1. An operator acting on identical bosons can be described in terms of N–particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

   In class, we defined the creation and annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$ in the occupation number basis according to

$$\hat{a}^\dagger_\alpha \left|\{n_\beta\}\right\rangle \;=\; \sqrt{n_\alpha+1}\,\left|\{n'_\beta = n_\beta + \delta_{\alpha\beta}\}\right\rangle, \tag{1}$$

$$\hat{a}_\alpha \left|\{n_\beta\}\right\rangle \;=\; \sqrt{n_\alpha}\,\left|\{n'_\beta = n_\beta - \delta_{\alpha\beta}\}\right\rangle \quad (\text{or } 0 \text{ when } n_\alpha = 0). \tag{2}$$

   We also wrote the wave functions of the $\left|\{n_\beta\}\right\rangle$ states: Let $N = \sum_\beta n_\beta$ is the number of particles and let $\left|\alpha_1,\ldots,\alpha_N\right\rangle = \left|\{n_\beta\}\right\rangle$; then

$$\varphi_{\alpha_1,\ldots,\alpha_N}(\mathbf{x}_1,\ldots,\mathbf{x}_N) \;=\; \frac{1}{\sqrt{C_{\alpha_1,\ldots,\alpha_N}}} \sum_{\substack{\text{distinct permutations}\\ (\tilde{\alpha}_1,\ldots,\tilde{\alpha}_N)\text{ of }(\alpha_1,\ldots,\alpha_N)}} \varphi_{\tilde{\alpha}_1}(\mathbf{x}_1)\cdots\varphi_{\tilde{\alpha}_N}(\mathbf{x}_N), \tag{3}$$

   where $C_{\alpha_1,\ldots,\alpha_N}$ is the number of distinct permutations.

   Our first task here is to derive the wave-function action of the creation and annihilation operators (1) and (2) using eq. (3).

   (a) Consider an N–particle state $|N,\Psi\rangle$ with a completely generic totally-symmetric wave function $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. Show that the $(N-1)$–particle state $|(N-1),\Psi'\rangle = \hat{a}_\gamma\,|N,\Psi\rangle$ has wave function

$$\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\mathbf{x}_N\,\varphi^*_\gamma(\mathbf{x}_N)\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{4}$$

   Hint: First verify this formula for Ψ of the form (3), and then generalize to arbitrary (but totally-symmetric) Ψ by linearity.

   (b) Next, show that the $(N+1)$–particle state $|(N+1),\Psi''\rangle = \hat{a}^\dagger_\gamma\,|N,\Psi\rangle$ has wave function

$$\Psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})) \;=\; \frac{1}{\sqrt{N+1}}\sum_{i=1}^{N+1}\varphi_\gamma(\mathbf{x}_i)\,\Psi(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots,\mathbf{x}_{N+1}). \tag{5}$$

   Hint: Use the fact that $\hat{a}^\dagger_\gamma$ is the hermitian conjugate of $\hat{a}_\gamma$.

Now consider a one-body operator $\hat{A}_1$. In the first-quantized formalism $\hat{A}_{\rm tot}$ acts on N–particle states according to

$$\hat{A}^{(1)}_{\rm tot} \;=\; \sum_{i=1}^{N} \hat{A}_1(i^{\underline{\rm th}} \text{ particle}) \tag{6}$$

while in the second-quantized formalism it becomes

$$\hat{A}^{(2)}_{\rm tot} \;=\; \sum_{\alpha,\beta} \langle\alpha|\, \hat{A}_1 \,|\beta\rangle \; \hat{a}^\dagger_\alpha \hat{a}_\beta \,. \tag{7}$$

(c) Use eq. (4) and/or eq. (5) to verify that for any two N–particle states $\langle N,\Psi_1|$ and $|N,\Psi_2\rangle$

$$\langle N,\Psi_1|\, \hat{A}^{(1)}_{\rm tot} \,|N,\Psi_2\rangle \;=\; \langle N,\Psi_1|\, \hat{A}^{(2)}_{\rm tot} \,|N,\Psi_2\rangle \,. \tag{8}$$

Hint: Use $\hat{A}_1 = \sum_{\alpha,\beta} |\alpha\rangle \langle\alpha|\, \hat{A}_1 \,|\beta\rangle \langle\beta|$.

