# Spin–Statistics Theorem

Relativistic causality requires quantum fields at two spacetime points $x$ and $y$ separated by a space-like interval $(x-y)^2 < 0$ to either commute or anticommute with each other. The **spin–statistics theorem** says that the fields of integral spins commute (and therefore must be quantized as bosons) while the fields of half-integral spin anticommute (and therefore must be quantized as fermions). The spin-statistics theorem applies to all quantum field theories which have:

1. Special relativity, *i.e.* Lorentz invariance and relativistic causality;
2. Positive energies of all particles;
3. Hilbert space with positive norms of all states.

The theorem is valid for both free or interacting quantum field theories, and in any spacetime dimension $D > 2$. In these notes I shall prove the theorem for the free fields in four dimensions and outline its generalization to $D \neq 4$; proving the theorem for the interactive fields is too complicated for this class.

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Consider a generic Lorentz multiplet $\phi_A(x)$ of complex quantum fields describing massive charged particles of spin $j$. In general, the multiplet could be reducible, $A \in (j_1^+, j_1^-) \oplus (j_2^+, j_2^-) \oplus \cdots$, but all the irreducible components must have

$$\left|j^+ - j^-\right| \;\le\; j \;\le\; (j^+ + j^-) \quad\text{and}\quad (-1)^{2j^+}(-1)^{2j^-} \;=\; (-1)^{2j}. \tag{1}$$

Free fields satisfy some kind of linear equations of motion which have plane-wave solutions with $p^2 = M^2$ corresponding to relativistic particles of mass $M$. Let $p^0 = +E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + M^2}$ and let

$$e^{-ipx} f_A(\mathbf{p}, s) \quad\text{and}\quad e^{+ipx} h_A(\mathbf{p}, s) \tag{2}$$

be respectively the positive-frequency and negative-frequency plane-wave solutions. By the

CPT theorem

$$h_A(\mathbf{p},s)\;=\;f_A(+\mathbf{p},-s)\times i^{2s}(-1)^{2j^-(A)} \tag{3}$$

where the $i^{2s}$ factor accompanies the spin reversal and the $(-1)^{2j^-(A)}$ sign is the $(j^+,j^-)$ representation of the proper-but-not-orthochronous Lorentz transform $\mathbf{PT}: x^\mu\to -x^\mu$.[⋆] For the complex conjugate plane waves, we have

$$h^*_{\bar{A}}(\mathbf{p},s)\;=\;f^*_{\bar{A}}(+\mathbf{p},-s)\times(-i)^{2s}(-1)^{2j^+(\bar{A})} \tag{4}$$

where the last factor is $(-1)^{2j^+(\bar{A})}=(-1)^{2j^-(A)}$ because the conjugation exchanges the $j^+$ and the $j^-$ of a Lorentz multiplet.

The relation between particle's spin and statistics follows from the spin sums

$$\mathcal{F}_{A\bar{B}}(p)\;\stackrel{\text{def}}{=}\;\sum_s f_A(\mathbf{p},s)\,f^*_{\bar{B}}(\mathbf{p},s)\quad\text{and}\quad \mathcal{H}_{A\bar{B}}(p)\;\stackrel{\text{def}}{=}\;\sum_s h_A(\mathbf{p},s)\,h^*_{\bar{B}}(\mathbf{p},s) \tag{5}$$

and the way they transform under Lorentz symmetries. The plane-wave solutions (2) themselves transform according to eqs. (18) of the homework set #6:

$$\begin{aligned} f_A(Lp,s') &= \sum_C\sum_s M_A^{\;C}(L)\,C^*_{s,s'}(L,p)\,f_C(p,s),\\ h_A(Lp,s') &= \sum_C\sum_s M_A^{\;C}(L)\,C_{s,s'}(L,p)\,h_C(p,s). \end{aligned} \tag{6}$$

where $C_{s,s'}(L,p)$ is a unitary $(2j+1)\times(2j+1)$ matrix of the Lorentz transform of particle

