1. Three exams ago, you saw that in three spacetime dimensions (one time, two space) one can make photons massive without breaking the gauge invariance. In this question we consider the non-abelian version of this mechanism known as the *topologically massive Yang–Mills theory.* The Lagrangian of this theory

$$\mathcal{L} \;=\; \frac{-1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{K}{8\pi}\,\epsilon^{\lambda\mu\nu}\,\mathrm{tr}\big(\mathcal{A}_\lambda\mathcal{F}_{\mu\nu} \;-\; \tfrac{2i}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\big) \tag{1}$$

comprises the usual Yang–Mills term as well as a parity-breaking Chern–Simons term. The Chern–Simons term is not gauge invariant but it leads to a gauge invariant theory, provided the coefficient K is an integer.

(a) Show that the action $S = \int \mathcal{L}\, d^3x$ is invariant under *infinitesimal* gauge transformations of the vector field $\mathcal{A}_\mu$.

(b) Under finite gauge transforms $U(x)$, the action is not invariant but changes by an $\mathcal{A}$–independent constant. Specifically,

$$\Delta S \;=\; -\frac{K}{12\pi}\int d^3x\,\epsilon^{\lambda\mu\nu}\,\mathrm{tr}\Big((U^\dagger\partial_\lambda U)(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)\Big). \tag{2}$$

Verify this formula.

It is often said that a symmetry of a field theory must preserve the action, but the actual symmetry requirements are less stringent. Classically, one needs a field-independent ΔS, which is sufficient to assure that the field equations are covariant under the symmetry in question. In a quantum theory, one needs a stronger condition: e^{iS} must be invariant to assure symmetry of the path integral. In other words, S must be invariant up to a *quantized* field-independent constant, *i.e.* $\Delta S = 2\pi \times$ an *integer* constant.

Fortunately, the integral (2) is topological in nature and takes quantized values: For any gauge symmetry $U(x)$, $\Delta S = 2\pi K \times$ an integer. Consequently, the path-integral $\iint \mathcal{D}[\mathcal{A}_\mu(x)]$ is gauge invariant *for integral values of the parameter K*.

Proving the topological "quantization" of the integral (2) is beyond the scope of this exam. Instead, let us focus on the classical consequences of ΔS being independent of the $\mathcal{A}_\mu(x)$.

(c) Derive the classical field equations of the topologically-massive Yang–Mills theory and write them down in a manifestly gauge-covariant form.

(d) Show that these equations imply

$$(D^2 + M^2)\mathcal{F}^\lambda \;=\; 2i\epsilon^{\lambda\mu\nu}\mathcal{F}_\mu\mathcal{F}_\nu \tag{3}$$

where in 3D $\mathcal{F}_{\mu\nu} = \epsilon_{\mu\nu\lambda}\mathcal{F}^\lambda$ and

$$M \;=\; \frac{Kg^2}{4\pi} \tag{4}$$

is the mass of the gauge field. (Note that in 3D, g has the dimension of $\sqrt{\text{mass}}$).

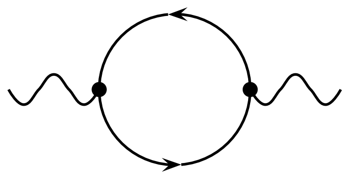
2. Now consider another parity-violating gauge theory in three spacetime dimensions, namely QCD_3 with massive fermions,

$$\mathcal{L} \;=\; \frac{-1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \overline{\Psi}(i\not{D} - m)\Psi. \tag{5}$$

For simplicity, assume the fermions form one fundamental multiplet of an $SU(N)$ gauge group (*i.e.*, there are N colors but only one flavor).

In odd spacetime dimensions, massive Dirac fermions have no Parity symmetry even at the tree level. At higher orders of perturbation theory, quark loop diagrams yield parity-violating amplitudes thanks to $\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu) = 2i\epsilon^{\lambda\mu\nu}$ in 3D (and similar formulæ in higher odd dimensions). Hence, even the gluon sector of the theory becomes parity-violating.

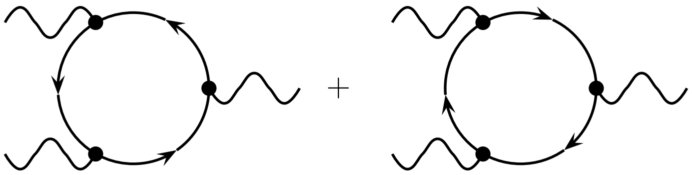
(a) Evaluate the one loop diagram



and show that for *small* gluon momentum $|p^2| \ll m^2$ it yields

$$\Sigma^{\mu\nu}_{\psi\,\mathrm{loop}}(k) \;=\; \frac{g^2}{8\pi}\left(ip_\lambda\epsilon^{\lambda\mu\nu} + \frac{p^\mu p^\nu - g^{\mu\nu}p^2}{3m} + O\left(\frac{p^3}{m^2}\right)\right). \tag{6}$$

(b) Similarly, show that for three external gluons with small momenta (compared to the fermions mass m), the one-loop amplitude is



$$= \frac{g^3}{8\pi} f^{abc} \epsilon^{\lambda\mu\nu} + O\left(\frac{p}{m}\right). \quad (7)$$

(c) Show that for quark loops with four or more external gluons with small momenta, all the one-quark-loop amplitudes are suppressed by negative powers of the quark mass m.

