;=\; 0 \quad \text{and} \quad \langle \sigma \rangle \;=\; f \;+\; \frac{\beta}{\lambda f^2} \;+\; O(\beta^2). \tag{2}$$

(b) Expand the fields around this minimum and show that the pions are light while the σ particle is much heavier. Specifically, $M_\pi^2 \approx (\beta/f)$ while $M_\sigma^2 \approx \lambda f^2$.

2. Our second exercise is about the Bogolyubov transform. Let $\hat{a}_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}^\dagger$ be annihilation and creation operators satisfying the bosonic commutation relations, and let

$$\hat{b}_{\mathbf{p}} \;=\; \cosh(t_{\mathbf{p}})\hat{a}_{\mathbf{p}} \;+\; \sinh(t_{\mathbf{p}})\hat{a}_{-\mathbf{p}}^\dagger\,, \quad \hat{b}_{\mathbf{p}}^\dagger \;=\; \cosh(t_{\mathbf{p}})\hat{a}_{\mathbf{p}}^\dagger \;+\; \sinh(t_{\mathbf{p}})\hat{a}_{-\mathbf{p}} \tag{3}$$

for some arbitrary real parameters $t_{\mathbf{p}} = t_{-\mathbf{p}}$.

(a) Show that the $\hat{b}_{\mathbf{p}}$ and the $\hat{b}_{\mathbf{p}}^\dagger$ satisfy the same bosonic commutation relations as the $\hat{a}_{\mathbf{p}}$ and the $\hat{a}_{\mathbf{p}}^\dagger$.

(b) Consider a Hamiltonian of the form

$$\hat{H} \;=\; \sum_{\mathbf{p}} A_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \;+\; \tfrac{1}{2} \sum_{\mathbf{p}} B_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}} \;+\; \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger \right) \tag{4}$$

where $A_{\mathbf{p}} = A_{-\mathbf{p}}$ and $B_{\mathbf{p}} = B_{-\mathbf{p}}$. Show that as long as $|B_{\mathbf{p}}| < A_{\mathbf{p}}\ \forall\mathbf{p}$, this Hamiltonian can be "diagonalized" by means of a Bogolyubov transform (3). That is, for a suitable choice of the $t_{\mathbf{p}}$ parameters,

$$\hat{H} \;=\; \sum_{\mathbf{p}} \omega_{\mathbf{p}} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \;+\; \text{const} \qquad \text{where } \omega_{\mathbf{p}} \;=\; \sqrt{A_{\mathbf{p}}^2 - B_{\mathbf{p}}^2}\,. \tag{5}$$

In particular, for the liquid helium,

$$A_{\mathbf{p}} \;=\; \frac{\mathbf{p}^2}{2M} + n_0\lambda(\mathbf{p}) \quad \text{and} \quad B_{\mathbf{p}} \;=\; n_0\lambda(\mathbf{p}) \quad \Longrightarrow \quad \omega_{\mathbf{p}} \;=\; \sqrt{\frac{\mathbf{p}^2}{2M}\left(\frac{\mathbf{p}^2}{2M} + n_0\lambda(\mathbf{p})\right)}. \tag{6}$$

3. The rest of this homework is about a charged relativistic scalar field $\Phi(x)$. A conserved charge implies a complex field with a $U(1)$ symmetry $\Phi(x) \mapsto e^{i\theta}\Phi(x)$ which gives rise to

a conserved Noether current

$$J^\mu \;=\; i\Phi^*\partial^\mu\Phi \;-\; i(\partial^\mu\Phi^*)\Phi. \tag{7}$$

For simplicity, let the Φ field be free, thus

$$\mathcal{L} \;=\; \partial^\mu\Phi^*\,\partial_\mu\Phi \;-\; m^2\,\Phi^*\Phi. \tag{8}$$

In the Hamiltonian formalism, we trade the time derivatives $\partial_0\Phi(x)$ and $\partial_0\Phi^*(x)$ for the canonically conjugate fields $\Pi^*(x)$ and $\Pi(x)$. (Note that for complex fields $\Pi(\mathbf{x})$ is canonically conjugate to the $\Phi^*(\mathbf{x})$ while $\Pi^*(\mathbf{x})$ is canonically conjugate to the $\Phi(\mathbf{x})$.) Canonical quantization of this system yields non-hermitian quantum fields $\hat\Phi(x) \neq \hat\Phi^\dagger(x)$ and $\hat\Pi(x) \neq \hat\Pi^\dagger(x)$ and the Hamiltonian operator

$$\hat H \;=\; \int\! d^3\mathbf{x}\,\left(\hat\Pi^\dagger\hat\Pi \,+\, \nabla\hat\Phi^\dagger\cdot\nabla\hat\Phi \,+\, m^2\,\hat\Phi^\dagger\hat\Phi\right). \tag{9}$$