⋆ For example, for the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ multiplet of Dirac spinor fields, the constant spinors $u_A(\mathbf{p},s)\equiv f_A(\mathbf{p},s)$ and $v_A(\mathbf{p},s)\equiv h_A(\mathbf{p},s)$ satisfy

$$v_A(\mathbf{p},s)\;=\;+\left(\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\right)_A\;=\;+\left(\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(i^{2s}\xi_{-s})\right)_A\;=\;+i^{2s}\,u_A(\mathbf{p},-s)$$

for $A\in(\frac{1}{2},0)$ (the left-handed Weyl spinor components), but

$$v_A(\mathbf{p},s)\;=\;-\left(\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\right)_A\;=\;-\left(\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(i^{2s}\xi_{-s})\right)_A\;=\;-i^{2s}\,u_A(\mathbf{p},-s)$$

for $A\in(0,\frac{1}{2})$ (the right-handed Weyl spinor components); in both cases, the $i^{2s}$ factor comes from $\eta_s=i^{2s}\xi_{-s}$ while the chirality-dependent sign between the $u_A$ and the $v_A$ components is the $(-1)^{2j^-(A)}$ factor.

spin states. Unitarity means $\sum_{s'} C^*_{s_1,s'} C_{s_2,s'} = \delta_{s_1,s_2}$, hence

$$\begin{aligned}
\mathcal{F}_{A\bar{B}}(Lp) \;&=\; \sum_{s'} \Biggl[ f_A(Lp,s') \;=\sum_{C}\sum_{s_1} M_A^{\ C}(L)\, C^*_{s_1,s'}(L,p)\, f_C(p,s_1) \Biggr] \times \\
&\qquad\qquad \times \Biggl[ f^*_{\bar{B}}(Lp,s') \;=\sum_{\bar{D}}\sum_{s_2} M^{*\bar{D}}_{\bar{B}}(L)\, C_{s_2,s'}(L,p)\, f^*_{\bar{D}}(p,s_2) \Biggr] \\
&= \sum_{C,\bar{D}} M_A^{\ C}(L)\, M^{*\bar{D}}_{\bar{B}}(L) \sum_{s_1,s_2} f_C(p,s_1)\, f^*_{\bar{D}}(p,s_2) \Biggl( \sum_{s'} C^*_{s_1,s'} C_{s_2,s'} = \delta_{s_1,s_2} \Biggr) \\
&= \sum_{C,\bar{D}} M_A^{\ C}(L)\, M^{*\bar{D}}_{\bar{B}}(L)\, \mathcal{F}_{C\bar{D}}(p)
\end{aligned} \tag{7}$$

and likewise

$$\mathcal{H}_{A\bar{B}}(Lp) \;=\; \sum_{C,\bar{D}} M_A^{\ C}(L)\, M^{*\bar{D}}_{\bar{B}}(L)\, \mathcal{H}_{C\bar{D}}(p). \tag{8}$$

In other words, the spin sums (5) are *Lorentz-covariant functions of the particle's momentum. And since we are only interested in the on-shell momenta with fixed $p^\mu p_\mu = M^2$, the functional form of any covariant function of the $p^\mu$ is determined by the* Spin(3,1) *analogue of the Wigner–Eckard theorem.*

In three Euclidean dimensions, the Wigner–Eckard theorem usually concerns the rotational properties of matrix elements of vector or tensor operators between states of given angular momenta, but it can be recast in terms of rotationally-covariant functions of a vector $\mathbf{v}$. Consider a covariant matrix of functions $Q_{a,b}(\mathbf{v})$ where the indices $a$ and $b$ run over components of some (possibly reducible) spin multiplet, $a,b \in (j_1)\oplus(j_2)\oplus\cdots$. According top the Wigner–Eckard theorem,

$$Q_{a,b}(\mathbf{v}=v\mathbf{n}) \;=\; \sum_{\ell=|j(a)-j(b)|}^{j(a)+j(b)} q_\ell(v) \sum_{m=-\ell}^{+\ell} v^\ell Y_{\ell,m}(\mathbf{n}) \times \mathrm{Clebbsch}(a,b|\ell,m), \tag{9}$$

where $q_\ell(v)$ depend only on $\ell$ and the magnitude $v$ of the vector and the spherical harmonics $v^\ell Y_{\ell,m}(\mathbf{n})$ are homogeneous polynomials (degree $\ell$) of the Cartesian components $v_x$, $v_y$ and $v_z$. For a vector of fixed magnitude $\mathbf{v}^2 = v^2$ the $q_\ell$ coefficients are constants, hence each $Q_{a,b}$ is effectively a polynomial of $(v_x,v_y,v_z)$ comprising terms of net degree $\ell$ ranging from $|j(a)-j(b)|$ to $j(a)+j(b)$.

