1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{1}$$

where $c=\hbar=1$, $F_{\mu\nu} \stackrel{\rm def}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad\text{and}\qquad (\partial^2+m^2)A^\mu = J^\mu. \tag{2}$$

Now, let us derive the Hamiltonian formalism for the massive vector field. As a first step, we need to identify the canonically conjugate "momentum" fields.

(c) Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

Thus, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H \;=\; -\int\! d^3\mathbf{x}\,\dot{\mathbf{A}}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;-\; L. \tag{3}$$

(d) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$ and A_0 fields and their *space* derivatives,

$$H \;=\int\! d^3\mathbf{x}\left\{\tfrac{1}{2}\mathbf{E}^2 \;+\; A_0\,(J_0-\nabla\cdot\mathbf{E}) \;-\; \tfrac{1}{2}m^2A_0^2 \;+\; \tfrac{1}{2}\,(\nabla\times\mathbf{A})^2 \;+\; \tfrac{1}{2}m^2\mathbf{A}^2 \;-\; \mathbf{J}\cdot\mathbf{A}\right\}. \tag{4}$$

Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x},t)$ to the values of other fields *at the same time t*.

Specifically, we have

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial A_0}\bigg|_{\mathbf{x}} - \nabla\cdot \frac{\partial \mathcal{H}}{\partial \nabla A_0}\bigg|_{\mathbf{x}} = 0. \tag{5}$$

At the same time, the vector fields $\mathbf{A}$ and $\mathbf{E}$ satisfy the Hamiltonian equations of motion,

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\frac{\delta H}{\delta \mathbf{E}(\mathbf{x})}\bigg|_t, \qquad \frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) = +\frac{\delta H}{\delta \mathbf{A}(\mathbf{x})}\bigg|_t. \tag{6}$$

(e) Write down the explicit form of all these equations.

(f) Finally, verify that the equations you have just written down are equivalent to the Euler–Lagrange equations you derived in question (a).

2. Next, consider the quantum electromagnetic fields. Canonical quantization of the massless vector field $A_\mu(x)$ is rather difficult because of the redundancy associated with the gauge symmetry, so let me simply state without proof a few key properties of the quantum tension fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$. In the absence of electric charges and currents, these fields satisfy time-independent operatorial identities

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t) = \nabla\cdot\hat{\mathbf{B}}(\mathbf{x},t) = 0 \tag{7}$$

(we assume free EM fields, *i.e.* no electric charges or currents), and have equal-time commutation relations

$$\begin{aligned}
\big[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{x}',t'=t)\big] &= 0, \\
\big[\hat{B}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\big] &= 0, \\
\big[\hat{E}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\big] &= -i\hbar c\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta^{(3)}(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{8}$$

(a) Verify that the commutation relations (8) are consistent with the time-independent Maxwell equations (7).

In the Heisenberg picture, the quantum EM fields also obey the time-dependent Maxwell equations

$$\begin{aligned} \frac{\partial \hat{\mathbf{B}}}{\partial \mathbf{t}} &= -\nabla \times \hat{\mathbf{E}}\,, \\ \frac{\partial \hat{\mathbf{E}}}{\partial \mathbf{t}} &= +\nabla \times \hat{\mathbf{B}}\,. \end{aligned} \tag{9}$$

(b) Derive eqs. (9) from the free electromagnetic Hamiltonian

$$\hat{H}_{EM} = \int d^3\mathbf{x}\,\left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \tfrac{1}{2}\hat{\mathbf{B}}^2\right) \tag{10}$$

and the equal-time commutation relations (8).

3. Finally, let us quantize a charged relativistic scalar field $\Phi(x)$. A conserved charge implies a complex field with a $U(1)$ symmetry $\Phi(x) \mapsto e^{i\theta}\Phi(x)$ which gives rise to a conserved Noether current

$$J^\mu = i\Phi^*\partial^\mu\Phi - i(\partial^\mu\Phi^*)\Phi. \tag{11}$$

For simplicity, let the Φ field be free, thus classically

$$\mathcal{L} = \partial^\mu\Phi^*\,\partial_\mu\Phi - m^2\,\Phi^*\Phi. \tag{12}$$

In the Hamiltonian formalism, we trade the time derivatives $\partial_0\Phi(x)$ and $\partial_0\Phi^*(x)$ for the canonically conjugate fields $\Pi^*(x)$ and $\Pi(x)$. (Note that for complex fields $\Pi(\mathbf{x})$ is canonically conjugate to the $\Phi^*(\mathbf{x})$ while $\Pi^*(\mathbf{x})$ is canonically conjugate to the $\Phi(\mathbf{x})$.) Canonical quantization of this system yields non-hermitian quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$ and the Hamiltonian operator

$$\hat{H} = \int d^3\mathbf{x}\,\left(\hat{\Pi}^\dagger\hat{\Pi} + \nabla\hat{\Phi}^\dagger\cdot\nabla\hat{\Phi} + m^2\,\hat{\Phi}^\dagger\hat{\Phi}\right). \tag{13}$$

(a) Derive the Hamiltonian (13) and write down the equal-time commutation relations between the quantum fields $\hat{\Phi}(\mathbf{x})$, $\hat{\Phi}^\dagger(\mathbf{x})$, $\hat{\Pi}(\mathbf{x})$ and $\hat{\Pi}^\dagger(\mathbf{x})$.