(a) Derive the Hamiltonian (9) and write down the equal-time commutation relations between the quantum fields $\hat\Phi(\mathbf{x})$, $\hat\Phi^\dagger(\mathbf{x})$, $\hat\Pi(\mathbf{x})$ and $\hat\Pi^\dagger(\mathbf{x})$.

The plane-wave modes

$$\hat\Phi_{\mathbf{p}} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{p}\mathbf{x}}\,\hat\Phi(\mathbf{x}), \qquad \hat\Pi_{\mathbf{p}} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{p}\mathbf{x}}\,\hat\Pi(\mathbf{x}), \tag{10}$$

of non-hermitian fields are completely independent of each other, thus $\hat\Phi^\dagger_{\mathbf{p}} \neq \hat\Phi_{-\mathbf{p}}$ and $\hat\Pi^\dagger_{\mathbf{p}} \neq \hat\Pi_{-\mathbf{p}}$. Consequently, we have two independent species of creation and annihilation operators; in the relativistic normalization

$$\begin{aligned} \hat a_{\mathbf{p}} &\stackrel{\text{def}}{=} E_{\mathbf{p}}\hat\Phi_{\mathbf{p}} + i\hat\Pi_{\mathbf{p}}\,, \qquad & \hat a^\dagger_{\mathbf{p}} &\stackrel{\text{def}}{=} E_{\mathbf{p}}\hat\Phi^\dagger_{\mathbf{p}} - i\hat\Pi^\dagger_{\mathbf{p}}\,, \\ \hat b_{\mathbf{p}} &\stackrel{\text{def}}{=} E_{\mathbf{p}}\hat\Phi^\dagger_{-\mathbf{p}} + i\hat\Pi^\dagger_{-\mathbf{p}}\,, \qquad & \hat b^\dagger_{\mathbf{p}} &\stackrel{\text{def}}{=} E_{\mathbf{p}}\hat\Phi_{-\mathbf{p}} - i\hat\Pi_{-\mathbf{p}}\,, \end{aligned} \tag{11}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

(b) Verify the bosonic commutation relations (at equal times) between the annihilation operators $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}$ and the corresponding creation operators $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{b}^\dagger_{\mathbf{p}}$.

(c) Show that the Hamiltonian of the free charged fields is

$$\hat{H} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\left(E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;+\; E_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right) \;+\; \text{const.} \tag{12}$$

Next, consider the charge operator $\hat{Q} = \int d^3\mathbf{x}\,\hat{J}_0(\mathbf{x})$.

(d) Show that for the system at hand

$$\hat{Q} \;=\; \int\!d^3\mathbf{x}\,\left(\tfrac{i}{2}\{\hat{\Pi}^\dagger,\hat{\Phi}\} - \tfrac{i}{2}\{\hat{\Pi},\hat{\Phi}^\dagger\}\right) \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right). \tag{13}$$

Actually, the classical formula (7) for the current $J_\mu(x)$ determines eq. (13) only up to ordering of the non-commuting operators $\hat{\Pi}(\mathbf{x})$ and $\hat{\Phi}^\dagger(\mathbf{x})$ (and likewise of the $\hat{\Pi}^\dagger(\mathbf{x})$ and $\hat{\Phi}(\mathbf{x})$). The anti-commutators in eq. (13) provide a solution to this ordering ambiguity, but any other ordering would be just as legitimate. The net effect of changing operator ordering in $\hat{J}_0$ amounts to changing the total charge $\hat{Q}$ by an infinite constant (prove this!). The specific ordering in eq. (13) provides for the neutrality of the vacuum state.