Next, consider a two-body operator $\hat{B}_2$ which acts in the first-quantized formalism according to

$$\hat{B}^{(1)}_{\rm tot} \;=\; \tfrac{1}{2} \sum_{i\neq j} \hat{B}_2(i^{\underline{\rm th}} \text{ and } j^{\underline{\rm th}} \text{ particles}) \tag{9}$$

and in the second-quantized formalism according to

$$\hat{B}^{(2)}_{\rm tot} \;=\; \tfrac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} (\langle\alpha| \otimes \langle\beta|) \hat{B}_2 (|\gamma\rangle \otimes |\delta\rangle) \, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \,. \tag{10}$$

(d) Again, show that for any two N–particle states $\langle N,\Psi_1|$ and $|N,\Psi_2\rangle$

$$\langle N,\Psi_1|\, \hat{B}^{(1)}_{\rm tot} \,|N,\Psi_2\rangle \;=\; \langle N,\Psi_1|\, \hat{B}^{(2)}_{\rm tot} \,|N,\Psi_2\rangle \,. \tag{11}$$

2. Next, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; 0, \qquad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \qquad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta} \,. \tag{12}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$.

(b) Consider three one-body operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$. Let us define the corresponding second-quantized operators $\hat{A}^{(2)}_{\rm tot}$, $\hat{B}^{(2)}_{\rm tot}$, and $\hat{C}^{(2)}_{\rm tot}$ according to eq. (7).

Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C}^{(2)}_{\rm tot} = [\hat{A}^{(2)}_{\rm tot}, \hat{B}^{(2)}_{\rm tot}]$.

(c) Next, calculate the commutator $[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]$.

(d) Finally, let $\hat{A}_1$ be a one-body operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}^{(2)}_{\rm tot}$, $\hat{B}^{(2)}_{\rm tot}$, and $\hat{C}^{(2)}_{\rm tot}$ be the corresponding second-quantized operators according to eqs. (7) and (10).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\underline{\rm st}}) + \hat{A}_1(2^{\underline{\rm nd}})\right), \hat{B}_2\right]$ then $\hat{C}^{(2)}_{\rm tot} = [\hat{A}^{(2)}_{\rm tot}, \hat{B}^{(2)}_{\rm tot}]$.

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$.

(a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\rm def}{=} \exp\bigl(\xi \hat{a}^\dagger - \xi^* \hat{a}\bigr)|0\rangle$. Show that

$$|\xi\rangle \;=\; e^{-|\xi|^2/2}\, e^{\xi \hat{a}^\dagger}\, |0\rangle \quad \text{and} \quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,. \tag{13}$$

(b) Calculate the uncertainties Δq and Δp for a coherent state $|\xi\rangle$ and verify their minimality: $\Delta q \Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\rm def}{=} \langle \hat{n}\rangle = |\xi|^2$.

Hint: use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$.

(c) Consider time-dependent coherent states $|\xi(t)\rangle$. Show that for $\xi(t) = \xi_0 e^{-i\omega t}$, the state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}|\xi(t)\rangle = \hat{H}\,|\xi(t)\rangle$.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap $\langle\eta|\xi\rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{14}$$

for any given classical complex field $\Phi(\mathbf{x})$.

(f) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \stackrel{\text{def}}{=} \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int\! d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{15}$$

(g) Let

$$\hat{H} \;=\; \int\! d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger\cdot\nabla\hat{\Psi} \;+\; V(\mathbf{x})\hat{\Psi}^\dagger\hat{\Psi}\right)$$

and show that for any classical field configuration $\Phi(\mathbf{x},t)$ that satisfies the classical field equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{16}$$

(h) Finally, show that the quantum overlap $|\langle\Phi_1|\Phi_2\rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.