Now consider the Functional Integral for the $d = 3$ QCD. Let us integrate $\iint D[\Psi(x)] \iint D[\overline{\Psi}(x)]$ over the quark fields for fixed gauge fields $A^a_\mu(x)$. The result of this integration is an effective quantum theory of the gauge fields with action

$$S[A^a_\mu] \;=\; S_{YM}[A^a_\mu] \;-\; i \log \mathrm{Det}(i\not{D} - m). \quad (8)$$

(d) Use the results of questions (a), (b) and (c) to show that in the large quark mass m limit,

$$-i \log \mathrm{Det}(i\not{D} - m) \;=\; \int d^3x \left\{ \frac{1}{8\pi} \epsilon^{\lambda\mu\nu} \operatorname{tr}\big(\mathcal{A}_\lambda \mathcal{F}_{\mu\nu} \;-\; \tfrac{2i}{3} \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu\big) \;+\; O\left(\frac{1}{m}\right) \right\} \quad (9)$$

and consequently, the effective low-energy quantum theory is precisely the topologically massive Yang–Mills theory discussed in the previous question.

$\star$ For extra credit: Suppose we have several flavors of massive quarks, some with $m_f > 0$ and some with $m_f < 0$ (in 3D, this makes a difference). Show that when we integrate out all these quarks, we end up with a Chern–Simons term with coefficient

$$K \;=\; \#(m_f > 0) \;-\; \#(m_f < 0). \quad (10)$$

3. The last problem is about the (semi-classical) Higgs mechanism. Consider a fermion-less $SU(N)\times SU(N)$ gauge theory with a bi-fundamental multiplet $(\mathbf{N},\overline{\mathbf{N}})$ of scalar fields $\Phi_i^{\,j}(x)$. In $N\times N$ matrix notations, we have a matrix-valued complex scalar field $\Phi(x)$ and two distinct matrix-valued vector fields $\mathcal{B}_\mu(x) = g_B B_\mu^a(x)\frac{\lambda^a}{2}$ and $\mathcal{C}_\mu(x) = g_C C_\mu^a(x)\frac{\lambda^a}{2}$. The classical Lagrangian of the theory is

$$\mathcal{L} \;=\; \frac{-1}{2g_B^2}\,\mathrm{tr}\big(\mathcal{B}^{\mu\nu}\mathcal{B}_{\mu\nu}\big) \;+\; \frac{-1}{2g_C^2}\,\mathrm{tr}\big(\mathcal{C}^{\mu\nu}\mathcal{C}_{\mu\nu}\big) \;+\; \mathrm{tr}\big(D_\mu\Phi^\dagger D^\mu\Phi\big) \;-\; V(\Phi^\dagger,\Phi). \tag{11}$$

Under the $SU(N)\times SU(N)$ local symmetry, the scalar fields transform according to

$$\Phi'(x) \;=\; U(x)\Phi(x)W^\dagger(x). \tag{12}$$

(a) Write down the transformation laws for the vector fields, spell out the covariant derivatives $D_\mu\Phi(x)$ and $D_\mu\Phi^\dagger(x)$, and check that they are indeed covariant.

Let the scalar potential have its deepest minimum when the scalar fields have vacuum expectation values (VEVs) of the form

$$\left\langle \Phi_i^j \right\rangle \;=\; h\times\delta_i^j \quad \big(i.\,e.,\ \langle\Phi\rangle \;=\; h\times\mathbf{1}_{N\times N}\big) \quad \textit{modulo symmetry}. \tag{13}$$

(b) Show that such VEVs break the $SU(N)\times SU(N)$ symmetry down to $SU(N)$ and write down the generators of the unbroken symmetry in terms of the T_B^a and T_C^a.

(c) Calculate the mass matrix for the canonically normalized vector fields B_μ^a and C_μ^a. Show that for some mixing angle θ,

$$\begin{aligned} X_\mu^a &= \cos\theta\times B_\mu^a \;-\; \sin\theta\times C_\mu^a \qquad &&\text{become massive,}\\ \text{but } A_\mu^a &= \sin\theta\times B_\mu^a \;+\; \cos\theta\times C_\mu^a \qquad &&\text{remain massless.}\end{aligned} \tag{14}$$

Calculate the angle θ and the mass of the X_μ^a fields.

(d) Finally, show that the massless vector fields $A_\mu^a(x)$ and their interactions with each other are described by the $SU(N)$ Yang–Mills Lagrangian

$$\mathcal{L}_A \;=\; \frac{-1}{2g_A^2}\,\mathrm{tr}\big(\mathcal{A}^{\mu\nu}\mathcal{A}_{\mu\nu}\big) \tag{15}$$

and calculate the gauge coupling g_A.