Now consider the stress-energy tensor of the charged field. Classically, Noether theorem gives

$$T^{\mu\nu} \;=\; \partial^\mu\Phi^*\,\partial^\nu\Phi \;+\; \partial^\mu\Phi\,\partial^\nu\Phi^* \;-\; g^{\mu\nu}\mathcal{L}. \tag{14}$$

Quantization of this formula is straightforward (modulo ordering ambiguity); for example, $\hat{\mathcal{H}} \equiv \hat{T}^{00}$ is precisely the integrand on the right hand side of eq. (9).

(e) Show that the total mechanical momentum operator of the fields is

$$\hat{\mathbf{P}}_{\text{mech}} \;\stackrel{\text{def}}{=}\; \int\!d^3\mathbf{x}\,\hat{T}^{0,\mathbf{i}} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\mathbf{p}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;+\; \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right) \tag{15}$$

Physically, eqs. (15), (12) and (13) show that a complex field $\Phi(x)$ describes both a particle and its antiparticle; they have exactly the same rest mass m but exactly opposite charges ± 1.

4. Finally, consider the time-dependence of the free charged fields $\hat{\Phi}(\mathbf{x}, t)$ and $\hat{\Phi}^\dagger(\mathbf{x}, t)$.

   (a) Compare the Schrödinger and the Heisenberg pictures of the creation and annihilation operators (11) and show that for the free Hamiltonian (12),

$$\hat{a}^H_{\mathbf{p}}(t) = e^{-itE_{\mathbf{p}}}\hat{a}^S_{\mathbf{p}}\,, \quad \hat{b}^H_{\mathbf{p}}(t) = e^{-itE_{\mathbf{p}}}\hat{b}^S_{\mathbf{p}}\,, \quad \hat{a}^{\dagger H}_{\mathbf{p}}(t) = e^{+itE_{\mathbf{p}}}\hat{a}^{\dagger S}_{\mathbf{p}}\,, \quad \hat{b}^{\dagger H}_{\mathbf{p}}(t) = e^{+itE_{\mathbf{p}}}\hat{b}^{\dagger S}_{\mathbf{p}}\,. \tag{16}$$

   (b) Now assemble the Heisenberg-picture quantum fields in terms of the operators (16) and show that in relativistic notations

$$\begin{aligned} \hat{\Phi}^H(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_{\mathbf{p}}} \left(e^{-ipx}\,\hat{a}^S_{\mathbf{p}} \;+\; e^{+ipx}\,\hat{b}^{\dagger S}_{\mathbf{p}} \right)_{p^0=E_{\mathbf{p}}}\,, \\ \hat{\Phi}^{\dagger H}(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_{\mathbf{p}}} \left(e^{-ipx}\,\hat{b}^S_{\mathbf{p}} \;+\; e^{+ipx}\,\hat{a}^{\dagger S}_{\mathbf{p}} \right)_{p^0=E_{\mathbf{p}}}\,, \end{aligned} \tag{17}$$

   where $px \stackrel{\text{def}}{=} p^\mu x_\mu = E_{\mathbf{p}}t - \mathbf{p}\mathbf{x}$.

Eqs. (17) allow us to derive the commutation relations between the quantum fields at un-equal times.

   (c) Show that $[\hat{\Phi}^H(x), \hat{\Phi}^H(x')] = 0$ and $[\hat{\Phi}^{\dagger H}(x), \hat{\Phi}^{\dagger H}(x')] = 0$ even for un-equal times $x_0 \neq x'_0$.

   (d) Show that for un-equal times $x_0 \neq x'_0$,

$$\left[\hat{\Phi}^H(x), \hat{\Phi}^{\dagger H}(x')\right] \;=\; D(x - x') \;-\; D(x' - x) \tag{18}$$

   where

$$D(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_{\mathbf{p}}}\; e^{-ipx}\Big|_{p^0=E_{\mathbf{p}}}\,. \tag{19}$$

In class, we shall see that $D(x - x')$ is invariant under orthochronous Lorentz transformations. Consequently, for space-like $(x - x')$, $D(x - x') = D(x' - x)$ and therefore $[\hat{\Phi}^H(x), \hat{\Phi}^{\dagger H}(x')] = 0$.

In general, for any consistent quantum field theory, for any measurable local operators $\hat{O}_1(x_1)$ and $\hat{O}_2(x_2)$ constructed from the quantum fields and their space or time derivatives at respective points x_1 and x_2, the commutator $[\hat{O}_1(x_1), \hat{O}_2(x_2)]$ must vanish for any space-like interval $(x_1 - x_2)^2 < 0$.