1. According to the Noether theorem, a translationally invariant system of classical fields ϕ_a has a conserved stress-energy tensor

$$T^{\mu\nu}_{\rm Noether} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{1}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu}=T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\rm Noether} \;+\; \partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}, \tag{2}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$,

$$\begin{aligned} \partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\rm Noether} \;=\text{(hopefully)}=\; 0 \\ P^\mu_{\rm net} \;&\equiv\; \int\! d^3\mathbf{x}\, T^{0\mu} \;=\; \int\! d^3\mathbf{x}\, T^{0\mu}_{\rm Noether}\,. \end{aligned} \tag{3}$$

For the scalar fields, real or complex, $T^{\mu\nu}_{\rm Noether}$ is properly symmetric and one simply has $T^{\mu\nu}=T^{\mu\nu}_{\rm Noether}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu,\partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{4}$$

where A_μ is a real vector field and $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$.

(b) Write down $T^{\mu\nu}_{\rm Noether}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\rm EM} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{5}$$

Show that this expression indeed has form (2) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

(d) Write down the components of the stress-energy tensor (5) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

2. Now consider the electromagnetic fields coupled to the electric current J^μ of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^\mu_{\rm EM}$ and $P^\mu_{\rm mat}$. Consequently, we should have

$$\partial_\mu T^{\mu\nu}_{\rm net} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\rm net} \;=\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \tag{6}$$

but generally $\partial_\mu T^{\mu\nu}_{\rm EM} \neq 0$ and $\partial_\mu T^{\mu\nu}_{\rm mat} \neq 0$.

(a) Use Maxwell's equations to show that

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda} J_\lambda \tag{7}$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda} J_\lambda . \tag{8}$$

Now consider a specific example a complex scalar field of charge $q \neq 0$ coupled to the EM fields:

$$\mathcal{L}_{\rm net} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{9}$$

where

$$D_\mu\Phi \;=\; (\partial_\mu \,+\, iqA_\mu)\Phi \quad \text{and} \quad D_\mu\Phi^* \;=\; (\partial_\mu \,-\, iqA_\mu)\Phi^* \tag{10}$$

are the *covariant* derivatives of the scalar fields.

(b) Verify the gauge invariance of the electric current

$$J^\mu \;=\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{11}$$

by expressing it in terms of the scalar fields and their covariant derivatives (10).

(c) Write down equations of motion for the scalar fields Φ and Φ^* and use them to verify the current conservation $\partial_\mu J^\mu = 0$.

(d) Write down the Noether stress-energy tensor for the whole field system and show that

$$T^{\mu\nu}_{\text{net}} \equiv T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{mat}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \tag{12}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is the same function of EM fields as in the free EM case (**1**.c), $T^{\mu\nu}_{\text{EM}}$ is exactly as in eq. (5), and

$$T^{\mu\nu}_{\text{mat}} = D^\mu\Phi^*\,D^\nu\Phi + D^\nu\Phi^*\,D^\mu\Phi - g^{\mu\nu}\big(D_\lambda\Phi^*\,D^\lambda\Phi - m^2\Phi^*\,\Phi\big). \tag{13}$$

Hint: In the presence of an electric current J^μ, the $\partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (13) for the scalar field.

(e) Use the scalar fields' equations of motion and non-commutativity of the covariant derivatives

$$[D_\mu, D_\nu]\Phi = iqF_{|mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* = -iqF_{|mu\nu}\Phi^* \tag{14}$$

to verify eq. (8).