In four Minkowski dimensions we have a similar situation, except for the spin group being $SL(2,\mathbf{C})$ instead of $SU(2)$, hence $A,\bar{B}\in(j_1^+,j_1^-)\oplus(j_2^+,j_2^-)\oplus\cdots$. Also, the Lorentz vector multiplet has $j^+=j^-=\frac{1}{2}$ (unlike the 3D vector multiplet which has $\ell=1$) and consequently the Minkowski analogues $\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)$ of the spherical harmonics do not have separate integer-valued indices $\ell^+$ and $\ell^-$ but rather a common index $J=j^+=j^-$ which takes both integer and half-integer values. Hence, the Wigner–Eckard theorem for Lorentz-covariant matrices $\mathcal{F}_{A\bar{B}}(p)$ and $\mathcal{H}_{A\bar{B}}(p)$ says:

$$\begin{aligned}
\mathcal{F}_{A\bar{B}}(p) &= \sum_{J=J_{\min}}^{\mathrm{J_{max}}} f_J(M)\sum_{\substack{-J\le m^+\le J\\ -J\le m^-\le J}} M^{2J}\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)\times \mathrm{Clebbsch}(A,\bar{B}|J,m^+,J,m^-),\\
\mathcal{H}_{A\bar{B}}(p) &= \sum_{J=J_{\min}}^{\mathrm{J_{max}}} h_J(M)\sum_{\substack{-J\le m^+\le J\\ -J\le m^-\le J}} M^{2J}\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)\times \mathrm{Clebbsch}(A,\bar{B}|J,m^+,J,m^-),
\end{aligned}\tag{10}$$

where $M$ is the particle's mass ($p^\mu p_\mu=M^2$), the indices $J$, $m^+$ and $m^-$ are all integral or all half-integral according to

$$(-1)^{2J} = (-1)^{2m^+} = (-1)^{2m^-} = (-1)^{2j^+(A)}(-1)^{2j^+(\bar{B})} = (-1)^{2j^-(A)}(-1)^{2j^-(\bar{B})},\tag{11}$$

and in the sum over $J$,

$$\begin{aligned}
J_{\min} &= \max\left(|j^+(A)-j^+(\bar{B})|,|j^-(A)-j^-(\bar{B})|\right),\\
J_{\max} &= \min\left((j^+(A)+j^+(\bar{B})),(j^-(A)+j^-(\bar{B}))\right).
\end{aligned}\tag{12}$$

Similar to their 3D counterparts, the 4D "spherical harmonics" $M^{2J}\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)$ are homogeneous polynomials of the 4–Momentum components $p^0,p^x,p^y,p^z$, although in 4D the polynomial degree is $2J$ rather than $\ell$. Consequently, for a fixed particle mass $M$, all spin sums $\mathcal{F}_{A\bar{B}}(p)$ and $\mathcal{H}_{A\bar{B}}(p)$ can be written as polynomials of the $p^0,p^x,p^y,p^z$.

Now, once we have written the spin sums (5) as polynomials of the $p^\mu$ components, we can analytically continue these polynomials to negative energies $p^0=-E_{\mathbf{p}}$ or even to complex 4–momenta satisfying $p^\mu p_\mu=M^2$. This analytic continuation allows us to compare the spin sums at opposite 4–momenta $+p^\mu=(+E,+\mathbf{p})$ and $-p^\mu=(-E,-\mathbf{p})$, and because every term in the same polynomial has the same degree $2J$ modulo 2, it follows that the whole polynomial is either odd or even according to eq. (12), thus

$$\begin{aligned} \mathcal{F}_{A\bar{B}}(-p^\mu) &= (-1)^{2j^-(A)}(-1)^{2j^-(\bar{B})}\mathcal{F}_{A\bar{B}}(+p^\mu), \\ \mathcal{H}_{A\bar{B}}(-p^\mu) &= (-1)^{2j^-(A)}(-1)^{2j^-(\bar{B})}\mathcal{H}_{A\bar{B}}(+p^\mu). \end{aligned} \tag{13}$$