Next, let us expand the quantum fields into plane-wave modes:

$$\hat{\Phi}(\mathbf{x}) \;=\; \sum_{\mathbf{p}} L^{-3/2} e^{i\mathbf{x}\mathbf{p}} \hat{\Phi}_b p, \qquad \hat{\Phi}_{\mathbf{p}} \;=\; \int d^3\mathbf{x}\, L^{-3/2} e^{-i\mathbf{p}\mathbf{x}}\, \hat{\Phi}(\mathbf{x}), \tag{14}$$

and ditto for the $\hat{\Phi}^\dagger(\mathbf{x})$, $\hat{\Pi}(\mathbf{x})$, and $\hat{\Pi}^\dagger(\mathbf{x})$ fields. Note that for the *non–hermitian* fields $\hat{\Phi}^\dagger_{\mathbf{p}} \neq \hat{\Phi}_{-\mathbf{p}}$ and $\hat{\Pi}^\dagger_{\mathbf{p}} \neq \hat{\Pi}_{-\mathbf{p}}$; instead, all the mode operators $\hat{\Phi}_{\mathbf{p}}$, $\hat{\Phi}^\dagger_{\mathbf{p}}$, $\hat{\Pi}_{\mathbf{p}}$, and $\hat{\Pi}^\dagger_{\mathbf{p}}$ are completely independent of each other. Consequently, we have two independent species of creation and annihilation operators, *i.e.* for each mode $\mathbf{p}$ we have independent operators

$$\begin{aligned} \hat{a}_{\mathbf{p}} &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{\mathbf{p}} + i\hat{\Pi}_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \qquad & \hat{a}^\dagger_{\mathbf{p}} &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}^\dagger_{\mathbf{p}} - i\hat{\Pi}^\dagger_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \\ &\text{and} \\ \hat{b}_{\mathbf{p}} &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}^\dagger_{-\mathbf{p}} + i\hat{\Pi}^\dagger_{-\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \qquad & \hat{b}^\dagger_{\mathbf{p}} &\stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{-\mathbf{p}} - i\hat{\Pi}_{-\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \end{aligned} \tag{15}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

(b) Verify the bosonic commutation relations (at equal times) between the annihilation operators $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}$ and the corresponding creation operators $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{b}^\dagger_{\mathbf{p}}$.

(c) Show that the Hamiltonian of the free charged fields is

$$\hat{H} \;=\; \int d^3\mathbf{x} \left(\Pi^\dagger\, \Pi \,+\, \nabla\Phi^\dagger \cdot \nabla\Phi \,+\, m^2 \Phi^\dagger\, \Phi \right) \;=\; \sum_{\mathbf{p}} \left(E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \,+\, E_{\mathbf{p}} \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \right) + \text{const.} \tag{16}$$

Next, consider the charge operator $\hat{Q} = \int d^3\mathbf{x}\, \hat{J}_0(\mathbf{x})$.

(d) Show that for the system at hand

$$\hat{Q} \;=\; \int d^3\mathbf{x} \left(\tfrac{i}{2}\{\hat{\Pi}^\dagger, \hat{\Phi}\} - \tfrac{i}{2}\{\hat{\Pi}, \hat{\Phi}^\dagger\} \right) \;=\; \sum_{\mathbf{p}} \left(\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \,-\, \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \right). \tag{17}$$

Actually, the classical formula (11) for the current $J_\mu(x)$ determines eq. (17) only up to ordering of the non-commuting operators $\hat{\Pi}(\mathbf{x})$ and $\hat{\Phi}^\dagger(\mathbf{x})$ (and likewise of the $\hat{\Pi}^\dagger(\mathbf{x})$ and $\hat{\Phi}(\mathbf{x})$). The anti-commutators in eq. (17) provide a solution to this ordering ambiguity, but any other ordering would be just as legitimate. The net effect of changing operator ordering in $\hat{J}_0$ amounts to changing the total charge $\hat{Q}$ by an infinite constant (prove this!). The specific ordering in eq. (17) provides for the neutrality of the vacuum state.

Finally, consider the stress-energy tensor of the charged field. Classically, Noether theorem gives

$$T^{\mu\nu} \;=\; \partial^\mu\Phi^*\,\partial^\nu\Phi \;+\; \partial^\mu\Phi\,\partial^\nu\Phi^* \;-\; g^{\mu\nu}\mathcal{L}. \tag{18}$$

Quantization of this formula is straightforward (modulo ordering ambiguity); for example, $\hat{\mathcal{H}} \equiv \hat{T}^{00}$ is precisely the integrand on the right hand side of eq. (13).

(e) Show that the total mechanical momentum operator of the fields is

$$\hat{\mathbf{P}}_{\text{mech}} \stackrel{\text{def}}{=} \int d^3\mathbf{x}\,\hat{T}^{0,\mathbf{i}} \;=\; \sum_{\mathbf{p}} \left(\mathbf{p}\,\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;+\; \mathbf{p}\,\hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right) \tag{19}$$

Physically, eqs. (19), (16) and (17) show that a complex field $\Phi(x)$ describes a relativistic particle together with its antiparticle; they have exactly the same rest mass m but exactly opposite charges ± 1.