1. Consider the matrix $\gamma^5 \stackrel{\rm def}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

   (a) Show that γ^5 anticommutes with each of the γ^μ matrices, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

   (b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

   (c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.
   (Sign convention: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$.)

   (d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

   (e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ and γ^5.

   Under continuous Lorentz symmetries, a Dirac spinor field $\Psi(x)$ transforms according to $\Psi'(x') = M(L)\Psi(x = L^{-1}x')$ where $M\big(L = e^\theta\big) = \exp\big(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}\big)$. Consider the transformation rules for the independent bilinear products $\overline{\Psi}\Gamma\Psi$ of a Dirac field and its conjugate $\overline{\Psi}(x)$, namely (*cf.* (e))

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi \quad \text{and} \quad P = \overline{\Psi}\gamma^5\Psi. \tag{1}$$

   (f) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors and the $T^{\mu\nu}$ as an antisymmetric tensor.

2. Under the *parity* symmetry $\mathcal{P} : (\mathbf{x}, t) \mapsto (-\mathbf{x}, t)$, Dirac spinor fields transform according to

$$\hat{\mathcal{P}}\,\hat{\Psi}(\mathbf{x},t)\,\hat{\mathcal{P}} \;\equiv\; \hat{\Psi}'(\mathbf{x},t) \;=\; \pm\gamma^0\,\hat{\Psi}(-\mathbf{x},t) \tag{2}$$

   where the overall sign depends on the so-called *intrinsic parity* of a particular Dirac field. Note: $\hat{\mathcal{P}}$ here is a unitary operator in the fermionic Fock space; by nature of the parity symmetry, $\hat{\mathcal{P}}^2 = 1$.

   (a) Verify the covariance of the Dirac equation under this symmetry.

(b) Find the transformation rules of the bilinears (1) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

3. Next, consider the charge-conjugation properties of Dirac bilinears $\overline{\Psi}\Gamma\Psi$. To avoid operator ordering problems, take $\Psi(x)$ and $\Psi^\dagger(x)$ to be "classical" fermionic fields which *anticommute* with each other, $\Psi_\alpha \Psi^\dagger\beta = -\Psi^\dagger\beta\Psi_\alpha$.

   (a) Show that $\hat{\mathcal{C}}\hat{\overline{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{C}} = \hat{\overline{\Psi}}\Gamma^c\hat{\Psi}$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

   (b) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are $\mathcal{C}$–even and which are $\mathcal{C}$–odd.

4. In theories involving both bosons and fermions, one often has to combine commutation and anti-commutation relations of various operators, depending on the overall statistics of the operators involved. For that purpose, it is useful to define a 'mixed' commutator bracket

$$[\hat{A}, \hat{B}\} \stackrel{\text{def}}{=} \hat{A}\hat{B} - (-1)^{AB}\hat{B}\hat{A} \tag{3}$$

   where $(-1)^{AB}$ is -1 if both $\hat{A}$ and $\hat{B}$ have overall Fermi statistics (*i.e.*, each comprises an odd number of *fermionic* creation/annihilation operators — the number of bosonic creation/annihilation operators does not matter) and $+1$ in all other cases.

   (a) Verify the Leibniz rule for the mixed brackets: $[\hat{A}, \hat{B}\hat{C}\} = [\hat{A}, \hat{B}\}\hat{C} + (-1)^{AB}\hat{B}[\hat{A}, \hat{C}\}$ and write down a similar rule for the $[\hat{A}\hat{B}, \hat{C}\}$.

   (b) Similarly, express $[\hat{A}\hat{B}, \hat{C}\hat{D}\}$ in terms of appropriate mixed brackets of $\hat{A}$ or $\hat{B}$ with $\hat{C}$ or $\hat{D}$.

   (c) Prove the 'mixed' Jacobi identity

$$(-1)^{CA}[\hat{A}, [\hat{B}, \hat{C}\}\} + (-1)^{AB}[\hat{B}, [\hat{C}, \hat{A}\}\} + (-1)^{BC}[\hat{C}, [\hat{A}, \hat{B}\}\} = 0. \tag{4}$$

In other words (and notations),

$$
\begin{aligned}
[\hat{B}_1,[\hat{B}_2,\hat{B}_3]] &+ [\hat{B}_2,[\hat{B}_3,\hat{B}_1]] + [\hat{B}_3,[\hat{B}_1,\hat{B}_2]] = 0,\\
[\hat{B}_1,[\hat{B}_2,\hat{F}]] &+ [\hat{B}_2,[\hat{F},\hat{B}_1]] + [\hat{F},[\hat{B}_1,\hat{B}_2]] = 0,\\
\{\hat{F}_1,[\hat{F}_2,\hat{B}]\} &- \{\hat{F}_2,[\hat{B},\hat{F}_1]\} + [\hat{B},\{\hat{F}_1,\hat{F}_2\}] = 0,\\
[\hat{F}_1,\{\hat{F}_2,\hat{F}_3\}] &+ [\hat{F}_2,\{\hat{F}_3,\hat{F}_1\}] + [\hat{F}_3,\{\hat{F}_1,\hat{F}_2\}] = 0,
\end{aligned}
\tag{5}
$$

where 'B' and 'F' indicate the overall statistics of the operator involved.

5. Finally, an exercise in fermionic creation and annihilation operators and their anticommutation relations,

$$\{\hat{a}_\alpha,\hat{a}_\beta\} = \{\hat{a}^\dagger_\alpha,\hat{a}^\dagger_\beta\} = 0, \quad \{\hat{a}_\alpha,\hat{a}^\dagger_\beta\} = \delta_{\alpha,\beta}\,. \tag{6}$$

   (a) Calculate the commutators $[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma\hat{a}_\delta]$.

   (b) Consider two one-body operators $\hat{A}_1$ and $\hat{B}_1$ and let $\hat{C}_1$ be their commutator, $\hat{C}_1 = [\hat{A}_1,\hat{B}_1]$. Let $\hat{A}$ be the second-quantized forms of $\hat{A}_{\text{tot}}$,

$$\hat{A} = \sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta\,, \tag{7}$$

   and ditto for the second-quantized $\hat{B}$ and $\hat{C}$.

   Verify that $[\hat{A},\hat{B}] = \hat{C}$.

   (c) Calculate the commutator $[\hat{a}^\dagger_\mu\hat{a}_\nu,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta]$.

   (d) The second quantized form of a two-body additive operator

$$\hat{B}_{\text{tot}} = \tfrac{1}{2}\sum_{i\neq j}\hat{B}_2(i^{\text{th}}\text{ and } j^{\text{th}}\text{ particles})$$

   acting on identical fermions is

$$\hat{B} = \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma\,. \tag{8}$$

   This expression is similar to its bosonic counterpart, but note the reversed order of the annihilation operators $\hat{a}_\delta$ and $\hat{a}_\gamma$.

Consider a one-body operator $\hat{A}_1$ and two two-body operators $\hat{B}_2$ and $\hat{C}_2$. Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\underline{\text{st}}}) + \hat{A}_1(2^{\underline{\text{nd}}})\right), \hat{B}_2\right]$, then the respective second-quantized operators in the fermionic Fock space satisfy $\hat{C} = [\hat{A}, \hat{B}]$.