Finally, for physical momenta (real $p^\mu = (+E_{\mathbf{p}}, \mathbf{p})$), the CPT theorem (*cf.* eqs. (3) and (4)) relates the positive and the negative frequency spin sums to each other according to

$$\mathcal{H}_{A\bar{B}}(p^\mu) = \mathcal{F}_{A\bar{B}}(p^\mu) \times (-1)^{2j^-(A)}(-1)^{2j^+(\bar{B})}. \tag{14}$$

Analytic continuation of the spin sums as polynomials of $p^\mu$ extends eq. (14) to any complex momenta, hence in light of eqs. (13),

$$\mathcal{H}_{A\bar{B}}(-p^\mu) = \mathcal{F}_{A\bar{B}}(+p^\mu) \times (-1)^{2j^-(\bar{B})}(-1)^{2j^+(\bar{B})}. \tag{15}$$

According to eq. (1), the sign factor in the above formula does not depend on a particular field component $\phi^\dagger_{\bar{B}}$ but only on the particle's spin:

$$\begin{aligned} \mathcal{H}_{A\bar{B}}(-p^\mu) &= +\mathcal{F}_{A\bar{B}}(+p^\mu) \quad \text{for particles of integral spin,} \\ \mathcal{H}_{A\bar{B}}(-p^\mu) &= -\mathcal{F}_{A\bar{B}}(+p^\mu) \quad \text{for particles of half-integral spin.} \end{aligned} \tag{16}$$

It turns out that this little red spin-dependent sign makes a big difference for the particles' statistics.

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A free quantum field is a superposition of plane-wave solutions with operatorial coefficients, thus

$$\begin{aligned} \hat{\phi}_A(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left[ e^{-ipx} f_A(\mathbf{p}, s)\, \hat{a}(\mathbf{p}, s) + e^{+ipx} h_A(\mathbf{p}, s)\, \hat{b}^\dagger(\mathbf{p}, s) \right]_{p^0=+E_{\mathbf{p}}}, \\ \hat{\phi}^\dagger_{\bar{B}}(y) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left[ e^{-ipy} h^*_{\bar{B}}(\mathbf{p}, s)\, \hat{b}(\mathbf{p}, s) + e^{+ipy} f^*_{\bar{B}}(\mathbf{p}, s)\, \hat{a}^\dagger(\mathbf{p}, s) \right]_{p^0=+E_{\mathbf{p}}}. \end{aligned} \tag{17}$$

(Without loss of generality we assume complex fields and charged particles; for the neutral particles we would have $\hat{b} \equiv \hat{a}$ and $\hat{b}^\dagger \equiv \hat{a}^\dagger$.) *Regardless of statistics*, positive particle energies require $\hat{a}^\dagger(p,s)$ and $\hat{b}^\dagger(p,s)$ to be creation operators while $\hat{a}(p,s)$ and $\hat{b}(p,s)$ are annihilation operators, thus

$$\hat{a}^\dagger(\mathbf{p},s)\,|0\rangle \;=\; |1(\mathbf{p},s,+)\rangle\,, \quad \hat{b}^\dagger(\mathbf{p},s)\,|0\rangle \;=\; |1(\mathbf{p},s,-)\rangle\,, \quad \hat{a}(\mathbf{p},s)\,|0\rangle \;=\; \hat{b}(\mathbf{p},s)\,|0\rangle \;=\; 0, \tag{18}$$

and hence, in a Fock space of positive-definite norm

$$\langle 0|\,\hat{a}(\mathbf{p},s)\,\hat{a}^\dagger(\mathbf{p}',s')\,|0\rangle \;=\; \langle 0|\,\hat{b}(\mathbf{p},s)\,\hat{b}^\dagger(\mathbf{p}',s')\,|0\rangle \;=\; +2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'} \tag{19}$$

while all the other "vacuum sandwiches" of two creation or annihilation operators vanish identically. Consequently, *regardless of particles' statistics*, vacuum expectation values of products of two fields at distinct points $x$ and $y$ are given by

$$\langle 0|\,\hat{\phi}_A(x)\,\hat{\phi}^\dagger_{\bar{B}}(y)\,|0\rangle \;=\; +\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\times\sum_s f_A(\mathbf{p},s)\,f^*_{\bar{B}}(\mathbf{p},s) \tag{20}$$

and

$$\langle 0|\,\hat{\phi}^\dagger_{\bar{B}}(y)\,\hat{\phi}_A(x)\,|0\rangle \;=\; +\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{+ip(x-y)}}{2E_{\mathbf{p}}}\times\sum_s h_A(\mathbf{p},s)\,h^*_{\bar{B}}(\mathbf{p},s). \tag{21}$$

And at this point, we can use the spin sums (5) and their polynomial dependence on the particle's 4–momenta to calculate

$$\langle 0|\,\hat{\phi}_A(x)\,\hat{\phi}^\dagger_{\bar{B}}(y)\,|0\rangle \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;e^{-ip(x-y)}\mathcal{F}_{A\bar{B}}(p)\Big|_{p^0=+E_{\mathbf{p}}} \;=\; \mathcal{F}_{A\bar{B}}(+i\partial_x)\,D(x-y) \tag{22}$$

where

$$D(x-y) \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;e^{-ip(x-y)}\Big|_{p^0=+E_{\mathbf{p}}}\,,$$

and likewise

$$\langle 0|\,\hat{\phi}^\dagger_{\bar{B}}(y)\,\hat{\phi}_A(x)\,|0\rangle \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;e^{+ip(x-y)}\mathcal{H}_{A\bar{B}}(p)\Big|_{p^0=+E_{\mathbf{p}}} \;=\; \mathcal{H}_{A\bar{B}}(-i\partial_x)\,D(y-x). \tag{23}$$

As explained in class, for a space-like distance between points $x$ and $y$, $D(y-x)=+D(x-y)$. At the same time, the differential operators $\mathcal{F}_{A\bar{B}}(+i\partial_x)$ and $\mathcal{H}_{A\bar{B}}(-i\partial_x)$ are related to each other according to eq. (16). Therefore, regardless of particles' statistics, for $(x-y)^2 < 0$

$$\begin{aligned}
\langle 0|\,\hat{\phi}_A(x)\,\hat{\phi}^\dagger_{\bar{B}}(y)\,|0\rangle \;&=\; +\,\langle 0|\,\hat{\phi}^\dagger_{\bar{B}}(y)\,\hat{\phi}_A(x)\,|0\rangle \quad \text{for particles of integral spin,}\\
\langle 0|\,\hat{\phi}_A(x)\,\hat{\phi}^\dagger_{\bar{B}}(y)\,|0\rangle \;&=\; -\,\langle 0|\,\hat{\phi}^\dagger_{\bar{B}}(y)\,\hat{\phi}_A(x)\,|0\rangle \quad \text{for particles of half-integral spin.}
\end{aligned} \tag{24}$$

On the other hands, $\boxed{\textit{relativistic causality}}$ requires for $(x-y)^2 < 0$

$$\left.\begin{aligned}
\hat{\phi}_A(x)\,\hat{\phi}^\dagger_{\bar{B}}(y) = &\; +\hat{\phi}^\dagger_{\bar{B}}(y)\,\hat{\phi}_A(x) \quad \text{for bosonic fields,}\\
\hat{\phi}_A(x)\,\hat{\phi}^\dagger_{\bar{B}}(y) = &\; -\hat{\phi}^\dagger_{\bar{B}}(y)\,\hat{\phi}_A(x) \quad \text{for fermionic fields,}
\end{aligned}\right\}\ \text{regardless of particle's spin.} \tag{25}$$

And the only way eqs. (24) and (25) can both hold true at the same time if *all particles of integral spin are bosons and all particles of half-integral spin are fermions*.

Indeed, for bosonic particles, the creation and annihilation operators commute with each other except for

$$\begin{aligned}
[\hat{a}(\mathbf{p},s),\hat{a}^\dagger(\mathbf{p}',s')] \;&=\; +2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,,\\
[\hat{b}^\dagger(\mathbf{p},s),\hat{b}(\mathbf{p}',s')] \;&=\; -2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,,
\end{aligned} \tag{26}$$

and therefore the quantum fields commute or do not commute according to

$$\begin{aligned}
\left[\hat{\phi}_A(x),\hat{\phi}^\dagger_{\bar{B}}(y)\right] \;&=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\left(e^{-ip(x-y)}f_A(p,s)f^*_{\bar{B}}(p,s)\,-\,e^{-ip(x-y)}h_A(p,s)h^*_{\bar{B}}(p,s)\right)\\
&=\; \mathcal{F}_{A\bar{B}}(i\partial_x)\,D(x-y)\,-\,\mathcal{H}_{A\bar{B}}(-i\partial_x)\,D(y-x)\\
&=\; \mathcal{F}_{A\bar{B}}(i\partial_x)\,\big(D(x-y)\,-\,(-1)^{2j}\,D(y-x)\big)
\end{aligned} \tag{27}$$

where $j$ is the particle's spin, *cf.* eq. (24). For particles of integral spin, this commutator duly vanishes when points $x$ and $y$ are separated by a space-like distance. But for particles of half-integral spin, the two terms on the last line of eq. (27) add up instead of canceling each other, and the fields $\hat{\phi}_A(x)$ and $\hat{\phi}^\dagger_{\bar{B}}(y)$ fail to commute — which violates relativistic causality. To avoid this violation, *bosonic particles must have integral spins only.*

Likewise, for fermionic particles, the creation and annihilation operators anticommute with each other except for

$$\begin{aligned} \{\hat{a}(\mathbf{p},s),\hat{a}^\dagger(\mathbf{p}',s')\} &= +2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \\ \{\hat{b}^\dagger(\mathbf{p},s),\hat{b}(\mathbf{p}',s')\} &= +2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \end{aligned} \tag{28}$$

and therefore the quantum fields anticommute or do not anticommute according to

$$\begin{aligned} \left\{\hat{\phi}_A(x),\hat{\phi}^\dagger_{\bar{B}}(y)\right\} &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\left(e^{-ip(x-y)}f_A(p,s)f^*_{\bar{B}}(p,s)\,+\,e^{-ip(x-y)}h_A(p,s)h^*_{\bar{B}}(p,s)\right) \\ &= \mathcal{F}_{A\bar{B}}(i\partial_x)\,D(x-y)\,+\,\mathcal{H}_{A\bar{B}}(-i\partial_x)\,D(y-x) \\ &= \mathcal{F}_{A\bar{B}}(i\partial_x)\,\big(D(x-y)\,+\,(-1)^{2j}\,D(y-x)\big)\,. \end{aligned} \tag{29}$$

This anticommutator vanishes when $(x-y)^2<0$ for half-integral $j$ but not for integral $j$. Hence, to maintain relativistic causality, *fermionic particles must have half-integral spins only.*

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I would like to conclude these notes with a few words about spin-statistics relations in spacetime dimensions other than four. In any dimension $D$, quantum fields form multiplets of the $\mathrm{Spin}(D-1,1)$ Lorentz symmetry while massive particles form multiplets of the spin symmetry $\mathrm{Spin}(D-1)$. For $D>4$, the multiplets are more complicated then in $D=4$, but they fall into the same two broad classes according to their behavior under rotations $R(2\pi)$ by $2\pi$ under any spatial axis: The single-valued tensor multiplets for which $R(2\pi)=+1$, and the double-valued spinor multiplets for which $R(2\pi)=-1$. The relation between spin sums (5) follows this distinction:

$$\mathcal{H}_{A\bar{B}}(-p^\mu) \;=\; \mathcal{F}_{A\bar{B}}(+p^\mu)\times R(2\pi), \tag{30}$$

although the proof is more complicated in higher dimensions. But but in any dimension, the statistics follow the sign in eq. (30), thus *particles invariant under $2\pi$ rotations must be bosons while particles for which $R(2\pi)=-1$ must be fermions*.

For $D = 3$ (two space dimensions) the situation is more complicated. The Lorentz symmetry $\mathrm{Spin}(2,1) = SL(2,\mathbf{R})$ has finite multiplets of quantized spin $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ but the space rotation group $SO(2)$ is abelian (1 generator only), so its multiplets are singlets of arbitrary, un-quantized $m_j$. **If** $m_j$ happens to be an integer or half-integer, **then** this particle species can be quantized as a free quantum field of definite $J = m_j$ modulo 1, and the spin–statistics theorem works as usual: *Particles with integral $m_j$ are bosons while particles with half-integral $m_j$ are fermions.* The particles with fractional spins $m_j$ are more difficult to quantize; they are neither bosons nor fermions but *anyons* obeying fractional statistics where $|\alpha, \beta\rangle = |\beta, \alpha\rangle \times e^{\pm 2\pi i m_j}$, depending on how the two particles are exchanged. But even in this case, the statistics follows the spin: When the spin is fractional, the statistics has the same fractional phase.